A geometry for the nuclear shell model

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Wigner-111, Budapest, November 2013

Independent-particle shell model

Independent motion of individual neutrons and protons in a mean-field potential.

Existence of shell structure with "magic numbers" 2, 8, 20, 28, 50, 82, 126 of increased stability.

Crucial ingredient: spin-orbit interaction (Fermi).

Nobel prize in 1963:

Mayer & Jensen: "…for their discoveries concerning shell structure."

Wigner: "…for his contributions to the theory of the atomic nucleus and the elementary particles…"

Nuclear shell model

Ingredients:

Mean-field potential.

Residual interaction between (some of) the nucleons.

Difficulties:

Nucleonic interactions from QCD (EFT).

Large-matrix diagonalization.

Issues of current interest:

Changing shell structure and three-body forces in exotic nuclei.

Continuum effects (nucleus = open quantum system).

Words of warning

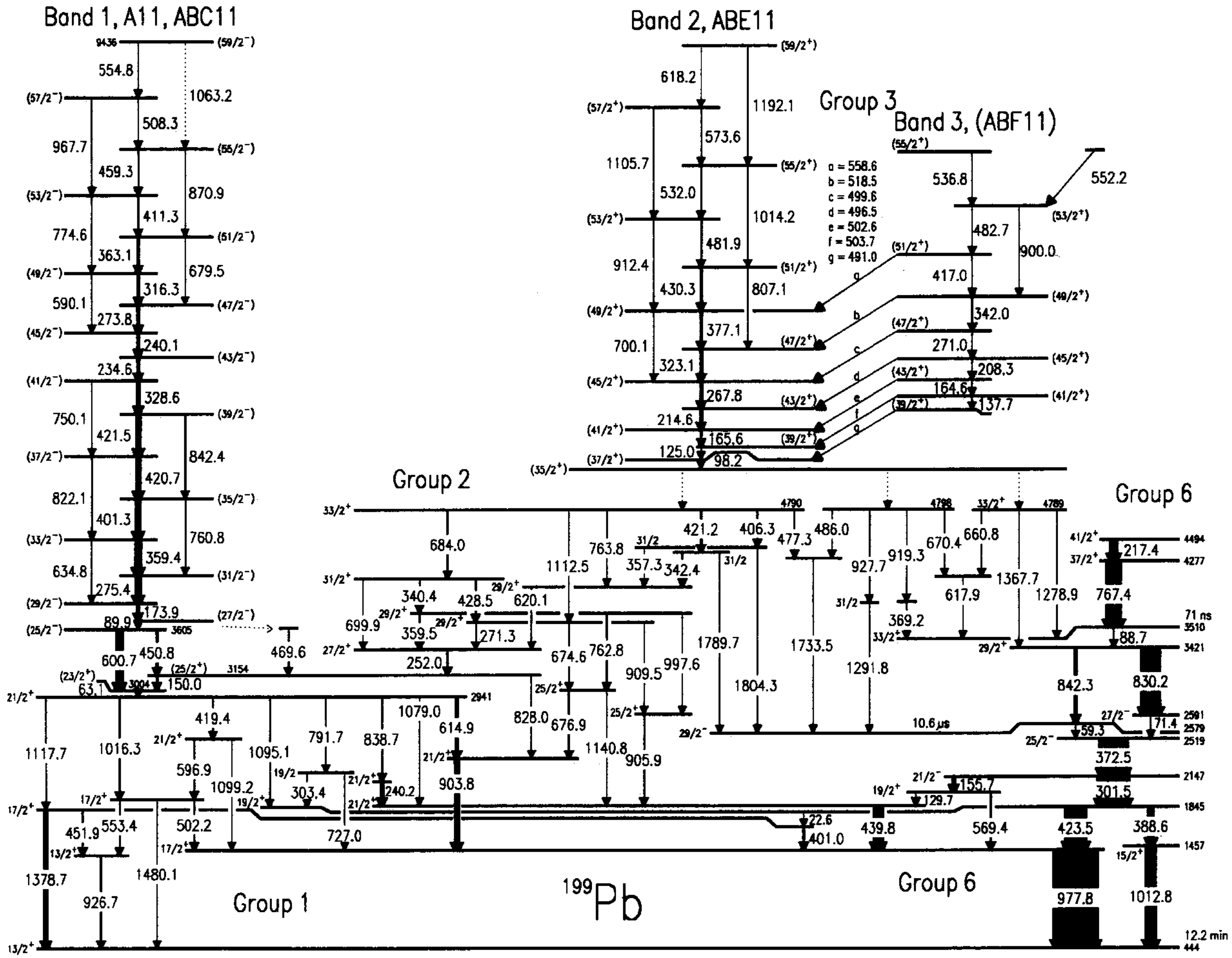
Bethe:

The complexity of the nuclear many-body problem is such that the shell-model wave functions cannot be the true eigenfunctions of the nuclear hamiltonian.

Wigner:

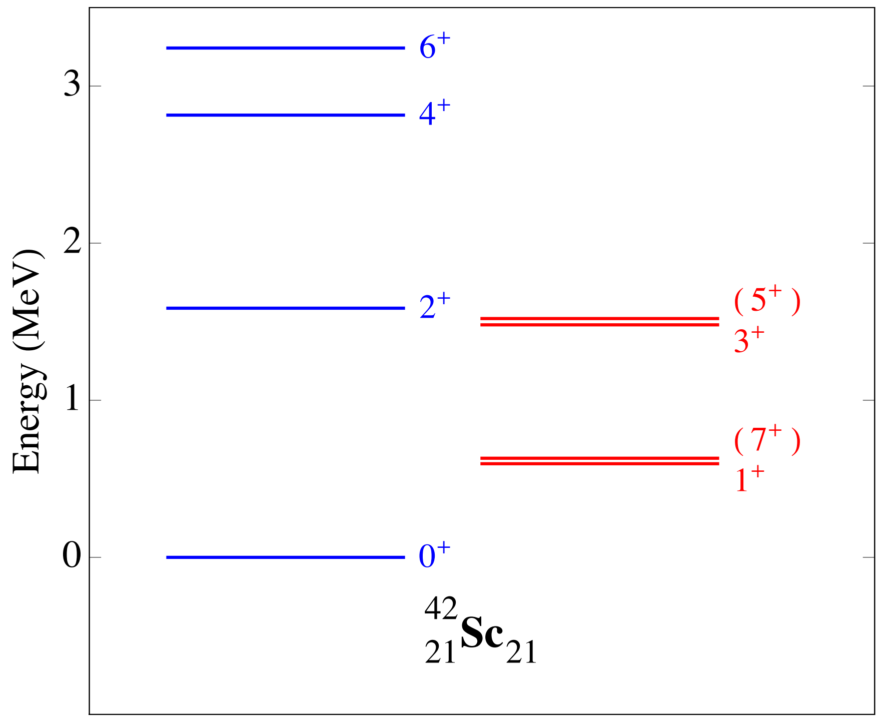
It is nice to know that the computer understands the problem. But I would like to understand it too.

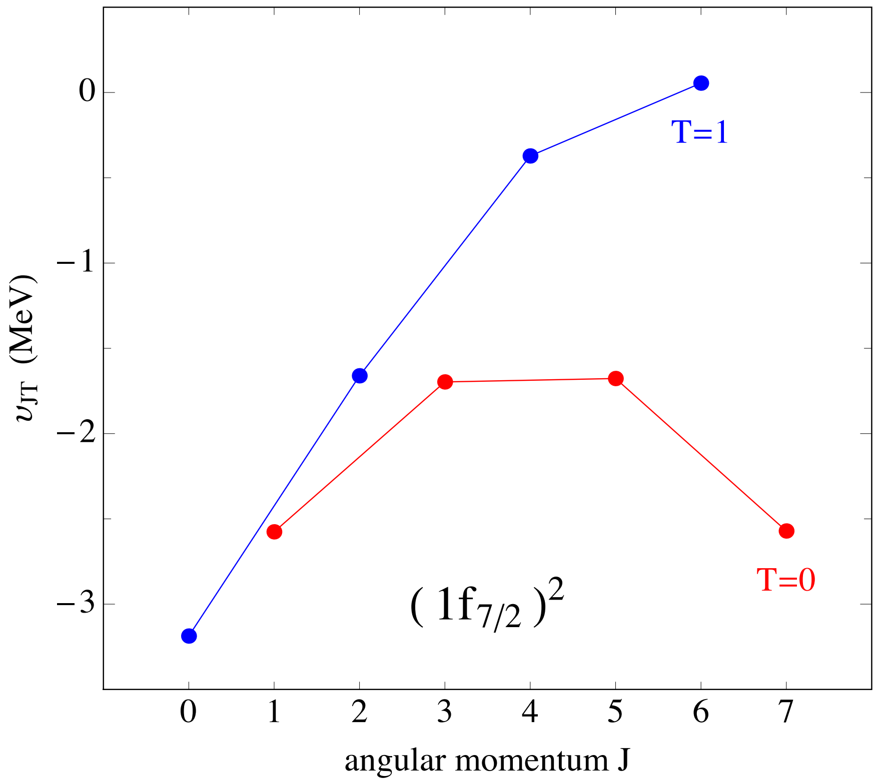
A complex nucleus: ^{199}Pb



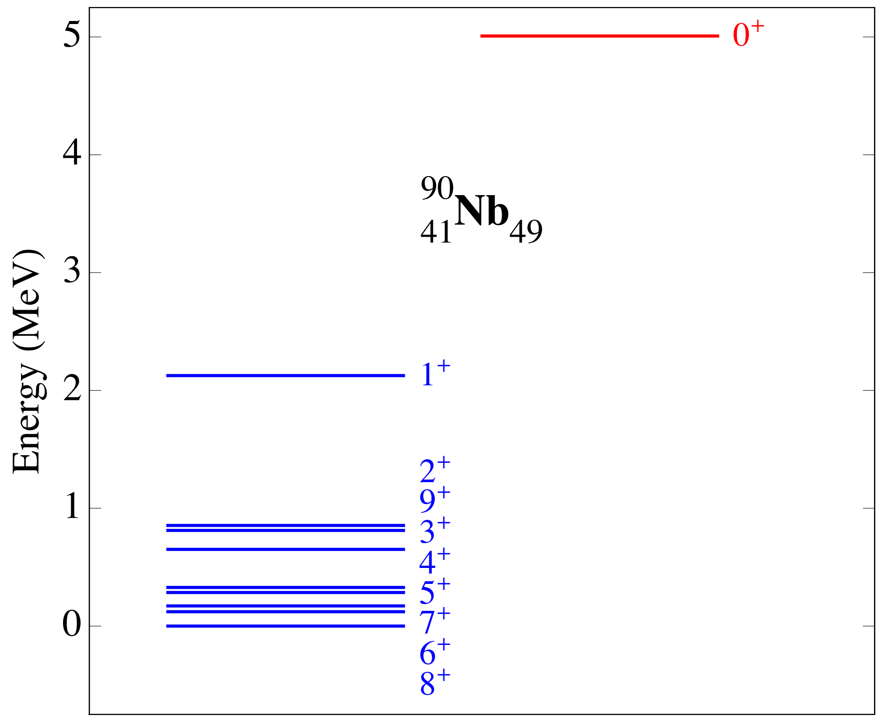
G. Baldsiefen *et al.*, Nucl. Phys. A **574** (1994) 521

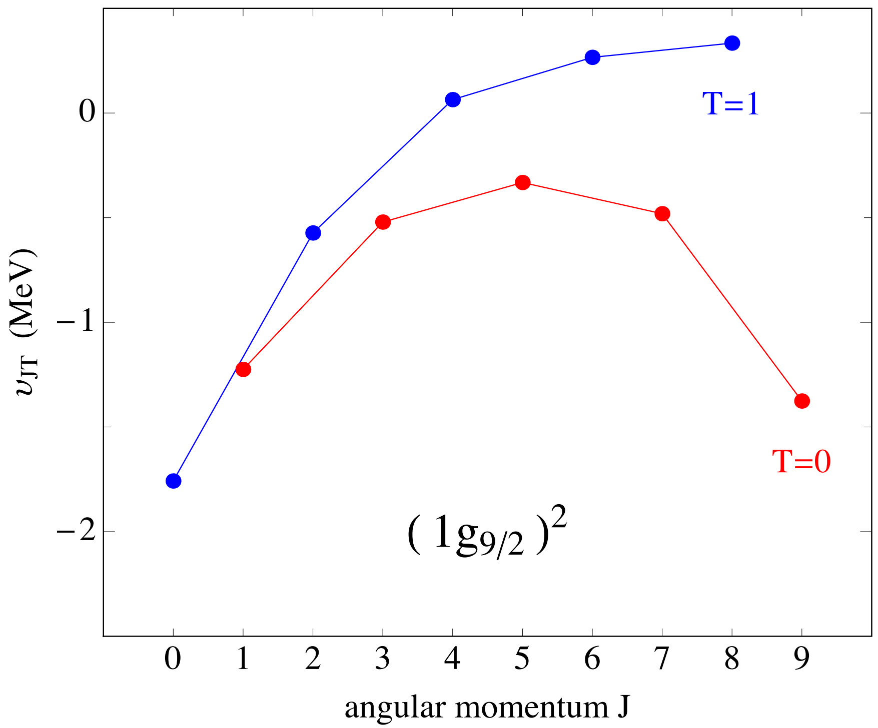
The effective nn interaction





The effective nn interaction





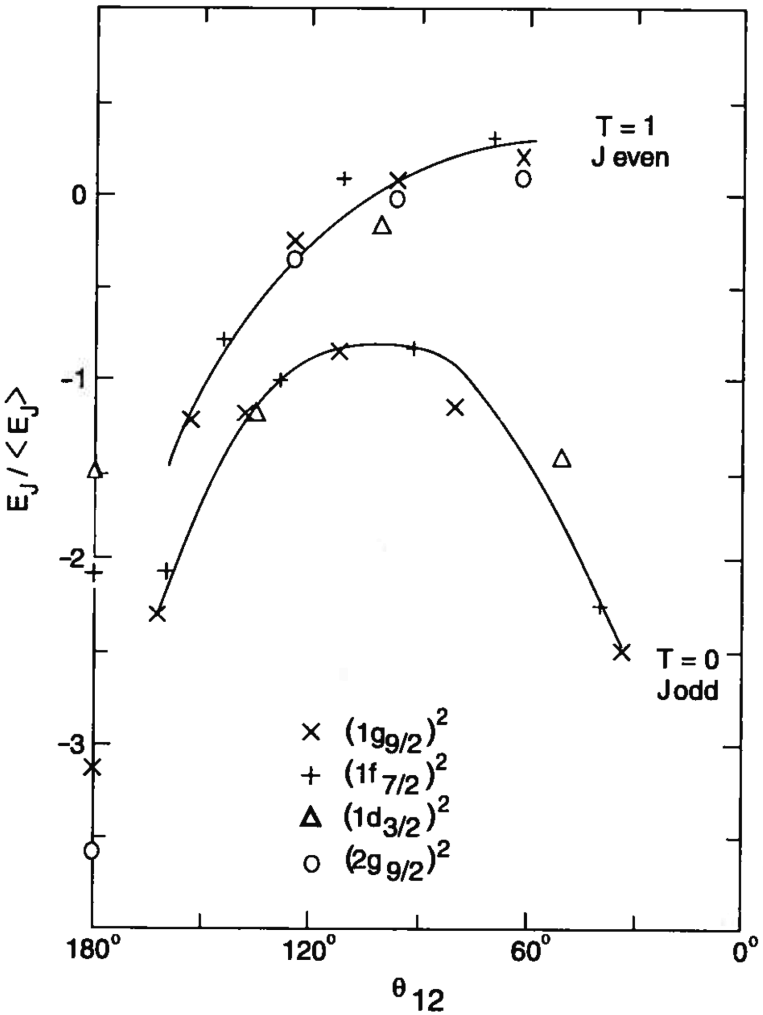
Geometric interpretation

Introduce the angle between the angular momentum vectors j_1 and j_2 of the two nucleons

$$\theta_{12} = \arccos \frac{J(J+1) - j_1(j_1+1) - j_2(j_2+1)}{2\sqrt{j_1(j_1+1)j_2(j_2+1)}}$$

The effective nn interaction can be represented as a "universal function" of θ_{12}.

Geometric interpretation



J.P. Schiffer & W.W. True, Rev. Mod. Phys. **48** (1976) 191

Short-range nn interaction

Delta interaction: $V(\bar{r}_1,\bar{r}_2) = a'_T \delta(\bar{r}_1 - \bar{r}_2)$

Its matrix elements are

$$\langle j_1 j_2; JT | a'_T \delta(\bar{r}_1 - \bar{r}_2) | j_1 j_2; JT \rangle / (2j_1+1)(2j_2+1)$$

$$= \frac{a_T}{2(1+\delta_{j_1 j_2})} \left[\alpha \begin{pmatrix} j_1 & j_2 & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2 + \beta \begin{pmatrix} j_1 & j_2 & J \\ 1/2 & 1/2 & -1 \end{pmatrix}^2 \right]$$

with

$$\alpha = 1 - (-)^{\ell_1+\ell_2+J+T}, \quad \beta = 1 + (-)^T$$

$$a_T = \frac{a'_T}{4\pi} \int R^4_{nl}(r) r^2 dr$$

I. Talmi, *Simple Models of Complex Nuclei* (1993)

Classical (large *j*) limit

Use Wigner's results:

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \approx \frac{1}{4\pi A} = \frac{1}{2\pi \ell_1 \ell_2 \sin\theta_{12}}$$

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}^2 \approx \frac{1}{24\pi V}$$

where *A* is the area of a triangle and *V* the volume of a tetrahedron with sides of length l_i+1/2.

E.P. Wigner, *Group Theory* (1959)
G. Ponzano & T. Regge, *Group Theoretical Methods in Physics* (1968)

Classical (large *j*) limit

The matrix element of the delta interaction in the large j limit equals

$$\langle j_1 j_2;JT|a'_T\delta(\overline{r}_1-\overline{r}_2)|j_1 j_2;JT\rangle \approx \frac{s_1}{\pi\sin\theta_{12}}+\frac{t_1}{\pi\tan\theta_{12}}$$

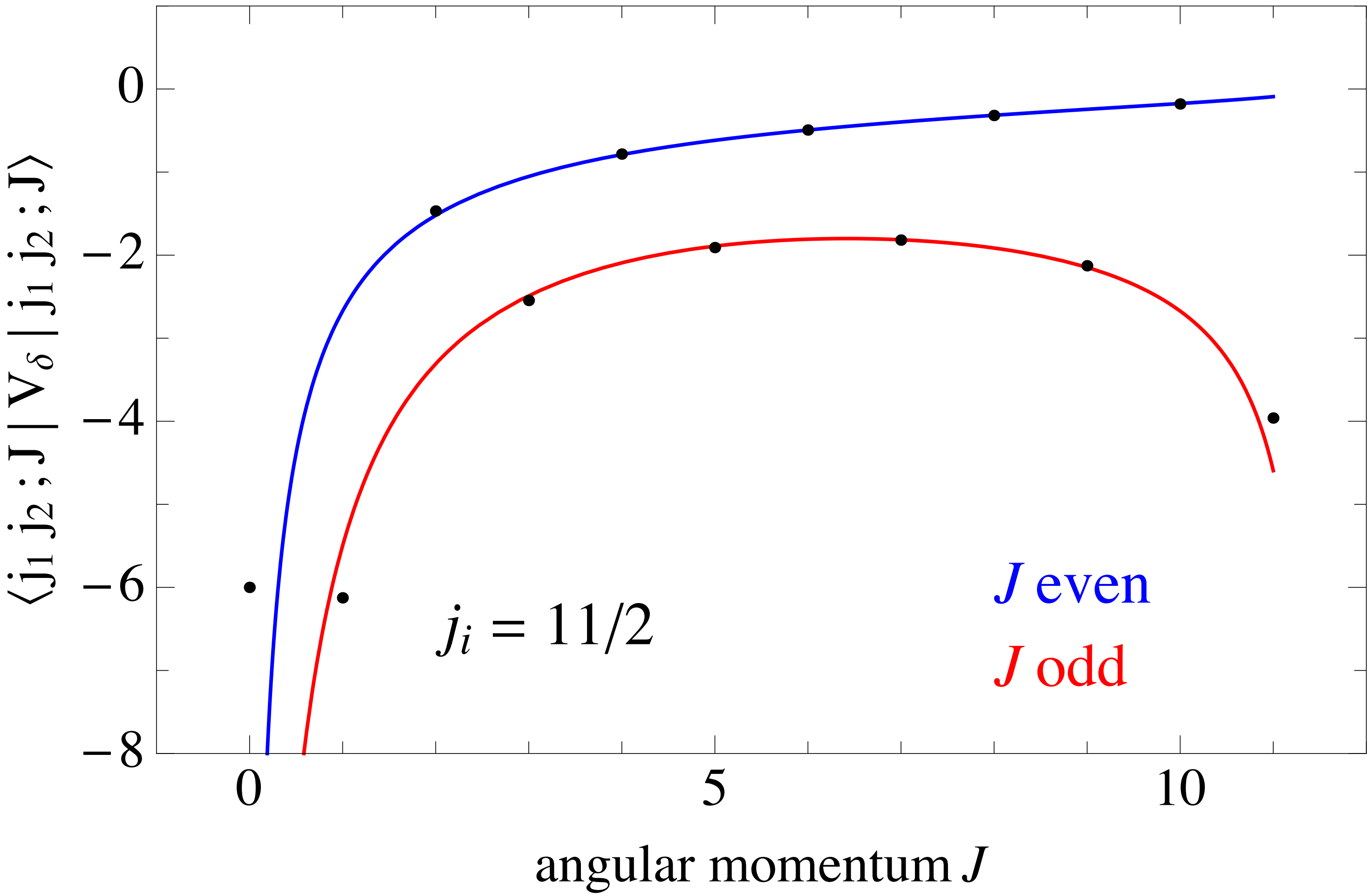
with

$$s_1=\frac{a_T}{1+\delta_{j_1j_2}}\left[\left(1-(-)^{\ell_1+\ell_2+J+T}\right)+\left(1+(-)^T\right)\left(1+(-)^{j_1+j_2+J}\right)\right]$$

$$t_1=\frac{a_T}{1+\delta_{j_1j_2}}(-)^{j_1+j_2+J}\left(1-(-)^{\ell_1+\ell_2+J+T}\right)$$

A. Molinari *et al.*, Nucl. Phys. A **239** (1975) 45

Classical (large j) limit



Generalized geometry

Assume two neutrons and two protons:

$$\left|j_\nu j'_\nu\left(J_\nu\right)j_\pi j'_\pi\left(J_\pi\right);J\right\rangle \equiv \left|J_\nu J_\pi;J\right\rangle$$

How do the energies depend on J, on the angular momenta of the single-particle orbits and on the angular momenta of the 'blades'?

Take $j_\nu=j'_\nu$ and $j_\pi=j'_\pi$.

The pp or hh matrix element

Consider a hamiltonian of the generic form

$$\hat{H} = \hat{H}_\nu + \hat{H}_\pi + \hat{V}_{\nu\pi}$$

The relative energies depend only on the neutron-proton interaction:

$$\frac{\langle J_\nu J_\pi;J|\hat{V}_{\nu\pi}|J_\nu J_\pi;J\rangle}{(2J_\nu+1)(2J_\pi+1)} = \frac{\langle J_\nu^{-1} J_\pi^{-1};J|\hat{V}_{\nu\pi}|J_\nu^{-1} J_\pi^{-1};J\rangle}{(2J_\nu+1)(2J_\pi+1)}$$

$$= 4\sum_R (2R+1) V^R_{j_\nu j_\pi, j_\nu j_\pi} \begin{bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ R & j_\pi & J & j_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{bmatrix}$$

The ph matrix element

The corresponding matrix element for a particle-hole configuration:

$$\frac{\left\langle J_\nu J_\pi^{-1};J\left|\hat{V}_{\nu\pi}\right|J_\nu J_\pi^{-1};J\right\rangle}{\left(2J_\nu+1\right)\left(2J_\pi+1\right)}$$

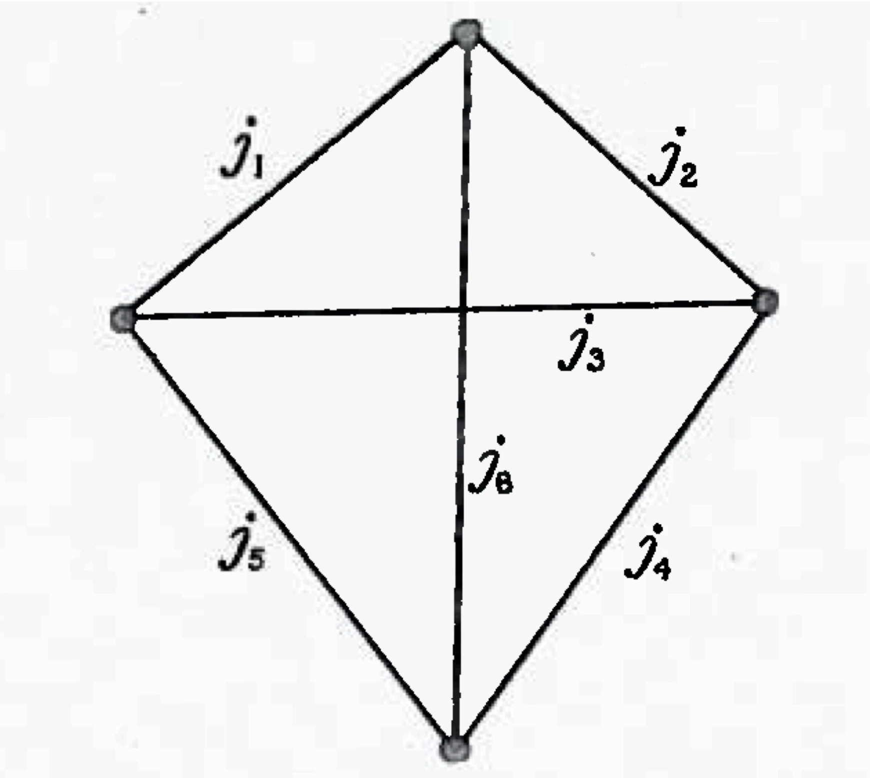
$$=-4\sum_R\left(2R+1\right)V^R_{j_\nu j_\pi,j_\nu j_\pi}\left\{\begin{array}{cccc} j_\nu & j_\pi & J_\pi & J_\nu \\ R & j_\pi & J & j_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{array}\right\}$$

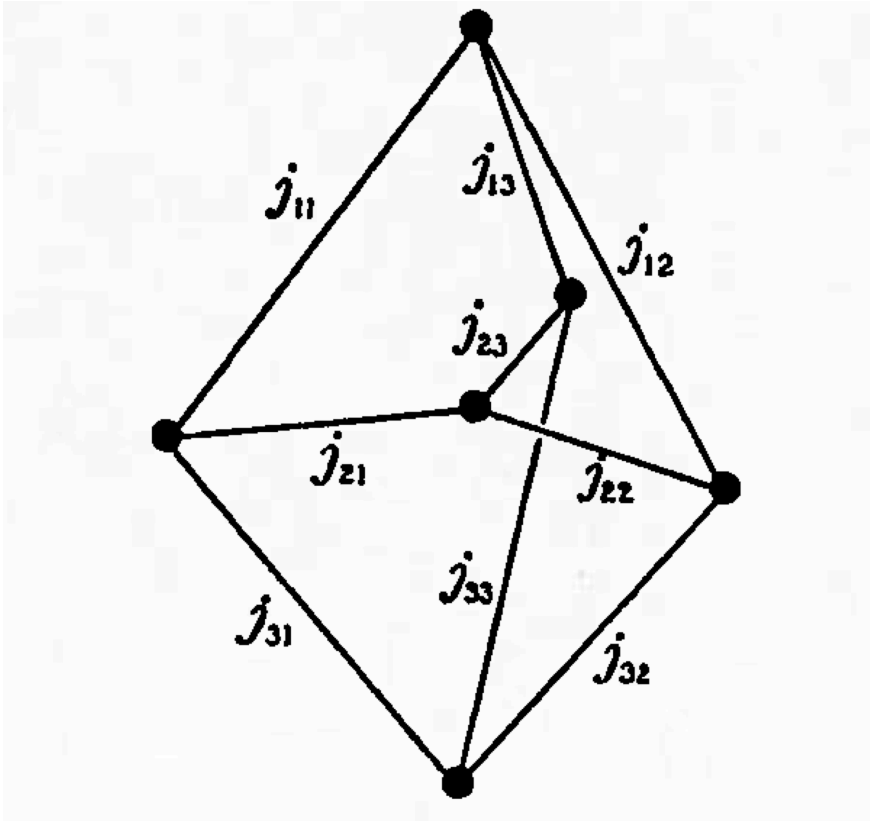
The large-j limit

Semi-classical expressions are known for Wigner ($3j$) and Racah ($6j$) coefficients but not for $3nj$ coefficients with $n > 2$.

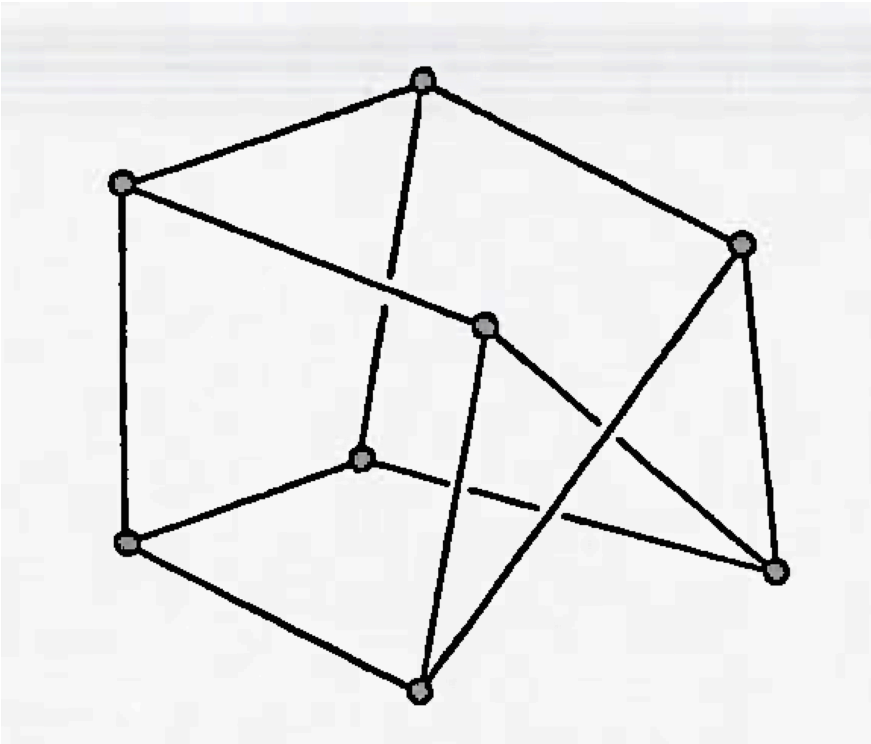
A field of active mathematical research with connections to graph theory, quantum gravity, spin networks…

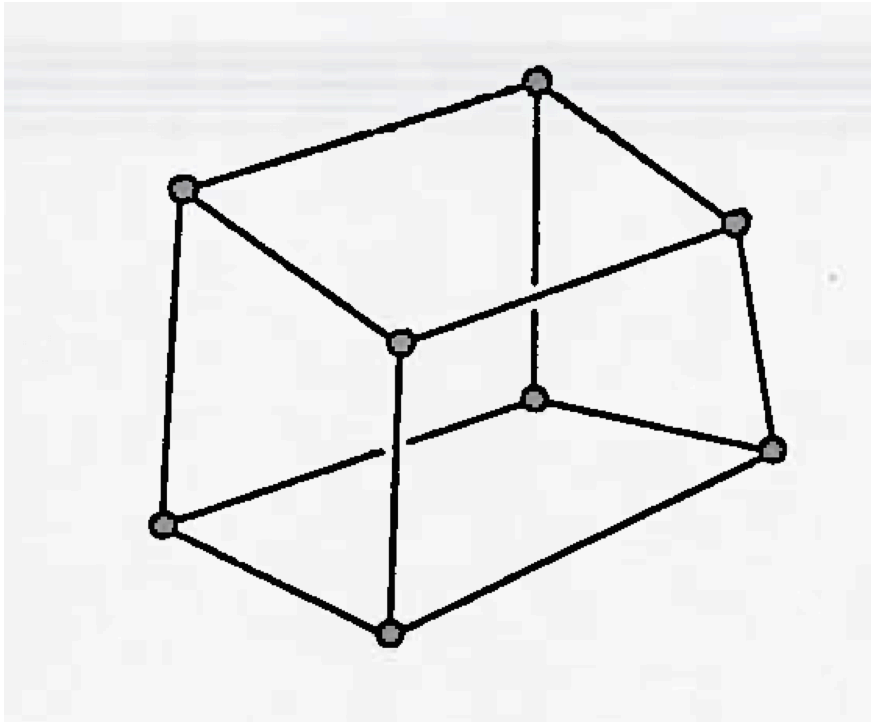
$3nj$ coefficients as graphs





Two kinds of $12j$ symbols





A simple sum

An exact result:

$$\sum_R (2R+1) \begin{Bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ R & j_\pi & J & j_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{Bmatrix} = \frac{1}{(2J_\nu+1)(2J_\pi+1)}$$

A.P. Yutsis *et al.*, *The Theory of Angular Momentum* (1962)

A more complicated sum (1)

An exact result:

$$S_n = \sum_R (2R+1) \begin{pmatrix} j_\nu & j_\pi & R \\ 1/2 & n-1/2 & -n \end{pmatrix}^2 \begin{Bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ R & j_\pi & J & j_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{Bmatrix}$$

$$= \sum_{\substack{m_\nu M_\nu \\ m_\pi M_\pi}} \begin{pmatrix} j_\nu & j_\nu & J_\nu \\ 1/2 & m_\nu & M_\nu \end{pmatrix}^2 \begin{pmatrix} j_\pi & j_\pi & J_\pi \\ -n+1/2 & m_\pi & M_\pi \end{pmatrix}^2 \begin{pmatrix} J_\nu & J_\pi & J \\ M_\nu & M_\pi & M \end{pmatrix}^2$$

A more complicated sum (2)

Wigner's classical approximation:

$$\begin{pmatrix} J_\nu & J_\pi & J \\ M_\nu & M_\pi & M \end{pmatrix}^2 \mapsto \frac{1}{4\pi A}$$

Therefore

$$S_n \approx \frac{1}{4\pi A} \sum \begin{pmatrix} j_\nu & j_\nu & J_\nu \\ 1/2 & m_\nu & M_\nu \end{pmatrix}^2 \begin{pmatrix} j_\pi & j_\pi & J_\pi \\ -n + 1/2 & m_\pi & M_\pi \end{pmatrix}^2$$

$$\approx \frac{1}{4\pi \left(2j_\nu + 1\right)\left(2j_\pi + 1\right) A}$$

$$\approx \frac{2}{\pi \left(2j_\nu + 1\right)\left(2j_\pi + 1\right)\left(2J_\nu + 1\right)\left(2J_\pi + 1\right)\sin\theta_{\nu\pi}}$$

Another sum

Another approximate result:

$$\bar{S}_0 = \sum_R (-)^R (2R+1) \begin{pmatrix} j_\nu & j_\pi & R \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 \begin{Bmatrix} j_\nu & j_\pi & J_\pi & J_\nu \\ R & j_\pi & J & j_\nu \\ j_\nu & j_\pi & J_\pi & J_\nu \end{Bmatrix}$$

$$\approx -(-)^{j_\nu + j_\pi} \frac{2}{\pi(2j_\nu+1)(2j_\pi+1)(2J_\nu+1)(2J_\pi+1)\tan\theta_{\nu\pi}}$$

Classical 2p-2h matrix element

We obtain for a delta interaction the following classical approximation:

$$\left\langle J_\nu J_\pi^{-1}; J \left| a_T' \delta\left(\overline{r}_1 - \overline{r}_2\right) \right| J_\nu J_\pi^{-1}; J \right\rangle \approx \frac{s_2}{\pi \sin\theta_{\nu\pi}} + \frac{t_2}{\pi \tan\theta_{\nu\pi}}$$

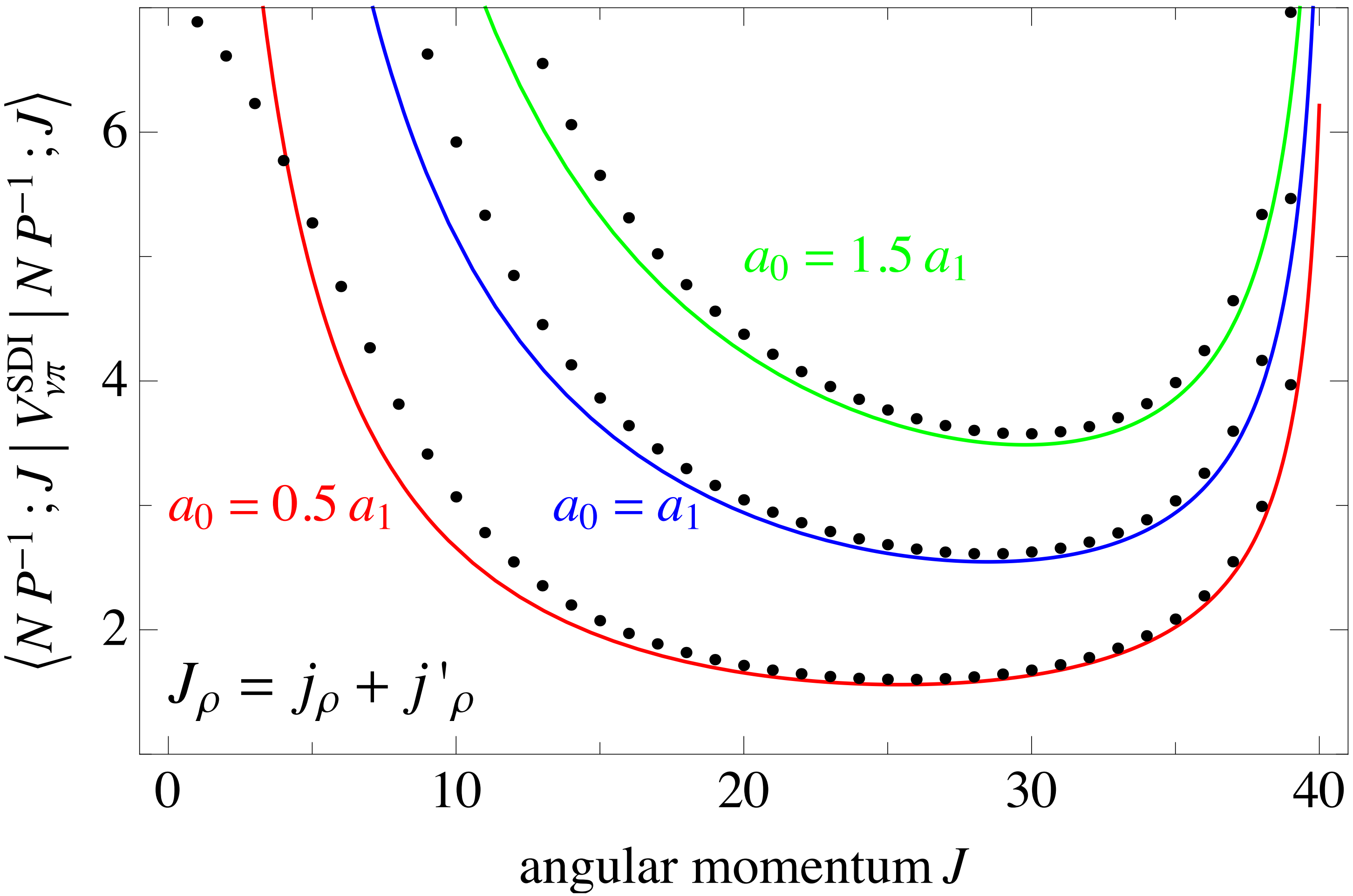
with

$$s_2 = 2\left(3a_0 + a_1\right), \quad t_2 = 2\left(a_0 - a_1\right)\varphi$$

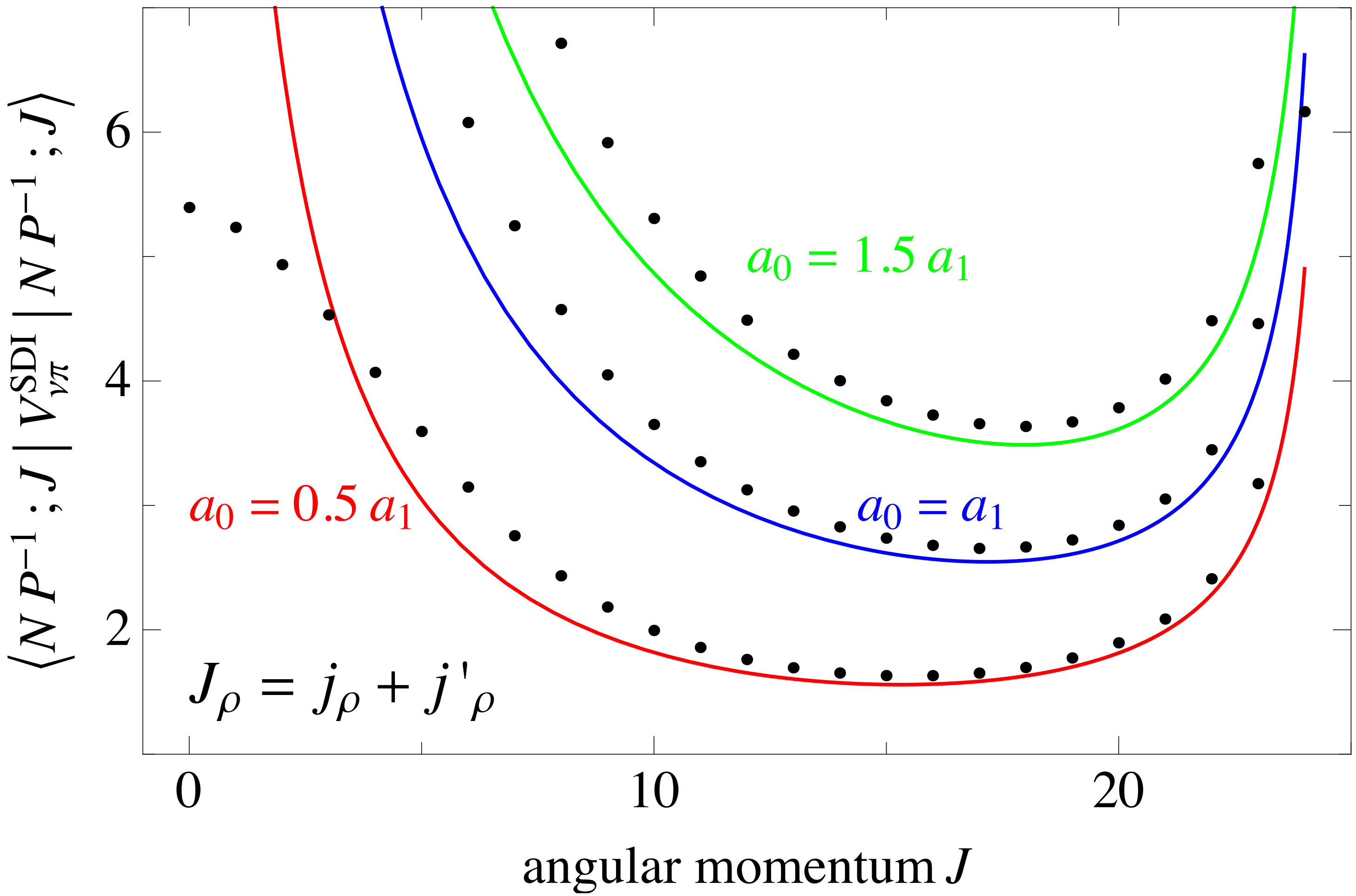
$$\varphi = \frac{1}{4}\left(\varphi_\nu \varphi_\pi + \varphi_\nu \varphi_\pi' + \varphi_\nu' \varphi_\pi + \varphi_\nu' \varphi_\pi'\right)$$

$$\varphi_\rho = \left(-\right)^{\ell_\rho + j_\rho}, \quad \varphi_\rho' = \left(-\right)^{\ell'_\rho + j'_\rho}$$

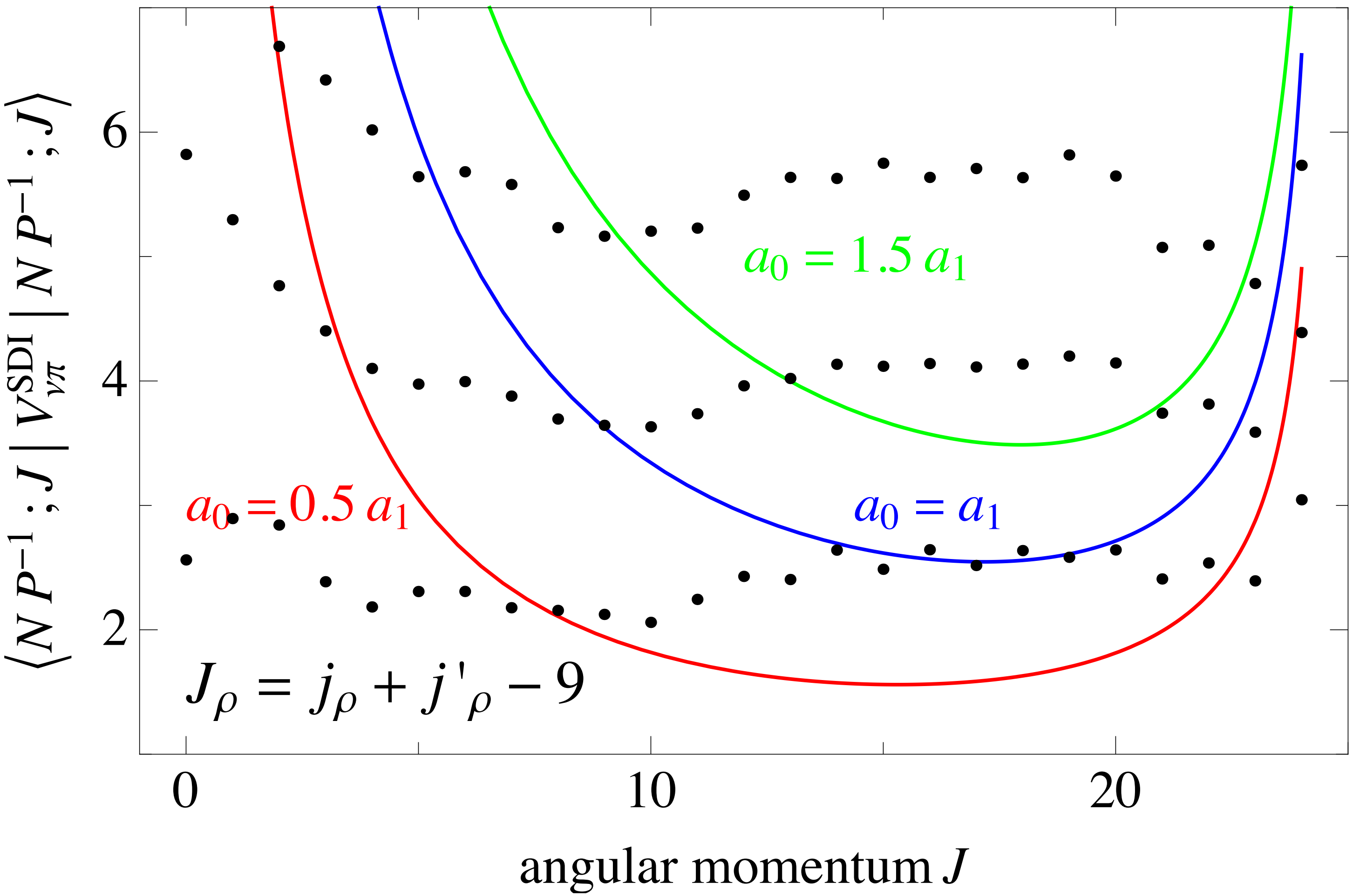
j_ϱ=19/2 & j'_ϱ=21/2 & J_ϱ=20



j_ϱ=11/2 & j'_ϱ=13/2 & J_ϱ=12



j_ϱ=21/2 & j'_ϱ=21/2 & J_ϱ=12



Wigner-111, Budapest, November 2013

Conjecture

Assume *near-aligned* neutrons and *near-aligned* protons:

$$\left|J_\nu\right\rangle \equiv \left|j_\nu j'_\nu j''_\nu \ldots; J_\nu\right\rangle \quad \& \quad \left|J_\pi^{-1}\right\rangle \equiv \left|j_\pi^{-1} j'^{-1}_\pi j''^{-1}_\pi \ldots; J_\pi\right\rangle$$

The nuclear force has an interaction energy in the coupled state which can be approximated as

$$\left\langle J_\nu J_\pi^{-1}; J \left| a'_T \delta(\bar{r}_1 - \bar{r}_2) \right| J_\nu J_\pi^{-1}; J \right\rangle \approx \frac{s_k}{\pi \sin\theta_{\nu\pi}} + \frac{t_k}{\pi \tan\theta_{\nu\pi}}$$

The coefficients s_k and t_k depend on the isoscalar and isovector interaction strengths.

Conclusions and outlook

The geometry of the effective nn interaction is generalized to more complex configurations.

This idea can be applied to shears-band states in nuclei (e.g. ^{199}Pb).

Outlook:

Proof of the np-nh conjecture.

Analysis of other interactions (tensor…).

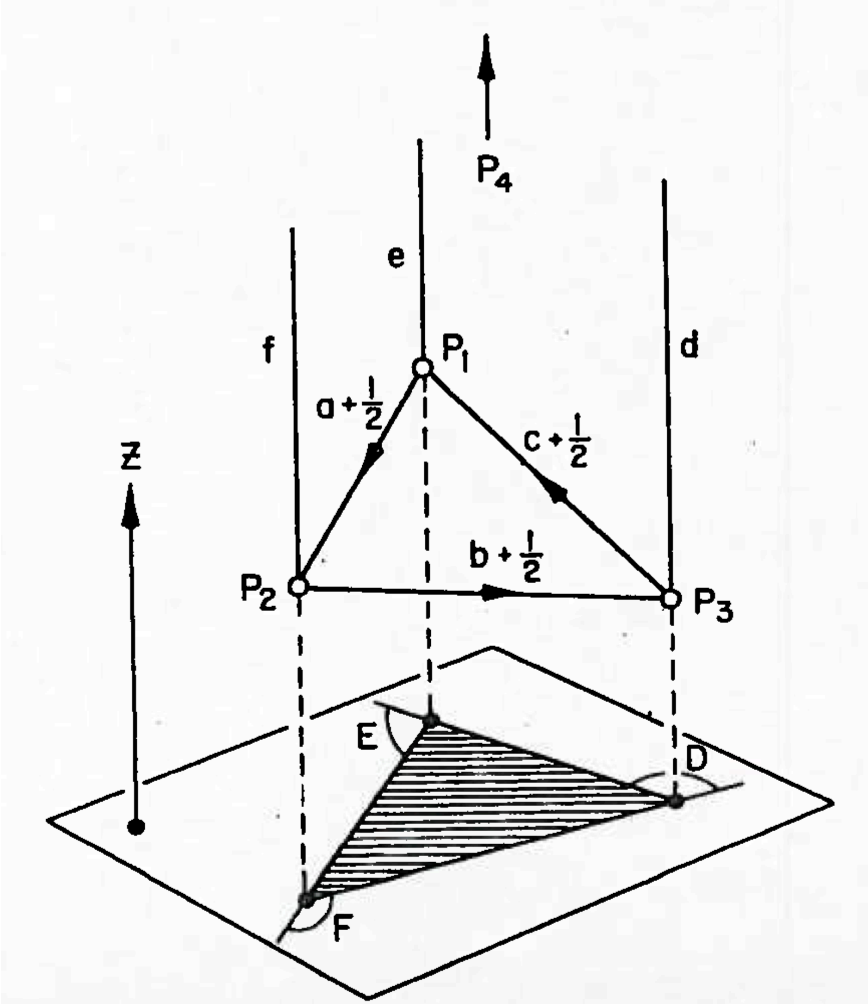
Treatment of mixed configurations.

P. Van Isacker & A.O. Macchiavelli, Phys. Rev. C **87** (2013) 061301(R)

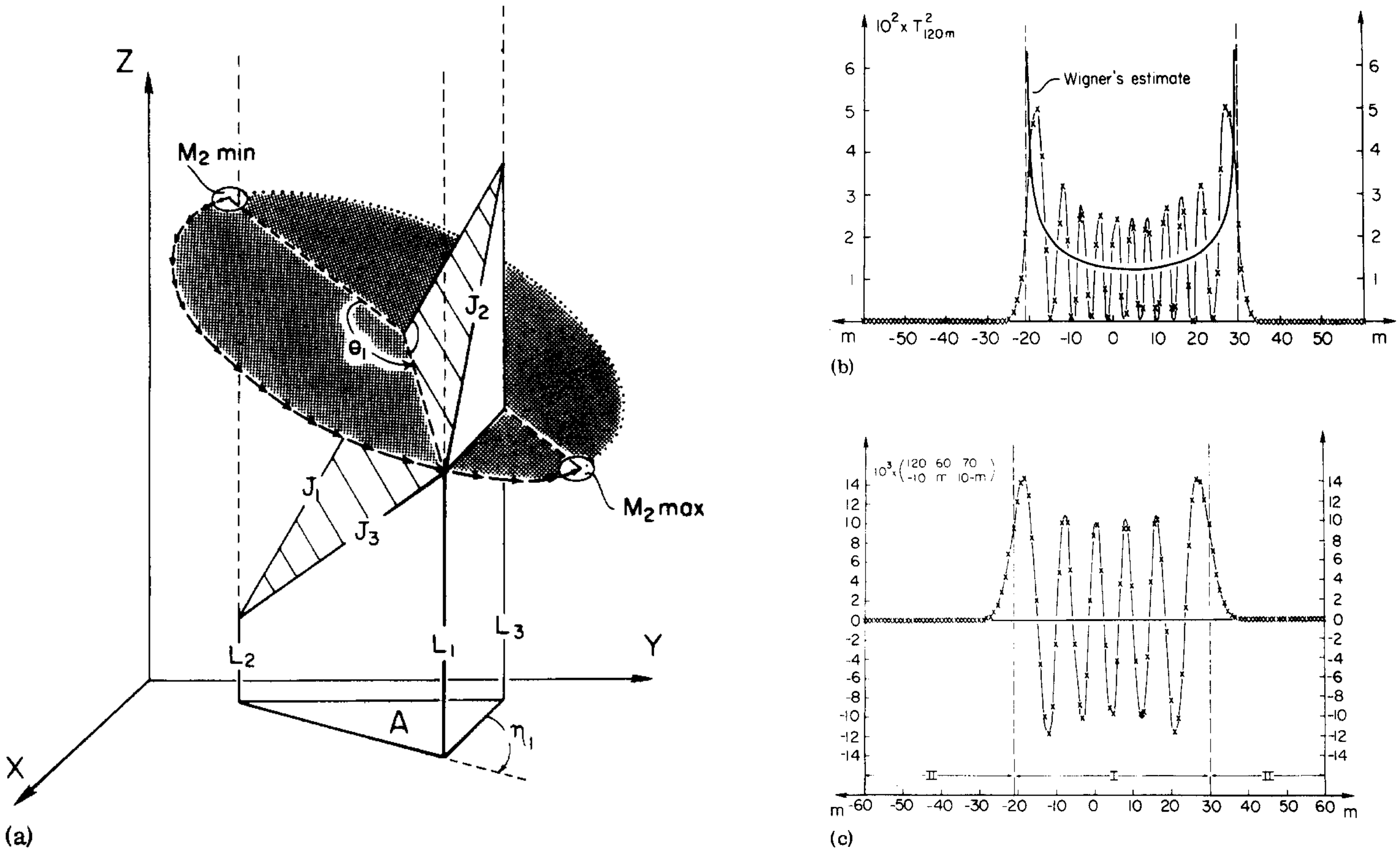
$3j$ as the limit of $6j$

The asymptotic formula:

$$\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} \xrightarrow{R\to\infty} \frac{(-)^{a+b+c+2(d+e+f)}}{\sqrt{2R}} \times \begin{pmatrix} a & b & c \\ e-f & f-d & d-e \end{pmatrix}$$



(Semi-)classical approximations



K. Schulten & R.G. Gordon, J. Math. Phys. **16** (1975) 1961 & 1971

Matrix elements of MSDI

Modified surface delta interaction:

$$\hat{V}^{\mathrm{MSDI}}(i,j) = -4\pi a'_T \delta(\bar{r}_i - \bar{r}_j)\delta(r_i - R_0) + b'\bar{\tau}_i \cdot \bar{\tau}_j + c'$$

Its matrix elements are

$$-\frac{(2j_\nu+1)(2j_\pi+1)}{2}\left[a_{01}\begin{pmatrix} j_\nu & j_\pi & R \\ 1/2 & -1/2 & 0 \end{pmatrix}^2 + a_0\begin{pmatrix} j_\nu & j_\pi & R \\ 1/2 & 1/2 & -1 \end{pmatrix}^2\right] - b + c$$

with

$$a_{01} = \frac{a_0 + a_1}{2} - (-)^{\ell_\nu + \ell_\pi + R}\frac{a_0 - a_1}{2}$$

$$a_T = a'_T C(R_0), \quad b = b'C(R_0), \quad c = c'C(R_0)$$

P.J. Brussaard & P.W.M. Glaudemans, *Shell-Model Applications* (1977)

1p-1h matrix element

Recall the well-known classical interpretation of a short-range nuclear matrix element.

For MSDI:

$$\left\langle j_\nu j_\pi^{-1};J\left|\hat{V}_{\nu\pi}^{\mathrm{MSDI}}\right|j_\nu j_\pi^{-1};J\right\rangle \approx (b-c)+\frac{\alpha_{\mathrm{s}}}{2\pi\sin\theta_{\nu\pi}}+\frac{\alpha_{\mathrm{t}}}{2\pi\tan\theta_{\nu\pi}}$$

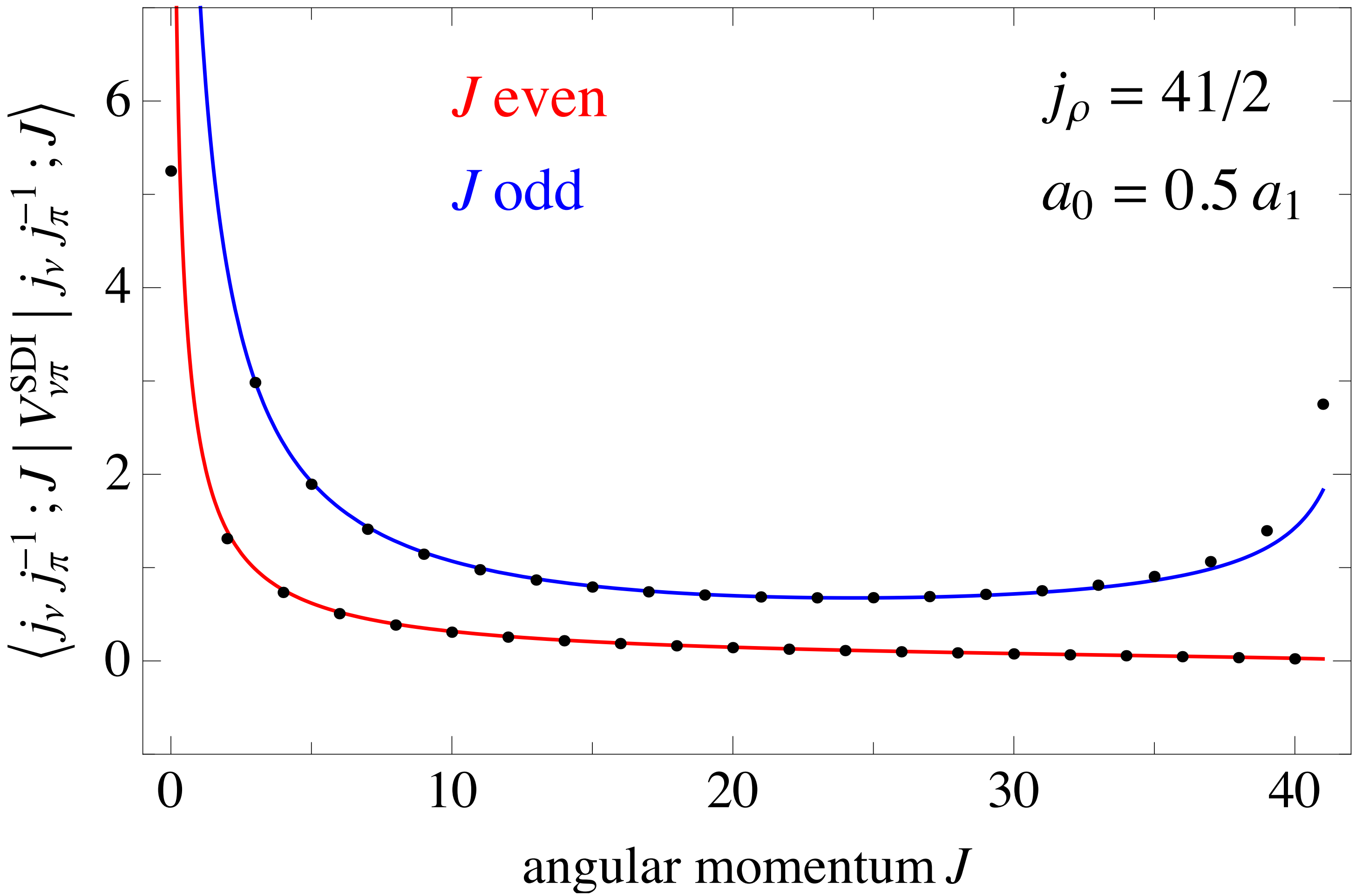
with

$$\alpha_{\mathrm{s}}=(a_0+a_1)\left[1+(-)^{j_\nu+j_\pi+J}\right]+2a_0+(-)^{\ell_\nu+\ell_\pi+J}(a_0-a_1)$$

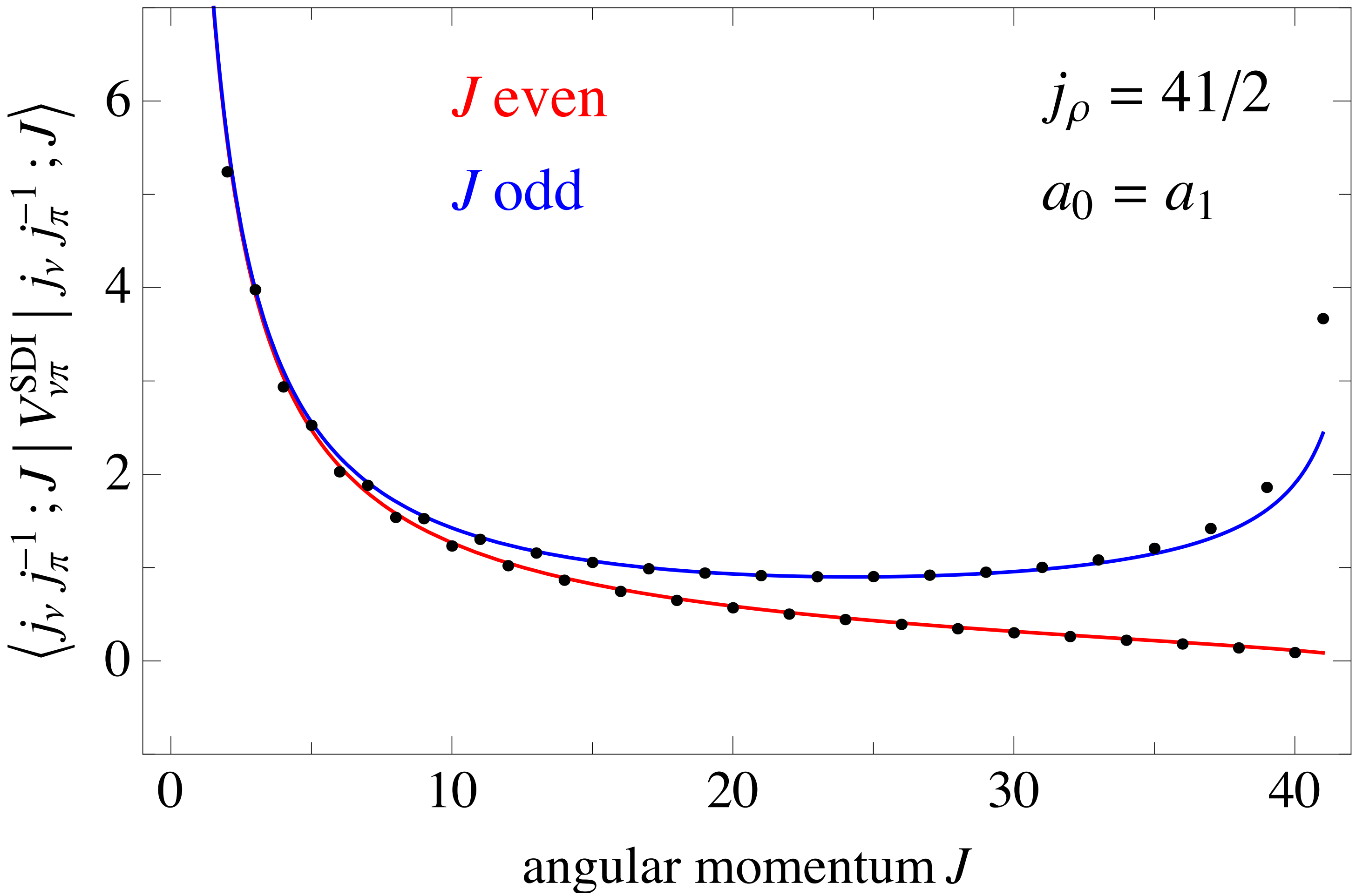
$$\alpha_{\mathrm{t}}=2(-)^{j_\nu+j_\pi+J}a_0+(-)^{\ell_\nu+\ell_\pi+j_\nu+j_\pi}(a_0-a_1)$$

J.P. Schiffer & W.W. True, Rev. Mod. Phys. **48** (1976) 191

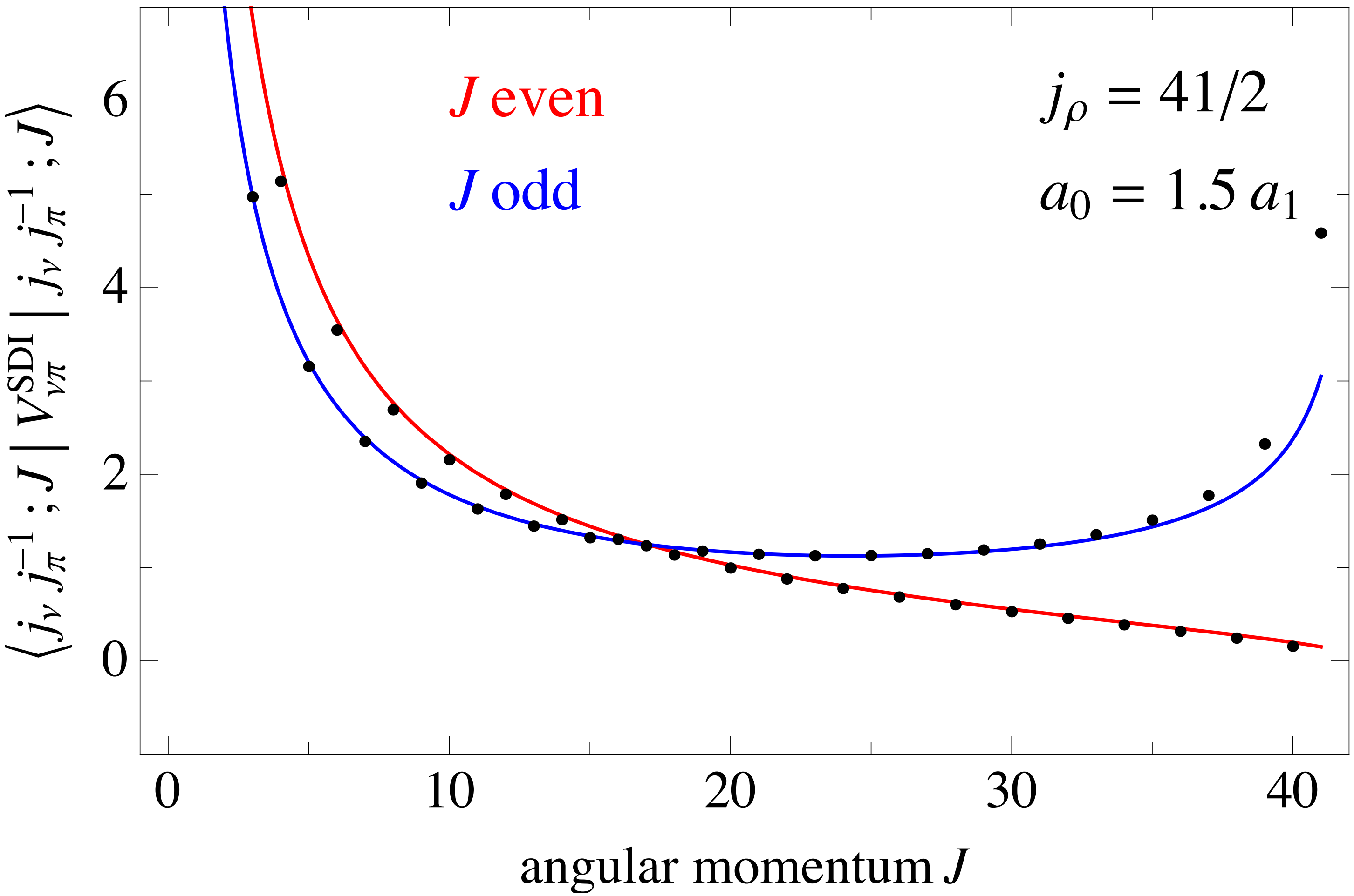
1p-1h matrix element



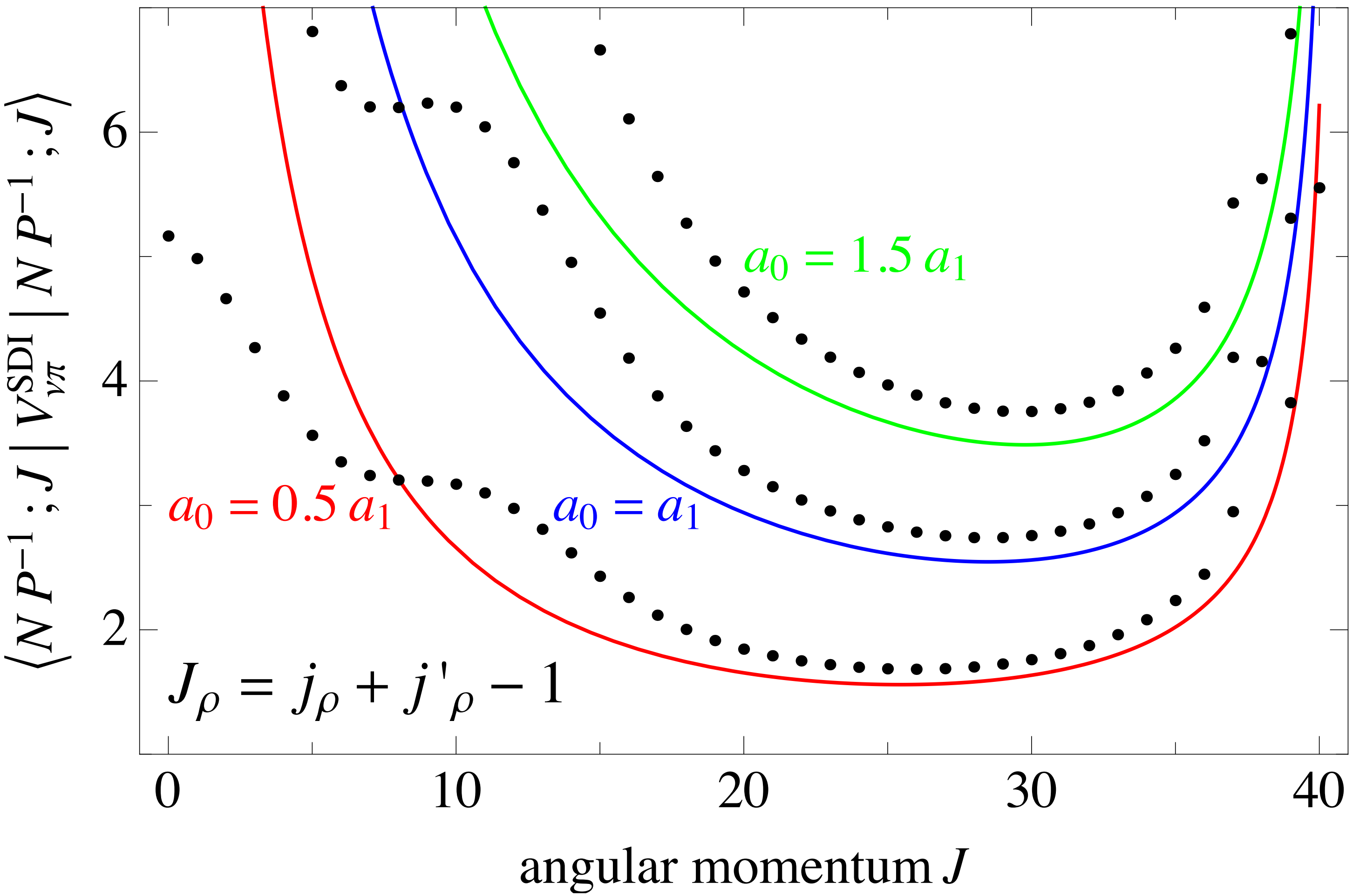
1p-1h matrix element



1p-1h matrix element



j_ϱ=21/2 & j'_ϱ=21/2 & J_ϱ=20



Bands without deformation

Regular sequences of levels (bands) are usually associated with nuclear collective behaviour.

In several regions of the nuclear chart in the neighbourhood of closed-shells nuclei regular bands are observed.

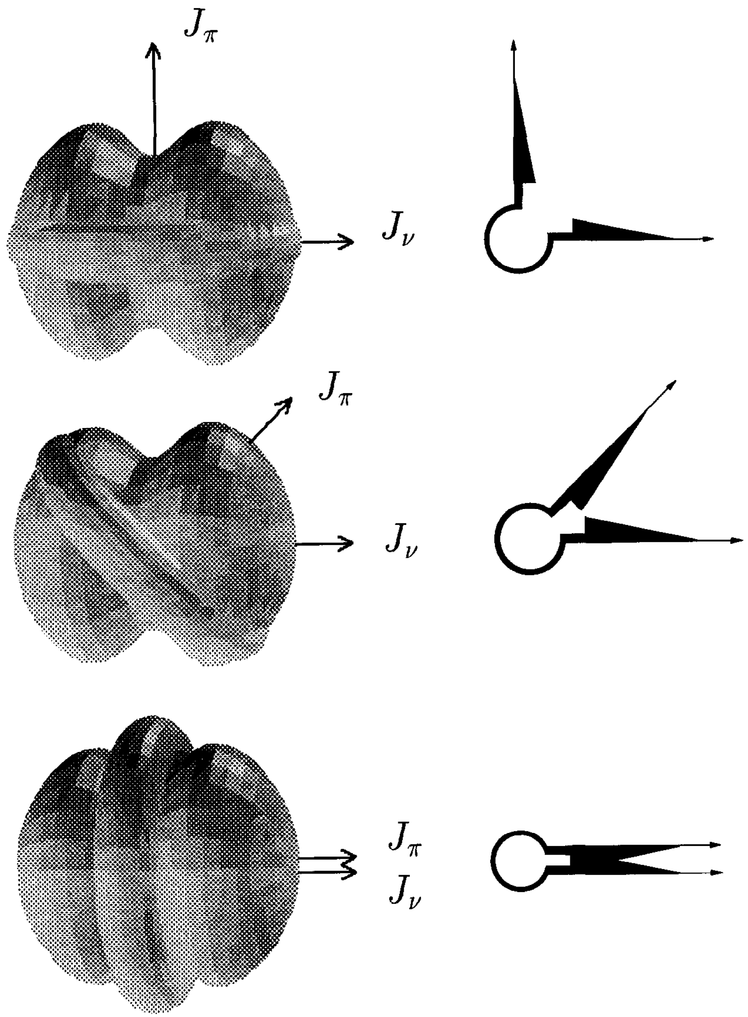
Shears bands

Question: How can sequences of levels appear rotational when deformation is weak?

Answer: Through the shears mechanism. This implies strong in-band M1 transitions.

S. Frauendorf, Nucl. Phys. A **557** (1993) 259c

The shears mechanism



S. Frauendorf *et al.*, Nucl. Phys. A **601** (1996) 41

A shell-model configuration

Assume a shears band in terms of two neutron particles and two proton holes:

$$\left|N\right\rangle \equiv \left|j_\nu j'_\nu;J_\nu\right\rangle \quad \& \quad \left|P^{-1}\right\rangle \equiv \left|j_\pi^{-1} j'^{-1}_\pi;J_\pi\right\rangle \Rightarrow \left|NP^{-1};J\right\rangle$$

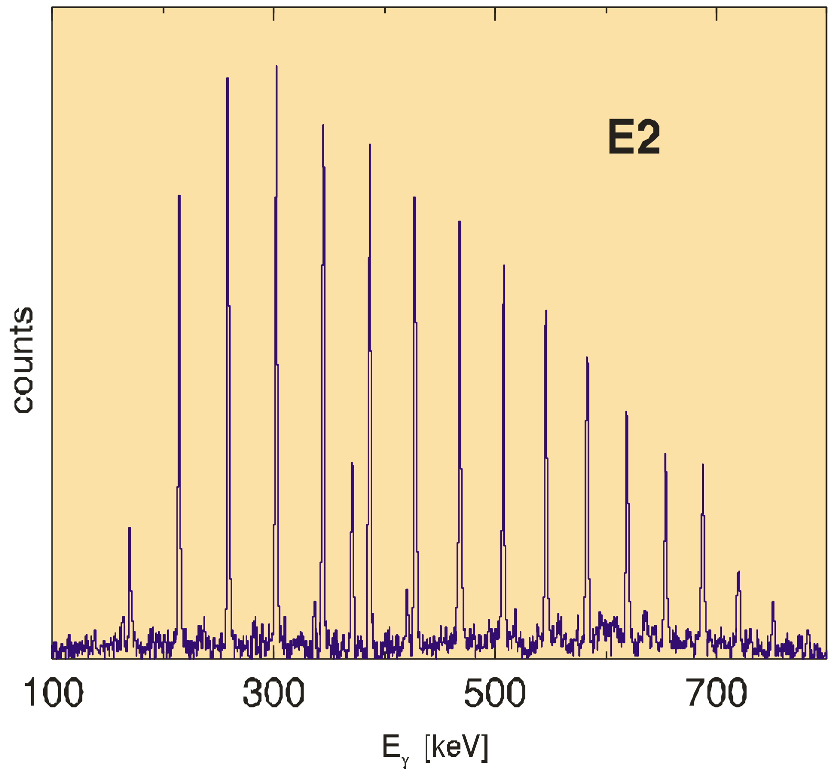
How do the energies of these states evolve as a function of J ?

How does this evolution depends on the angular momenta of the single-particle orbits and on the angular momenta of the 'blades'?

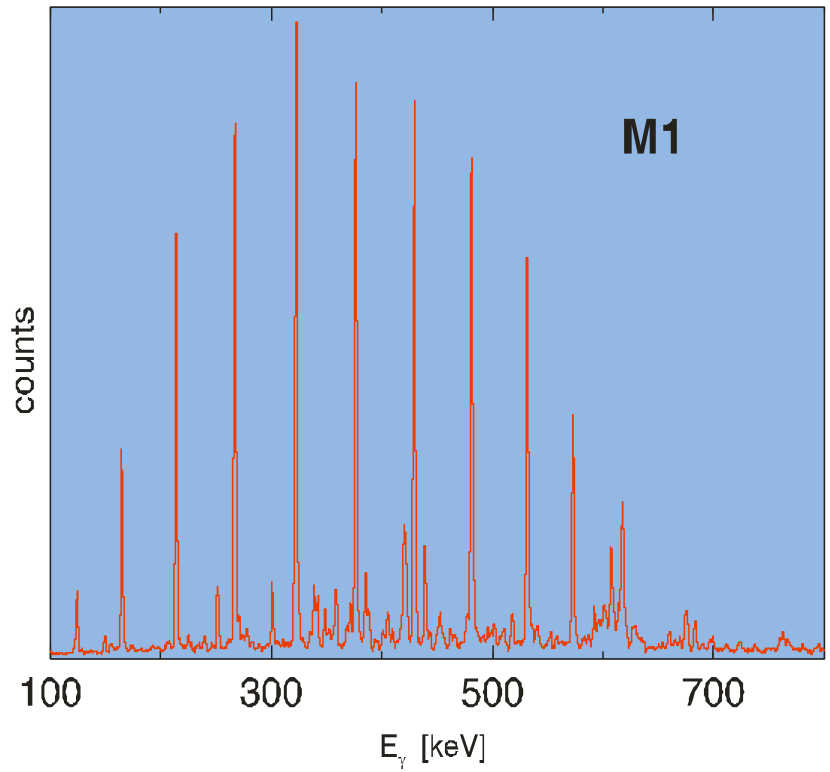
Take $j_\nu=j'_\nu$ and $j_\pi=j'_\pi$.

Regular sequences

rotational band in superdeformed ^{194}Pb



magnetic rotation in spherical ^{199}Pb



In terms of the shears angle

The shears angle is the angle between the angular momentum vectors of neutron particles and the proton holes:

$$\theta_{\nu\pi} = \arccos \frac{J(J+1) - J_\nu(J_\nu+1) - J_\pi(J_\pi+1)}{2\sqrt{J_\nu(J_\nu+1)J_\pi(J_\pi+1)}}$$

We have

$$S_n \approx \frac{2}{\pi(2j_\nu+1)(2j_\pi+1)(2J_\nu+1)(2J_\pi+1)\sin\theta_{\nu\pi}}$$

Semi-classical interpretation

Schematic model in terms of the coupling of two vectors J_ν and J_π and a 'shears' angle

$$\theta_{\nu\pi} = \arccos \frac{J(J+1) - J_\nu(J_\nu+1) - J_\pi(J_\pi+1)}{2\sqrt{J_\nu(J_\nu+1)J_\pi(J_\pi+1)}}$$

An effective interaction of the form

$$V(\theta_{\nu\pi}) = V_0 + V_2 P_2(\cos\theta_{\nu\pi}) + \cdots$$

➜ Can this geometry of the shears mechanism be derived from the spherical shell model?

A.O. Macchiavelli *et al.*, Phys. Rev. C **57** (1998) R1073
A.O. Macchiavelli *et al.*, Phys. Rev. C **58** (1998) R621

A simple application

Let's accept the expression for the shears energy

$$E(J) = \frac{\alpha_s}{2\pi \sin\theta_{\nu\pi}} + \frac{\alpha_t}{2\pi \tan\theta_{\nu\pi}}$$

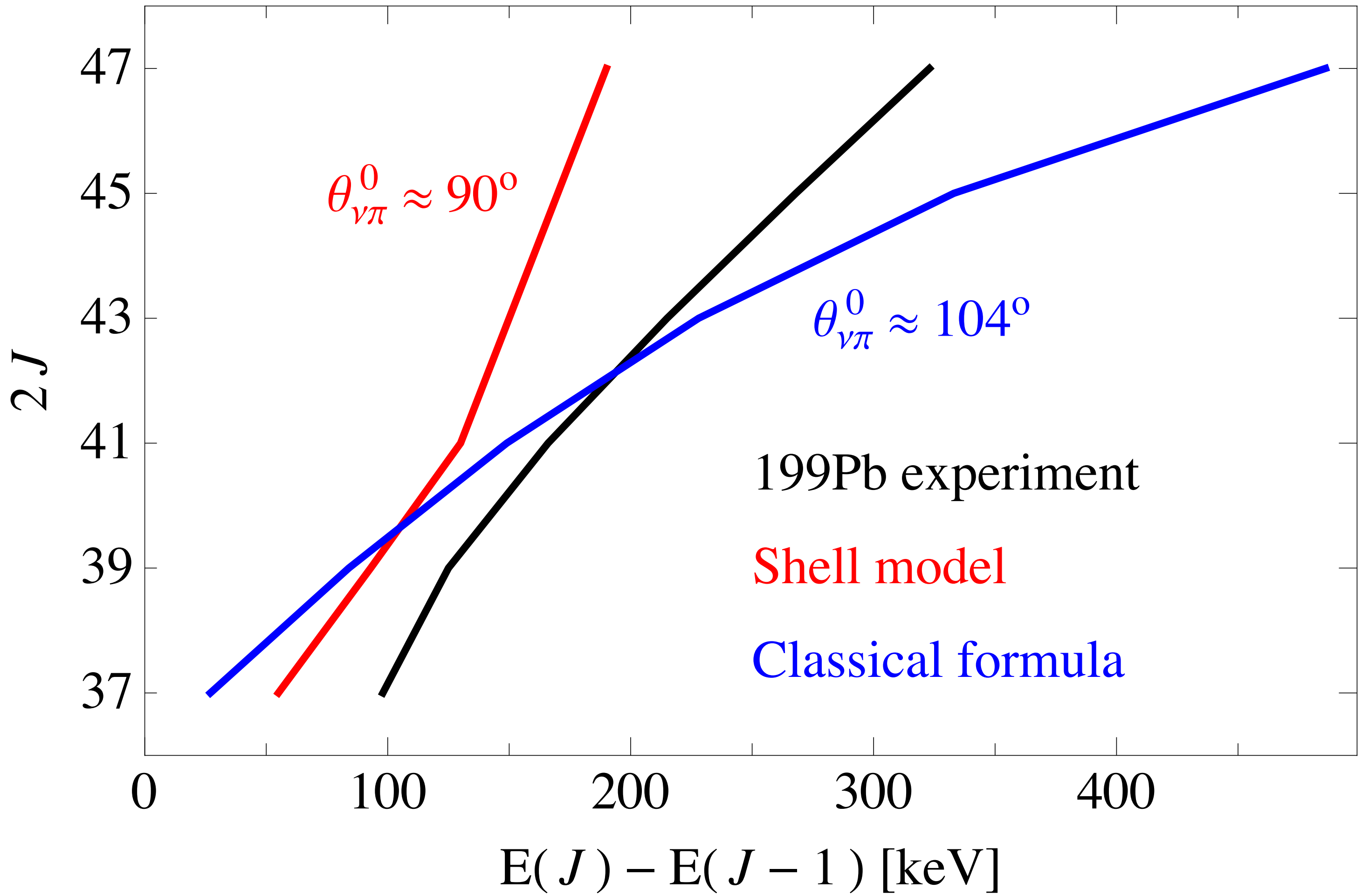
The head of the shears band follows from

$$\left.\frac{\partial E}{\partial \theta_{\nu\pi}}\right|_{\theta_{\nu\pi}=\theta^0_{\nu\pi}} = 0 \Rightarrow \cos\theta^0_{\nu\pi} = -\frac{\alpha_t}{\alpha_s} \quad \left(= \frac{a_0 - a_1}{3a_0 + a_1} \right)$$

The excitation energies of the shears-band members are given as

$$E_x(J) = \frac{\alpha_s}{2\pi \sin\theta_{\nu\pi}}\left(1 - \cos\theta^0_{\nu\pi} \cos\theta_{\nu\pi}\right) - \frac{\alpha_s \sin\theta^0_{\nu\pi}}{2\pi}$$

A simple application: ^{199}Pb



S. Frauendorf *et al.*, Nucl. Phys. A **601** (1996) 41

M1 transitions

Exact result for np-nh configurations:

$$B(\mathrm{M1}; J \to J-1)$$

$$= \frac{3}{4\pi}\left(g_{J_\nu} - g_{J_\pi}\right)^2 \frac{(C'+1)(C'-2J_\nu)(C'-2J_\pi)(C'-2J+1)}{4J(2J+1)}$$

with $C' = J_\nu + J_\pi + J$.

Classical approximation:

$$B(\mathrm{M1}; J \to J-1) \approx \frac{3}{4\pi}\left(g_{J_\nu} - g_{J_\pi}\right)^2 \frac{(2J_\nu+1)^2(2J_\pi+1)^2}{16J(2J+1)} \sin^2\theta_{\nu\pi}$$

M1 transitions in ^{199}Pb

Proposed configuration of states in band 1:

$$\left[\nu\left(1i_{13/2}^{-3}\right)^{33/2}\times\pi\left(1h_{9/2}1i_{13/2}\right)^{11}\right]^{(J)}$$

Calculation of *g* factors:

$$\nu\left(1i_{13/2}^{-3}\right)^{33/2}: g_{J_\nu} = g_{1i_{13/2}}^{\nu} = -0.29$$

$$\pi\left(1h_{9/2}1i_{13/2}\right)^{(11)}: g_{J_\pi} = \frac{9}{22}g_{1h_{9/2}}^{\pi} + \frac{13}{22}g_{1i_{13/2}}^{\pi} = 1.03$$

M1 transitions in ^{199}Pb