>> *The Cellular Automaton Perspective* <<
>> *on the* <<
>> *Linearity of Quantum Mechanics* <<

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Linearity $\longrightarrow$ superposition principle $\longrightarrow$ interference

$\longrightarrow$ entanglement: *Schrödinger's Cat & Wigner's Friend*

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Linearity ...

- Def.: linearity $\Longleftrightarrow$ dynamics maps states linearly on states.
- Theorem: QM is linear.

  proof: E.P. Wigner and V. Bargmann
  – assumption: dynamics does not change $|\langle\psi'|\psi\rangle|$
  – e.g., unitary, $\exp(-i\hat{H}t)$; but, if $\hat{H}$ state dependent **...**

  proof: T.F. Jordan
  – assumption: No Influences Without Interactions
  " ... dynamics does not depend on anything outside the system, but the system can be described as part of a larger composite system together with another separate system."

- Experiments testing linearity, test also these assumptions!

... we shall replace Jordan's "No Influences Without Interactions"

Based on synthesis of three ingredients ...

- deterministic discrete mechanics – T.D. Lee *et al.*
- sampling theory for discrete structures on / of spacetime – A. Kempf *et al.*
- QM in terms of classical notions of observables & phase space – A. Heslot; HTE, N. Buric *et al.*

... three ingredients, which lead us to:

- deterministic discrete mechanics

  $\longrightarrow$ **Hamiltonian cellular automata (CA), action principle**

- sampling theory for discreteness on spacetime

  $\longrightarrow$ **map: CA $\leftrightarrow$ continuum QM + corrections**

- QM in terms of classical notions of observables & phase space

  $\longrightarrow$ **“oscillator representation”**

... $\Longrightarrow$ **from CA perspective linearity of QM unavoidable.**

Assumptions

- fundamental discreteness to be incorporated in dynamics
- there is a fundamental length or time scale l – "such that in a $(d+1)$-dimensional spacetime volume Ω maximally Ω/l^{d+1} measurements can be performed or maximally this number of events take place"
- time is a discrete dynamical variable

Hamiltonian Cellular Automata (CA) – *"bit machines"*

- classical CA with denumerable degrees of freedom
- state described by **integer valued** coordinates x_n^α, τ_n and momenta p_n^α, π_n

  $\alpha \in \mathbf{N}_0$: different degrees of freedom
  $n \in \mathbf{Z}$: successive states
- finite differences, $\Delta f_n := f_n - f_{n-1}$, all dynamical variables
- $\mathcal{A}_n := \Delta\tau_n(H_n + H_{n-1}) + c_n\pi_n$,

  $H_n := \frac{1}{2}S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}p_n^\alpha x_n^\beta + R_n(x,p)$.

The CA Action Principle

- $\mathcal{A}_n := \Delta\tau_n(H_n + H_{n-1}) + c_n\pi_n$,

  $H_n := \frac{1}{2}S_{\alpha\beta}(p_n^\alpha p_n^\beta + x_n^\alpha x_n^\beta) + A_{\alpha\beta}p_n^\alpha x_n^\beta + R_n(x,p)$,

  constants c_n, sym. $\hat{S} \equiv \{S_{\alpha\beta}\}$, antisym. $\hat{A} \equiv \{A_{\alpha\beta}\}$, remainder R_n, all integer valued parameters

- $\mathcal{S} := \sum_n[(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - \mathcal{A}_n]$.

- Postulate (Action Principle): $\delta\mathcal{S} \stackrel{!}{=} 0 \;\Rightarrow$ CA updating rules, for arbitrary integer valued variations of the dynam. variables, $\delta g(f_n) := [g(f_n + \delta f_n) - g(f_n - \delta f_n)]/2$. *remarks* ... $R_n \equiv 0$.

CA equations of motion (e.o.m.)

- $\mathcal{S} := \sum_n [(p_n^\alpha + p_{n-1}^\alpha)\Delta x_n^\alpha + (\pi_n + \pi_{n-1})\Delta\tau_n - \mathcal{A}_n]$,
  with $\delta\mathcal{S} \overset{!}{=} 0 \;\Rightarrow$ CA finite difference e.o.m.:

$$\dot{x}_n^\alpha = \dot{\tau}_n(S_{\alpha\beta}p_n^\beta + A_{\alpha\beta}x_n^\beta) \; ,$$

$$\dot{p}_n^\alpha = -\dot{\tau}_n(S_{\alpha\beta}x_n^\beta - A_{\alpha\beta}p_n^\beta) \; ,$$

$$\dot{\tau}_n = c_n \; , \quad \dot{\pi}_n = \dot{H}_n \; , \quad \text{with } \dot{O}_n := O_{n+1} - O_{n-1} \; .$$

- e.o.m. time reversal invariant, $(n \mp 1, n) \rightarrow (n \pm 1)$
- $\Longrightarrow \quad \dot{\psi}_n^\alpha = \dot{\tau}_n \hat{H}_{\alpha\beta}\psi_n^\beta$, discrete "Schrödinger equation"
  with $\hat{H} := \hat{S} + i\hat{A}$, self-adjoint, $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, CA "time" n

CA conservation laws

- discrete "Schrödinger equation", $\dot{\psi}_n^\alpha = \dot{\tau}_n \hat{H}_{\alpha\beta}\psi_n^\beta \implies$
- Theorem: For any $\hat{G}$ with $[\hat{G}, \hat{H}] = 0$ there is a discrete conservation law: $\psi_n^{*\alpha} G_{\alpha\beta}\dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G_{\alpha\beta}\psi_n^\beta = 0$.

  For self-adjoint $\hat{G}$, with complex integer elements $\rightarrow$ real integer quantities.

  For $\hat{G} := \hat{1} \implies$ constraint: $\psi_n^{*\alpha}\dot{\psi}_n^\alpha + \dot{\psi}_n^{*\alpha}\psi_n^\alpha = 0$.
  For $\hat{G} := \hat{H} \implies$ "energy conservation" .
- conservation laws not "integrable", since Leibniz rule modified, e.g.: $(O_n \dot{} O'_n) = \frac{1}{2}(\dot{O}_n[O'_{n+1} + O'_{n-1}] + [O_{n+1} + O_{n-1}]\dot{O}'_n)$.

How to obtain more of QM ...

- recall $\psi_n^\alpha := x_n^\alpha + ip_n^\alpha$, CA "time" n

  introduce fundamental scale $l \longrightarrow n \cdot l$, physical time?

  Problem: continuum limit, $l \to 0$, does not work
  – integer valuedness $\Rightarrow$ time derivatives diverge!

- Idea: **invertible MAP between discrete integer valued and continuous (differentiable ...) quantities.** – G. 't Hooft

- we are familiar with the problem – *digital audio and video!* – but maybe not with the solution: *information can be simultaneously continuous & discrete* – C.E. Shannon

The Sampling Theorem

- Consider square integrable bandlimited functions f :

  $$f(t) = (2\pi)^{-1} \int_{-\omega_{max}}^{\omega_{max}} d\omega \; e^{-i\omega t} \tilde{f}(\omega) \; , \;\; \text{bandwidth } \omega_{max} \; .$$

- Shannon's Theorem:

  Given $\{f(t_n)\}$ for set $\{t_n\}$ of equidistantly spaced times (spacing π/ω_{max}), function f is obtained for all t by:

  $$f(t) = \sum_n f(t_n) \frac{\sin[\omega_{max}(t-t_n)]}{\omega_{max}(t-t_n)} \quad \text{(reconstruction formula)} \; .$$

- CA "time" n $\sim$ discrete time $t_n := nl$ $\rightarrow$ continuous time t
  bandwidth $\omega_{max} := \pi/l$ (Nyquist rate)

MAP: discrete CA $\leftrightarrow$ continuous QM

- applying Shannon's *reconstruction formula* ...
- the discrete "Schrödinger equation", $\dot{\psi}^\alpha_n = \dot{\tau}_n \hat{H}_{\alpha\beta}\psi^\beta_n$ ...
- is mapped to continuous time Schrödinger equation:

$$\frac{\hat{D}_l - \hat{D}_{-l}}{2}\psi^\alpha(t) = \sinh(l\partial_t)\psi^\alpha(t) = \frac{1}{i}H_{\alpha\beta}\psi^\beta(t) \quad ,$$

with $\hat{D}_T f(t) := f(t+T)$ and employing $\underline{\dot{\tau}_n \equiv 2}$.

- correction terms, $|\partial^k\psi/\partial t^k| \overset{?}{\ll} l^{-k} = (\omega_{max}/\pi)^k$
- stationary states, $\sin(E_\alpha l) = \epsilon_\alpha$, for $\hat{H} \to \mathrm{diag}(\epsilon_0, \epsilon_1, \dots)$
  $\Rightarrow$ spectrum cut off by $|E_\alpha| \leq \pi/2l = \omega_{max}/2$

Conservation Laws: discrete CA $\leftrightarrow$ continuous QM

- CA & QM equations are both **linear**! – Then, CA $\rightarrow$ QM by $\dot{\psi}_n := \psi_{n+1} - \psi_{n-1} \longrightarrow \frac{1}{l}\sin(il\partial_t)\psi(t)$, *suggests* ...
- Theorem: For any $\hat{G}$ with $[\hat{G},\hat{H}] = 0$, it holds that

  $$\psi^{*\alpha} G_{\alpha\beta} \sin(il\partial_t)\psi^\beta + [\sin(il\partial_t)\psi^{*\alpha}]G_{\alpha\beta}\psi^\beta = 0 \quad .$$

  In particular, wave function normalization conserved

  $$\psi^{*\alpha} \sin(il\partial_t)\psi^\alpha + [\sin(il\partial_t)\psi^{*\alpha}]\psi^\alpha = 0 \quad .$$

- **same commutator** $[\hat{G},\hat{H}] = 0$ $\Rightarrow$ CA & QM conserv. laws!
- in *all* modified QM equations, continuum limit $l \rightarrow 0$ works.

Some things to find out ...

- CA properties that become unitary symmetries in QM?
- CA Hamiltonians have interesting spectra?
- QM approximation scheme with bandwidth limited wave fct.s?
- What is relativistic/QFT version of CA $\leftrightarrow$ QM map?
- How gauge fields come in?

On the relation between Hamiltonian CA and QM ...

- **MAP: CA** $\leftrightarrow$ **QM** based on
  - **integer valued action principle** with **arbitrary variations**
  - **sampling theory**, i.e. bandwidth limited wave functions

  $\rightarrow$ replaces Jordan's separability assumption ($\Rightarrow$ linearity)
- **Schrödinger equation** with correction terms, $\sim (l\partial_t)^k$,
  - incorporating discreteness scale l
- $[\hat{G}, \hat{H}] = 0 \rightarrow$ corresp. **conservation laws** for CA & QM

$= \bullet =$