Spontaneous Photon Emission in Cavities

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Gernot Alber

Nils Griebe

Institut für Angewandte Physik, Technische Universität Darmstadt, Darmstadt

in collaboration with

- **Luis Sanchez-Soto** (Max-Planck Institut für die Physik des Lichts, Erlangen)
- **Gerd Leuchs** (Max-Planck Institut für die Physik des Lichts, Erlangen)
- **Magdalena Stobinska** (University of Gdansk)

Problem - Motivation

- strong coupling between single photons (flying qubits) and elementary material quantum systems (stationary qubits)

  optimal transfer of quantum information $\longrightarrow$ quantum information processing

  e.g. quantum memories, quantum repeaters,...

- qubits interacting with few modes of the radiation field (e.g. Jaynes-Cummings-Paul model)

here:
qubits interacting with a structured continuum of modes, e.g. half open cavities?

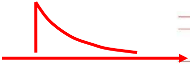
photon-atom coupling

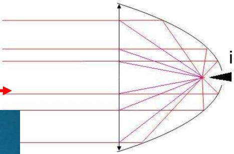
demonstration of the
time reversal of spontaneous emission

plan:
M. Sondermann et al.,
Appl. Phys. B 89, 489 (2007)

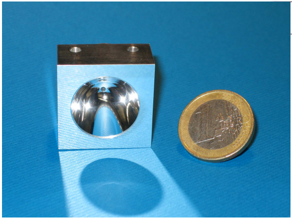
G.L., M. Sondermann,
Phys. Scr. 85, 058101 (2012)
"Time reversal symmetry in optics"

one photon wave packet





ion trap



Robert Maiwald
Simon Heugel
Andrea Golla
Marianne Bader
Martin Fischer
Daniel Weber
Bénoît Chalopin
Markus Sondermann

Gernot Alber
Luis L. Sanchez-Soto

Contents

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- spontaneous emission of a photon:
  a two-level system in the focus of a parabolic cavity
  $\longrightarrow$ modifications originating from ideally conducting metallic boundary
  - modifications of spontaneous decay rate ?
  - semiclassical photon-path representation
    $\longrightarrow$ separation between free-space phenomena and effects of boundary

    'beyond the Weisskopf-Wigner (pole) approximation'
    [V. Weisskopf and E. Wigner, Z. Phys.63, 54 (1930)]
- spontaneous emission of a photon:
  two two-level systems in the foci of an elliptic cavity
- time evolution of field fluctuations (normally ordered field density)?

A single qubit in the focus of a parabolic cavity: the dynamical system

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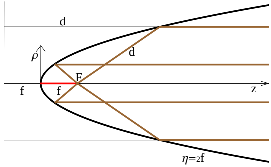
- a single qubit located at the focal point **F** of parabolic cavity with ideally conducting walls

$$\hat{H}_A = E_g|g\rangle\langle g| + E_e|e\rangle\langle e|$$

- quantized radiation field with Hamiltonian



$$\hat{H}_F = \sum_n \int_0^\infty d\omega\ \hbar\omega\ \hat{a}^\dagger_{\omega,n}\hat{a}_{\omega,n}$$

$$\hat{\mathbf{E}}^{(+)}(\mathbf{x}) = \sum_n \int_0^\infty d\omega\ i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\ \mathbf{g}_{\omega,n}(\mathbf{x})\ \hat{a}_{\omega,n}, \quad \text{transversality } (\nabla\cdot\mathbf{g}_{\omega,n})(\mathbf{x}) = 0$$

- optical transition frequency $\omega_0 = (E_e - E_g)/\hbar$ (largest parameter)
  $\longrightarrow$ dipole and rotating wave approximation
- dipole **d** aligned along symmetry axis of parabola

A single qubit in the focus of a parabolic cavity: the dynamical system

- solve time dependent Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle^{(\pm)} = \hat{H}|\psi(t)\rangle^{(\pm)}, \qquad |\psi(t=0)\rangle^{(\pm)} = |e\rangle \otimes |0\rangle$$

$+$: retarded solution ($t \geq 0$), $-$: advanced solution ($t \leq 0$)

- Hamiltonian in the dipole- and rotating wave approximation

$$\hat{H} = \hat{H}_A + \hat{H}_F - \left(|e\rangle\langle g|\langle e|\hat{\mathbf{d}}|g\rangle \cdot \hat{\mathbf{E}}^{(+)}(\mathbf{x} = \mathbf{F}) + \text{h.c.}\right)$$

$$\hat{\mathbf{E}}^{(+)}(\mathbf{x}) = \sum_n \int_0^\infty d\omega \, i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} \, \mathbf{g}_{\omega,n}(\mathbf{x}) \, \hat{a}_{\omega,n}$$

-

$$\longrightarrow \quad |\psi(t)\rangle^{(\pm)} = A_e^{\pm)}(t)|e\rangle \otimes |0\rangle + \sum_n \int_0^\infty d\omega \, A_{\omega,n}^{(\pm)}(t)|g\rangle \otimes \hat{a}_{\omega,n}^\dagger|0\rangle$$

Dipole-excited field modes inside the parabola

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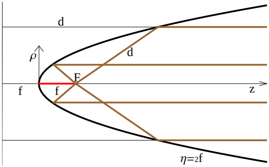
solve Helmholtz equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \mathbf{g}_{\omega,n}(\mathbf{x}) \quad = \quad 0$$



with

- transversality condition for radiation field $(\nabla \cdot \mathbf{g}_{\omega,n})(\mathbf{x}) = 0$
- boundary condition of a parabolic cavity with ideally conducting wall

$$\mathbf{t} \cdot \mathbf{g}_{\omega,n}(\mathbf{x}) \mid_{\partial V} \quad = \quad 0, \quad \mathbf{n} \cdot \nabla \wedge \mathbf{g}_{\omega,n}(\mathbf{x}) \mid_{\partial V} = 0$$

- orthonormality condition

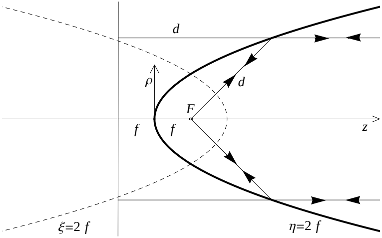
$$\int_V d^3\mathbf{x}\, \mathbf{g}^*_{\omega,n}(\mathbf{x}) \cdot \mathbf{g}_{\omega',n'}(\mathbf{x}) \quad = \quad \delta(\omega - \omega')\delta_{n,n'}$$

- only modes with $\mathbf{g}_{\omega,n}(\mathbf{x} = \mathbf{F}) || \mathbf{d}$ can be excited by the dipole $\mathbf{d}$ at the focus $\mathbf{F}$

Dipole-excited field modes inside the parabola

- transverality condition $(\nabla \cdot \mathbf{g}_{\omega,n})(\mathbf{x}) = 0 \quad \longrightarrow \quad \mathbf{g}_{\omega,n}(\mathbf{x}) = \nabla \wedge \mathbf{G}_{\omega,n}(\mathbf{x})$
- separation of Helmholtz equation in parabolic coordinates

$$x = \sqrt{\xi\eta}\cos\varphi,\ y = \sqrt{\xi\eta}\sin\varphi,\ z = (\xi - \eta)/2$$



Dipole-excited field modes inside the parabola

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- $$\mathbf{G}_{\omega,n}(\mathbf{x}) = (\omega\mathcal{N}_{\omega,n})^{-1/2}\frac{\chi_{\omega,n}(\xi)}{\sqrt{\xi}}\frac{\chi_{\omega,n}(\eta)}{\sqrt{\eta}}\frac{\mathbf{e}_\varphi}{\sqrt{2\pi}}$$

$$\left\{\frac{d^2}{d\eta^2} + \left(\frac{\omega}{2c}\right)^2 + \frac{\alpha}{\eta}\right\}\chi_{\omega,n}(\eta) = 0, \quad \left\{\frac{d^2}{d\xi^2} + \left(\frac{\omega}{2c}\right)^2 - \frac{\alpha}{\xi}\right\}\chi_{\omega,n}(\xi) = 0$$

$$\longrightarrow \chi_{\omega,n}(\xi) = \sqrt{\frac{4c}{\pi\omega}}F_{L=0}\left(\frac{\alpha}{\omega/c}, \frac{\omega\xi}{2c}\right), \ \chi_{\omega,n}(\eta) = \sqrt{\frac{4c}{\pi\omega}}F_{L=0}\left(-\frac{\alpha}{\omega/c}, \frac{\omega\eta}{2c}\right)$$

- boundary conditions at $\eta = 2f \longrightarrow$ quantization of separation constant α

$$\frac{d\chi_{\omega,n}}{d\eta}(\eta = 2f) = 0 \longrightarrow \alpha_n(\omega)$$

- normalization factor for frequency normalizaton

$$\mathcal{N}_{\omega,n} = \int_0^{2f} d\eta \, \frac{\chi^2_{\omega,n}(\eta)}{\eta}$$

The spontaneous decay rate $\Gamma(\omega_0)$: relation to free-space decay rate $\Gamma_s(\omega_0)$

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perturbation theory (golden rule)

$$\longrightarrow \quad \Gamma(\omega_0) = \frac{2\pi}{\hbar^2}\sum_n \left| \mathbf{d}\cdot\sqrt{\frac{\hbar\omega_0}{2\epsilon_0}}\mathbf{g}_{\omega_0,n}(\mathbf{x}=\mathbf{F}) \right|^2 = \Gamma_s(\omega_0)\underbrace{\frac{6c}{\pi\omega_0}\sum_n \frac{1}{\mathcal{N}_{\omega_0,n}}\left(\frac{x_n(\omega_0)}{\sinh x_n(\omega_0)}\right)^2}_{\longrightarrow\ 1\ \text{in free space}}$$

with $x_n(\omega_0) := \pi\alpha_n(\omega_0)/(\omega_0/c)$ and with the free-space spontaneous decay rate

$$\Gamma_s(\omega_0) = \frac{|\langle e|\mathbf{d}|g\rangle|^2\,\omega_0^3}{3\pi\epsilon_0\hbar c^3}, \quad \omega_0 = (E_e - E_g)/\hbar \gg \Gamma_s(\omega_0)$$

The spontaneous decay rate $\Gamma(\omega_0)$: semiclassical approach

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$$\left\{ \frac{d^2}{d\eta^2} + \left(\frac{\omega}{2c}\right)^2 + \frac{\alpha}{\eta} \right\} \chi_{\omega,n}(\eta) = 0 \ , \eta \in [0,\infty), \quad \frac{d\chi_{\omega,n}}{d\eta}(\eta = 2f) = 0$$

semiclassical regular solution ('Langer substitution')

$$\chi_{\omega,n}(\eta) = \sqrt{\frac{2}{\pi k(\eta)}} \sin\left(\int_{\eta_0}^{\eta} d\eta' \ k(\eta') + \pi/4 \right), \quad k(\eta) = \sqrt{\left(\frac{\omega}{2c}\right)^2 + \frac{\alpha}{\eta} - \frac{1}{4\eta^2}}$$

classical Eikonal and quantization condition $\alpha_n(\omega)$ — local wave number

$$W(\omega,\alpha) := \int_{\eta_0}^{2f} d\eta \ k(\eta) = \pi(n(\omega,\alpha) + 1/2), \quad n(\omega,\alpha) \in \mathbb{N}_0 \quad \longrightarrow \quad \alpha_n(\omega)$$

classical Eikonal and normalization factor $\mathcal{N}_{\omega,n}$

$$\mathcal{N}_{\omega,n} := \int_0^{2f} d\eta \frac{\chi^2_{\omega,n}(\eta)}{\eta} = \int_{\eta_0}^{2f} d\eta \frac{\frac{1}{2}\left(\frac{2}{\pi k(\eta)}\right)}{\eta} = 2\frac{\partial n}{\partial \alpha}(\omega,\alpha)$$

The spontaneous decay rate $\Gamma(\omega_0)$: semiclassical linearization approximation

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- exact solution for $\alpha = 0$ ($x := \pi\alpha c/\omega$):

$$\chi_{\omega,\alpha=0}(\eta) = \sqrt{\frac{4c}{\pi\omega}} \sin\left(\frac{\omega\eta}{2c}\right) \longrightarrow \pi(n(\omega, x=0) + \frac{1}{2}) = \frac{f\omega}{c} - \frac{\pi}{2}$$

- exact normalization factor at $\alpha = 0 \longrightarrow \partial n/\partial x(\omega, x = 0)$:

$$\mathcal{N}_{\omega,n} := \int_0^{2f} d\eta \frac{\chi^2_{\omega,\alpha=0}(\eta)}{\eta} = \frac{2\pi c}{\omega}\frac{\partial n}{\partial x}(\omega, x=0)$$

$\longrightarrow$ Eikonal in the linearization approximation

$$n(\omega, x) = \frac{1}{\pi}\left(\frac{f\omega}{c} - \frac{\pi}{2}\right) + x\,\frac{\partial n}{\partial x}(\omega, x=0)$$

The spontaneous decay rate $\Gamma(\omega_0)$: semiclassical linearization approximation

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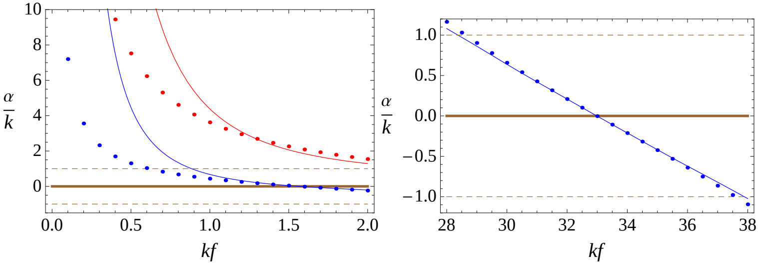
comparison between exact quantization of separation constant $\alpha_n(\omega)$ (dots) and semiclassical linearization approximation (full) $(k = \omega/c)$



$$\Gamma(\omega) \quad = \quad \Gamma_s(\omega_0)\,\frac{6c}{\pi\omega_0}\sum_n \frac{1}{\mathcal{N}_{\omega_0,n}}\left(\frac{x_n(\omega_0)}{\sinh x_n(\omega_0)}\right)^2, \quad x_n(\omega) := \frac{\pi c \alpha_n(\omega)}{\omega} = \frac{n + 1/2 - f\omega/(c\pi)}{\frac{\partial n}{\partial x}(\omega, x=0)}$$

The spontaneous decay rate $\Gamma(\omega_0)$: linearization approximation

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- semiclassical linearization approximation

$$n(\omega, x) = \underbrace{\frac{1}{\pi}\left(\frac{f\omega}{c} - \frac{\pi}{2}\right)}_{:=n_0} + x \underbrace{\frac{\partial n}{\partial x}(\omega, x=0)}_{:=n_{0x}}, \quad \mathcal{N}_{\omega,n} = \frac{2\pi c}{\omega} n_{0x}$$

- Poisson summation formula

$$\frac{\Gamma(\omega_0)}{\Gamma_s(\omega_0)} = \sum_n \frac{1}{\mathcal{N}_{\omega_0,n}} \frac{6c}{\pi\omega_0} \frac{x_n^2(\omega_0)}{\sinh^2 x_n(\omega_0)} = \sum_{M=-\infty}^{\infty} \frac{3}{\pi^2} \int_{-\infty}^{\infty} dx \, \frac{x^2}{\sinh^2 x} e^{iM2\pi n(\omega_0,x)}$$

- Poisson summation formula and linearization approximation

$$\frac{\Gamma(\omega_0)}{\Gamma_s(\omega_0)} = 1 + 6\sum_{M=1}^{\infty} \cos(M2\pi n_0) \frac{M\pi^2 n_{0x}\coth(M\pi^2 n_{0x}) - 1}{\sinh^2(M\pi^2 n_{0x})}$$

The spontaneous decay rate $\Gamma(\omega_0)$: semiclassical path representation

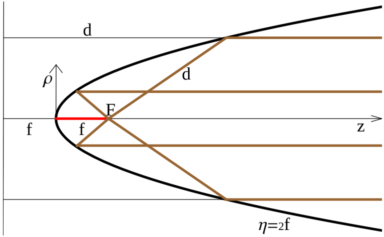
- semiclassical path representation

$$\frac{\Gamma(\omega_0)}{\Gamma_s(\omega_0)} = \underbrace{1 + 6\sum_{M=1}^{\infty} \cos(M2\pi n_0)\frac{M\pi^2 n_{0x}\coth(M\pi^2 n_{0x}) - 1}{\sinh^2(M\pi^2 n_{0x})}}_{\text{effects of boundary}}$$

$2\pi n(\omega, x=0) = 2\pi n_0$ classical Eikonal of periodic photon path $F \to S \to F$

$n_{0x} = \partial n/\partial x(\omega_0, x=0)$ stability property

M number of reflections at the boundary



The spontaneous decay rate $\Gamma(\omega_0)$: effects of the parabolic boundary

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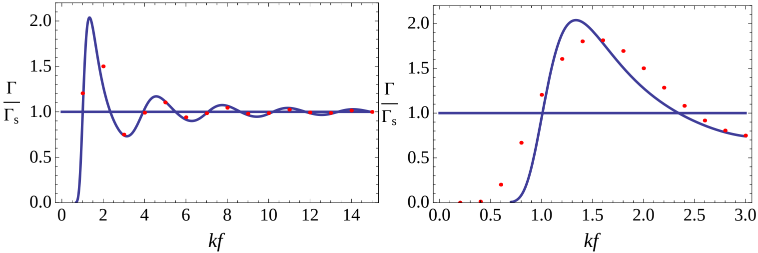
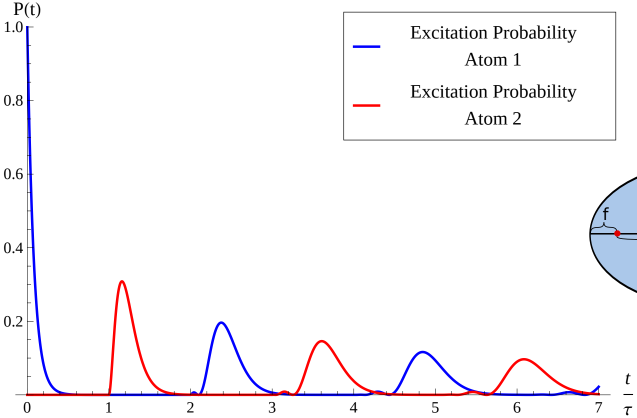
comparison between the exact spontaneous decay rate (dots) and the semiclassical path representation (full curve) ($k = \omega_0/c$)

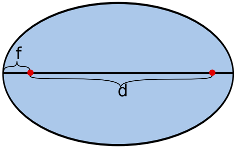
$$\frac{\Gamma(\omega_0)}{\Gamma_s(\omega_0)} = 1 + 6\sum_{M=1}^{\infty} \cos(M2\pi n_0)\frac{M\pi^2 n_{0x}\coth(M\pi^2 n_{0x}) - 1}{\sinh^2(M\pi^2 n_{0x})}$$



Time evolution of spontaneous decay process: two two-level atoms in an elliptic cavity

$\tau = (d + 2f)/c = 12.5/\Gamma_s(\omega_0)$ with eccentricity $\epsilon = 0.5 \rightarrow d/f = 2\epsilon/(1-\epsilon) = 1 \;\; f \gg \lambda$





Distribution of field fluctuations: two parabolic cavities with two two-level atoms

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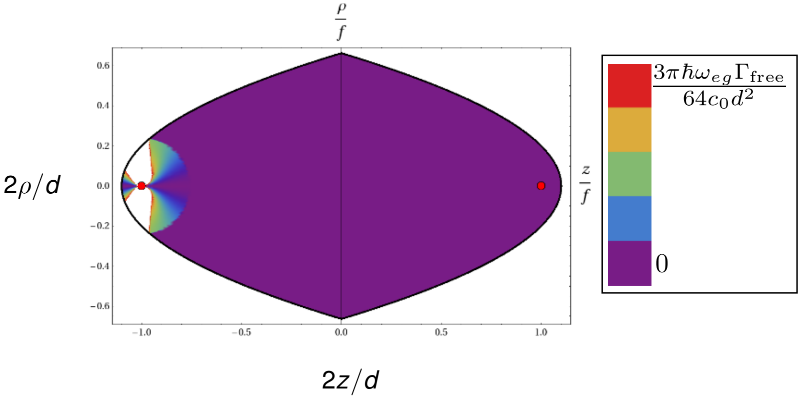
$f = 0.05d \gg \lambda,\ \tau = (4f + d)/c = 16/\Gamma_{free},\ t/\tau = 0.1 \quad \langle\psi(t)| : \frac{\epsilon_0}{2}\left(\hat{\mathbf{E}}^2(\mathbf{x}) + c^2\hat{\mathbf{B}}^2(\mathbf{x})\right) : |\psi(t)\rangle$



Distribution of field fluctuations: two parabolic cavities with two two-level atoms

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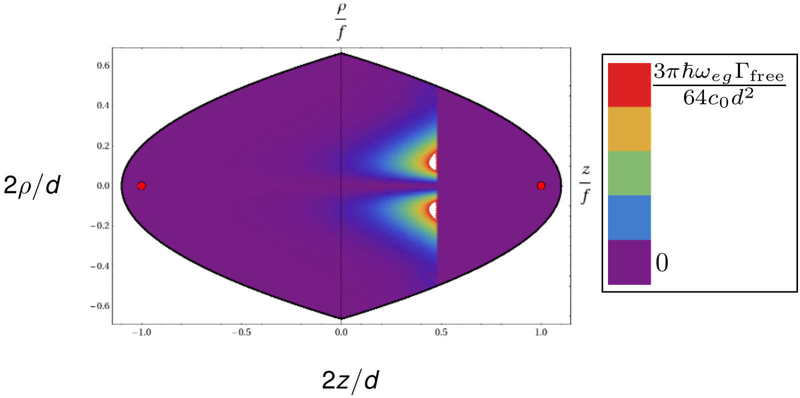
$f = 0.05d \gg \lambda, \ \tau = (4f + d)/c = 16/\Gamma_{free}, \ t/\tau = 0.7 \quad \langle\psi(t)| : \frac{\epsilon_0}{2}\left(\hat{\mathbf{E}}^2(\mathbf{x}) + c^2\hat{\mathbf{B}}^2(\mathbf{x})\right) : |\psi(t)\rangle$



Distribution of field fluctuations: two parabolic cavities with two two-level atoms

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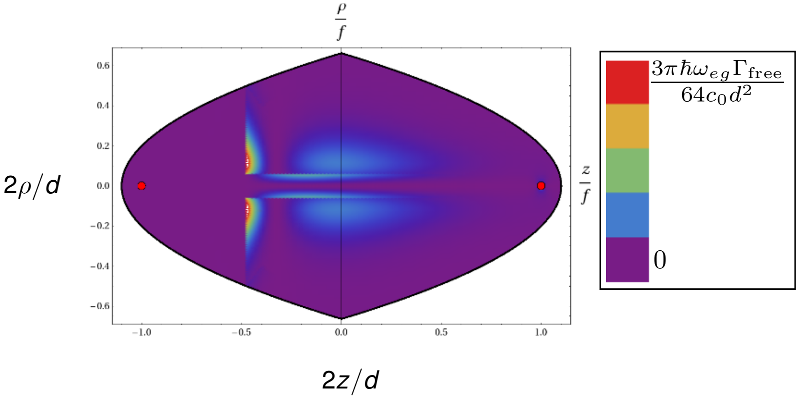

Distribution of field fluctuations: two parabolic cavities with two two-level atoms

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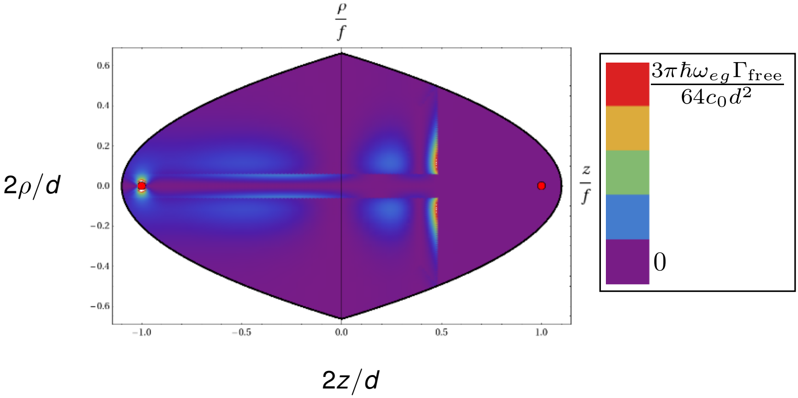
$f = 0.05d \gg \lambda,\ \tau = (4f + d)/c = 16/\Gamma_{free},\ t/\tau = 2.7 \quad \langle\psi(t)| : \frac{\epsilon_0}{2}\left(\hat{\mathbf{E}}^2(\mathbf{x}) + c^2\hat{\mathbf{B}}^2(\mathbf{x})\right) : |\psi(t)\rangle$



Conclusions

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- spontaneous photon emission of two-level systems in cavities
- theoretical description based on semiclassical photon path representations
  $\longrightarrow$ separation between free-space properties and effects of boundaries
  (separable problem $\longrightarrow$ beyond multidimensional semiclassical approximation)
- repeated reflections of photon wave packet at boundary $\longrightarrow$ re-excitations
- characteristic phenomena
  - modification of spontaneous decay rate $\Gamma(\omega_0)$
    small for $f\omega_0/c \gg 1$ and significant for $f\omega_0/c \ll 1$
  - decay and re-excitations separated in time
  - modulations of field fluctuations

[Phys.Rev. A **88**, 023825 (2013)]

Perfect excitation of a stationary qubit by a single photon in free space

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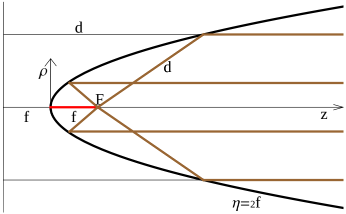
- perfect excitation of a qubit by a single photon is possible in free space!
  exploit time-reversal of spontaneous one-photon emission

  but

  preparation of appropriate one-photon state difficult
  (spherically incoming wave with appropriate polarization properties)

- excitation of a qubit in the focus of a parabolic cavity
  $\longrightarrow$ changes plane asymptotically incoming wave into spherical wave converging to the focal point



Time evolution of spontaneous decay process: two two-level atoms in an elliptic cavity

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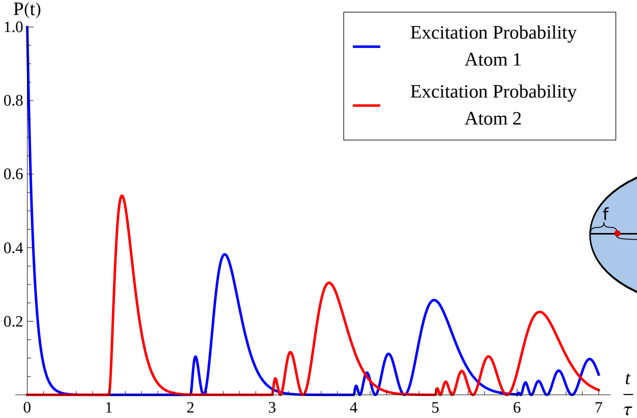
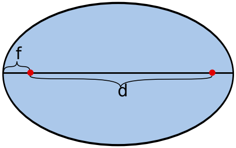
$\tau = (d + 2f)/c = 12.5/\Gamma_s(\omega_0)$ with eccentricity $\epsilon \ll 1 \rightarrow d/f = 2\epsilon/(1-\epsilon) \ll 1 \quad f \gg \lambda$





Asymptotic distribution of field fluctuations

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planar electromagnetic field energy density in asymptotic plane with z constant

$$\int_{-f}^{\infty} dz \, \langle \psi(t) | : \frac{\epsilon_0}{2} \left(\hat{\mathbf{E}}^2(\mathbf{x}) + c^2 \hat{\mathbf{B}}^2(\mathbf{x}) \right) : | \psi(t) \rangle \quad = \quad \hbar\omega_0 H(y), \quad y = (\rho/(2f))^2$$

semiclassical path representation for $f \ll c/\Gamma_s(\omega_0)$

$$H(y) \quad = \quad \frac{\Gamma_s(\omega_0)}{\Gamma(\omega_0)} \frac{1}{(2f)^2 \pi} \left(6 \frac{y}{(1+y)^4} + 12 \sum_{M=1}^{\infty} \cos(2\pi M n_0) \frac{y e^{2Mu}}{(1+y)^2 (y + e^{2Mu})^2} \right)$$

$$\longrightarrow \quad \int_0^{2\pi} d\varphi \int_0^{\infty} d\rho \, \rho \, H(y) = 1$$

$u = \pi^2 n_{0x}$ stability of periodic photon path

Asymptotic distribution of field fluctuations

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planar electromagnetic field energy density in asymptotic plane with z constant

$$H(y) = \frac{\Gamma_s(\omega_0)}{\Gamma(\omega_0)} \frac{1}{(2f)^2\pi} \underbrace{\left(6\frac{y}{(1+y)^4} + 12\sum_{M=1}^{\infty} \cos(2\pi M n_0) \frac{y e^{2Mu}}{(1+y)^2 (y + e^{2Mu})^2} \right)}_{:=h(y)}$$

$f\omega_0/c = \pi/2 \cdots$ dashed

$f\omega_0/c = 3\pi/2 \cdots$ full