Relativistic Wigner functions

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On the Quantum Correction For Thermodynamic Equilibrium

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The probability of a configuration is given in classical theory by the Boltzmann formula $\exp[-V/hT]$ where V is the potential energy of this configuration. For high temperatures this of course also holds in quantum theory. For lower temperatures, however, a correction term has to be introduced, which can be developed into a power series of h. The formula is developed for this correction by means of a probability function and the result discussed.

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In classical statistical mechanics the relative probability for the range p_1 to p_1+dp_1; p_2 to p_2+dp_2; $\cdots$; p_n to p_n+dp_n for the momenta and x_1 to x_1+dx_1; x_2 to x_2+dx_2; $\cdots$; x_n to x_n+dx_n for the coordinates is given for statistical equilibrium by the Gibbs-Boltzmann formula

$$P(x_1, \cdots, x_n; p_1, \cdots, p_n)dx_1 \cdots dx_n dp_1 \cdots dp_n = e^{-\beta\epsilon} dx_1 \cdots dx_n dp_1 \cdots dp_n \quad (1)$$

where ϵ is the sum of the kinetic and potential energy V

Two fathers of the Wigner function

Leó Szilárd (1898-1964)

Eugene Wigner (1902-1995)

Footnote in Wigner's paper:

"This expression was found by L. Szilard and the present author some years ago for another purpose"

Outline

- **Wigner function in nonrelativistic QM**
- **Wigner function for the Dirac particles**
- **Wigner function for the photons**
- **Wigner function and the number of quanta**
- **Wigner function for the thermal state**
- **Electromagnetic field as a huge oscillator**
- **Wigner functional for the vacuum state**
- **Wigner functional at finite temperature**

Standard Wigner function

$$W(\vec{r},\vec{p},t) = \int \frac{d^3\eta}{(\pi\hbar)^3} e^{2i\vec{p}\cdot\vec{\eta}/\hbar}\psi(\vec{r}-\vec{\eta},t)\psi^*(\vec{r}+\vec{\eta},t)$$

Wigner function for the state ρ is the expectation value of the hermitian operator $\mathcal{W}_{\vec{r}\vec{p}}$

$$W(\vec{r},\vec{p},t) = \mathrm{Tr}\{\rho(t)\,\mathcal{W}_{\vec{r}\vec{p}}\}$$

$$\mathcal{W}_{\vec{r}\vec{p}} = \frac{1}{(\pi\hbar)^3}\int d^3\eta |\vec{r}+\vec{\eta}\rangle e^{2i\vec{p}\cdot\vec{\eta}/\hbar}\langle\vec{r}-\vec{\eta}| = \frac{1}{(\pi\hbar)^3}\sqrt{\mathbb{1}}$$

Time evolution is "almost" classical

$$\partial_t W(\vec{r},\vec{p},t) = -\left(\tfrac{1}{m}\vec{p}\cdot\vec{\nabla} + \vec{F}\cdot\vec{\partial}_p + \mathcal{O}(\hbar^2)\right)W(\vec{r},\vec{p},t)$$

Wigner function for the Dirac particle

In a review paper† we read "It (*the Wigner function*) is non-relativistic in nature because it is not invariant under the Lorentz group; also configuration space quantum mechanics for more than one particle would be difficult to formulate relativistically" Despite these warnings we proceeded in 1991 to show that the original concept of Szilard and Wigner can be applied to relativistic electrons

†M. Hillert, R. F. O'Connell, M, O. Scully, and E. P. Wigner Physics Reports **106**, 121 (1984)

Wigner function for the Dirac particle

The difficulties announced by Wigner et. al. are solved by using the idea that goes back to Kadanoff and Baym[†]
They noted (in the non-relativistic theory) that one may replace *wave functions* **by the expectation values of** *field operators*

$$W_{\alpha\beta}(\vec{r},\vec{p},t) = \int d^3\eta e^{2i\vec{p}\cdot\vec{\eta}/\hbar}\langle\Phi|\hat{\psi}_\alpha(\vec{r}-\vec{\eta},t)\hat{\psi}^\dagger_\beta(\vec{r}+\vec{\eta},t)|\Phi\rangle$$

Different choices of $|\Phi\rangle$ lead to different Wigner functions

[†]Leo Kadanoff and Gordon Baym Quantum Statistical Mechanics 1962

Some refinements

The identity $\hat{\psi}\hat{\psi}^\dagger = \frac{1}{2}[\hat{\psi}, \hat{\psi}^\dagger] + \frac{1}{2}\{\hat{\psi}, \hat{\psi}^\dagger\}$ **enables one to replace the product of the field operators by the commutator since the anticommutator is a number**

The second refinement is the introduction of the line integral to make $W_{\alpha\beta}$ **gauge invariant**

Our final definition of $W_{\alpha\beta}$ **was**[†]

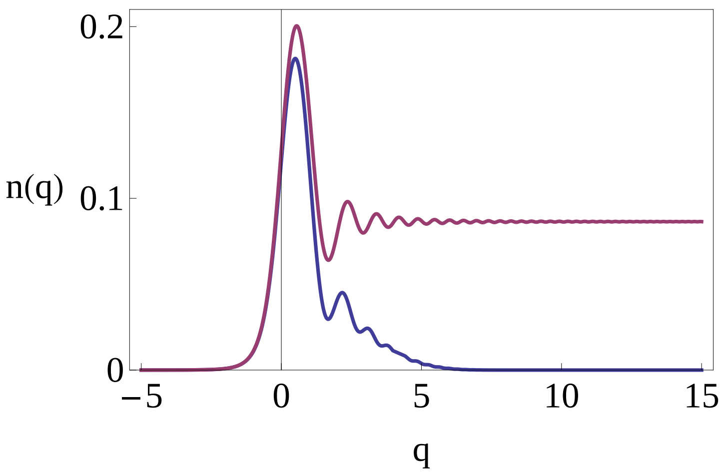
$$W_{\alpha\beta}(\vec{r}, \vec{p}, t) = -\frac{1}{2} \int d^3\eta \, e^{2i\vec{p}\cdot\vec{\eta}/\hbar}$$

$$\times \langle \Phi | e^{-ie \int_{-1}^{1} d\lambda \, \vec{\eta}\cdot\vec{A}(\vec{r}+\lambda\vec{\eta},t)} \left[\hat{\psi}_\alpha(\vec{r} - \vec{\eta}, t), \hat{\psi}^\dagger_\beta(\vec{r} + \vec{\eta}, t) \right] |\Phi\rangle$$

[†]IBB, P. Górnicki and J. Rafelski, Phys. Rev. D **44**, 1825 (1991)

Simple application

The evolution equations for the Wigner function can be solved in some simple cases, e. g. Pair production by a time-dependent electric field



The pair density $n(q)$ as a function of the particle momentum for the constant field (upper curve) and for the switched on field (lower curve)

Wigner function for photons

Natural replacement for the electron field operator is the Riemann-Silberstein vector $\vec{F}$

$$\vec{F}(\vec{r},t) = \frac{\vec{D}(\vec{r},t)}{\sqrt{2\epsilon}} + i\frac{\vec{B}(\vec{r},t)}{\sqrt{2\mu}} \qquad i\partial_t\vec{F}(\vec{r},t) = c\nabla\times\vec{F}(\vec{r},t)$$

The Wigner function is now a 3×3 matrix

$$W_{ij}(\vec{r},\vec{k},t) = -\tfrac{1}{2}\int d^3\eta\, e^{2i\vec{k}\cdot\vec{\eta}/\hbar}$$

$$\times\langle\Phi|\left\{\hat{F}_i(\vec{r}-\vec{\eta},t),\hat{F}_j^\dagger(\vec{r}+\vec{\eta},t)\right\}|\Phi\rangle$$

Useless in an inhomogeneous medium
Except when medium is a gravitational field

Wigner function(al) for the whole electromagnetic field

The concept of the Wigner function can be extended from particles to fields

W **becomes the functional and its arguments are the electric and magnetic field vectors**

$$W(\vec{r}, \vec{p}, t) \to W[\vec{B}, \vec{D}, t]$$

$W[\vec{B}, \vec{D}, t]$ **can be constructed by analogy with the one-dimensional harmonic oscillator without any reference to quantum field theory**

Wigner function for the harmonic oscillator

Ground state of the harmonic oscillator in 1D

$$\psi_G(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$W_G(x,p) = \frac{1}{\pi\hbar} \exp\left(-\frac{2H(p,x)}{\hbar\omega}\right) = \frac{1}{\pi\hbar} \exp\left(-2N(p,x)\right)$$

$$H(p,x) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Number of quanta $N(p,x) = \frac{H(p,x)}{\hbar\omega}$

Wolfgang Schleich, Quantum Optics in Phase Space 2001

Wigner function for the coherent state and for the n-th state

The coherent state

$$W_C(x,p) = \frac{1}{\pi\hbar}\exp\left[-\frac{2H(p-\langle p\rangle, x-\langle x\rangle)}{\hbar\omega}\right]$$

Wigner function for the coherent state is the generating function for all eigenstates

$$\begin{aligned}\langle n|A|n\rangle &= \frac{1}{n!}(\partial_\alpha \partial_\alpha^*)^n \exp\left(|\alpha|^2\right)\langle\alpha|A|\alpha\rangle\Big|_{\alpha=0} \\ &= \frac{1}{n!}(\partial_\alpha \partial_\alpha^*)^n \langle 0|\exp(\alpha^* a) A \exp(\alpha a^\dagger)|0\rangle\Big|_{\alpha=0}\end{aligned}$$

Wigner function for the thermal state of an oscillator

The result involves Laguerre polynomials

$$W_n(x,p) = \frac{1}{\pi\hbar}(-1)^n \exp\left[-2N(p,x)\right] L_n\left[4N(p,x)\right]$$

The thermal state $\rho = \frac{e^{-\frac{\hat{H}}{k_B T}}}{\mathrm{Tr}\{e^{-\frac{\hat{H}}{k_B T}}\}}$

$$W_T(x,p) = \frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)\exp\left[-2\tanh\left(\frac{\hbar\omega}{2k_BT}\right)\frac{H(p,x)}{\hbar\omega}\right]$$

Electromagnetic field as a huge harmonic oscillator

Translation from 1D to ∞D

$$H(p,x) \to H[\vec{D},\vec{B}] = \frac{1}{2}\int d^3r \left[\frac{\vec{D}(\vec{r})\cdot\vec{D}(\vec{r})}{\epsilon} + \frac{\vec{B}(\vec{r})\cdot\vec{B}(\vec{r})}{\mu}\right]$$

$$N[\vec{D},\vec{B}] = \frac{1}{4\pi^2\hbar c}\int d^3r \int d^3r' \left[\frac{\vec{D}(\vec{r})\cdot\vec{D}(\vec{r}')}{\epsilon|\vec{r}-\vec{r}'|^2} + \frac{\vec{B}(\vec{r})\cdot\vec{B}(\vec{r}')}{\mu|\vec{r}-\vec{r}'|^2}\right]$$

$$\text{``}\frac{1}{\hbar\omega}\text{''} = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{\hbar ck} = \frac{1}{2\pi^2\hbar c|\vec{r}-\vec{r}'|^2}$$

Wigner functionals for the ground state and for the coherent states[†]

Ground state (vacuum)

$$W_G[\vec{B}, \vec{D}] = \exp(-2N[\vec{D}, \vec{B}])$$

Note the change in normalization!

Coherent state

$$W_C[\vec{B}, \vec{D}] = \exp(-2N[\vec{D} - \langle \vec{D} \rangle, \vec{B} - \langle \vec{B} \rangle])$$

[†]IBB, Optics Communications, **179**, 237 (2000)

Wigner functional for the thermal state

$$\text{“}\frac{1}{\hbar\omega}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)\text{”} = \int\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{\hbar ck}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)$$

$$\frac{1}{2\pi^2\hbar c|\vec{r}-\vec{r}'|^2} \to \frac{kT}{\pi\hbar^2c^2|\vec{r}-\vec{r}'|\sinh(2\pi kT|\vec{r}-\vec{r}'|/\hbar c)}$$

Field correlations

Using the formula

$$\langle x^k x^l \rangle = \frac{\int d^n x \; x^k x^l \exp(-\frac{1}{2} A_{ij} x^i x^j)}{\int d^n x \; \exp(-\frac{1}{2} A_{ij} x^i x^j)} = (A^{-1})_{kl}$$

we obtain field correlation functions, for example

$$\langle B_i(\vec{r}_1) B_j(\vec{r}_2) \rangle = P_{ij} \, \frac{1}{2|\vec{r}_1 - \vec{r}_2|} \frac{\pi \mu \hbar c \, \cosh\left(\frac{\pi |\vec{r}_1 - \vec{r}_2|}{\lambda_T}\right)}{\left[\lambda_T \sinh\left(\frac{\pi |\vec{r}_1 - \vec{r}_2|}{\lambda_T}\right)\right]^3}$$

$$P_{ij} = \delta_{ij} + \frac{1}{4\pi} \partial_i \partial_j \int d^3 r' \, \frac{1}{|\vec{r} - \vec{r}\,'|} \qquad \lambda_T = \frac{\hbar c}{k_B T}$$

Thermal length λ_T

The same field correlations are obtained by the standard field-theoretic methods Wigner functional gives us a better insight into the statistical properties of the EM field

Long range correlations are wiped out by thermal fluctuations and they have the range λ_T At $T = 2.7$K the range is merely 0.85 millimeter At room temperature λ_T=7.6 microns

Wigner functional is nonlocal

Instantaneous measurements of correlation functions inside a bounded region give information about the system outside the boundary

Of course this does not ruin causality
All changes in the system configuration propagate with the velocity of light