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WIGNER

The role of the Wigner function in charged particle dynamics

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Outline

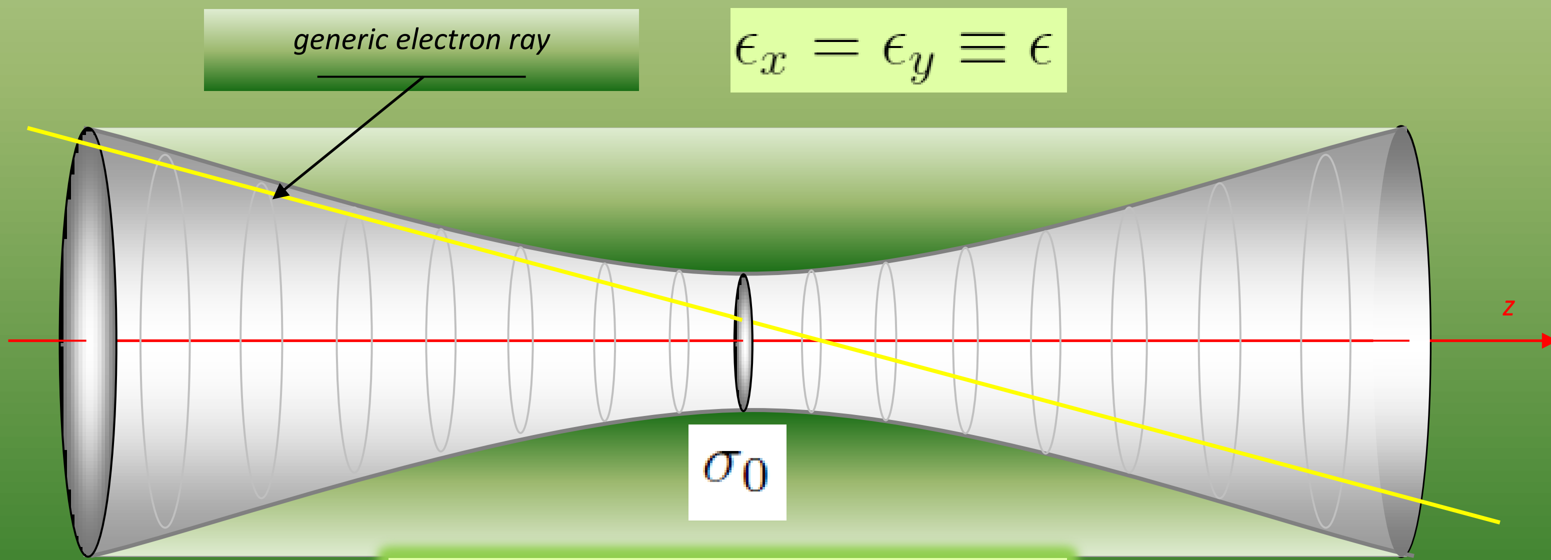
- **Motion of relativistic non laminar charged particle beams (thermal regime):**
  - ***classical picture (geometrical optics)***
  - ***quantum-like picture (paraxial approximation)***
- **Quantum-like description of particle beam dynamics in high-energy accelerating machines (configuration space)** ***vs*** **conventional description**
- **Quantum-like description of particle beam dynamics in high-energy accelerating machines with the use of the Wigner function (phase space)** ***vs*** **conventional description**
- **Conclusions and remarks**

Motion of relativistic non laminar charged particle beams: classical picture of thermal spreading

Qualitative rappresentation of the free envelope motion (paraxial approximation) of a cylindrically-symmetric beam travelling in vacuo.

$$\varepsilon_j(z) = 2[\sigma_j^2(z)\,\sigma_{pj}^2(z) - \sigma_{jpj}^2(z)]^{1/2}$$ (**beam emittance**)

$$\epsilon_x = \epsilon_y \equiv \epsilon$$



σ_0

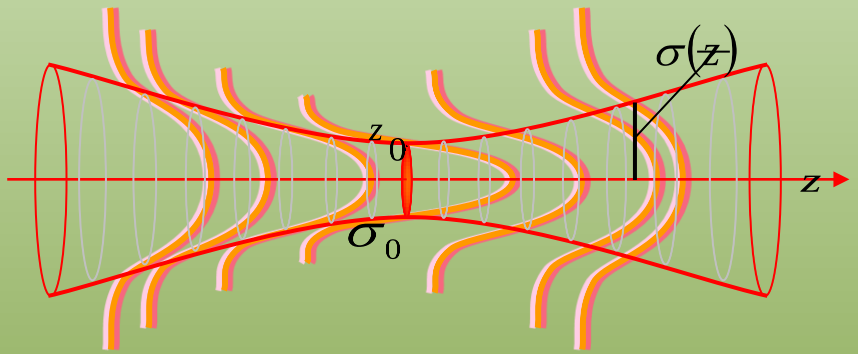
$\sigma(z)$

$$\sigma(z) = \sqrt{\sigma_0^2 + \frac{\epsilon^2}{\sigma_0^2}(z - z_0)^2}$$

Motion of relativistic non laminar charged particle beams: quantum-like picture

FREE PROPAGATION

$$i\varepsilon \frac{\partial \Psi}{\partial z} = -\frac{\varepsilon^2}{2} \nabla_\perp^2 \Psi$$



$$\Psi(x,y,z) = \text{beam wave function (BWF)}$$

$$|\Psi(x,y,z)|^2 \propto \text{transverse density beam profile}$$

$$\Psi(x,y,z) = \sqrt{n(x,y,z)} \exp\left[\frac{i}{\epsilon}\theta(x,y,z)\right]$$

$$\int_{-\infty}^{\infty} |\Psi(x,y,z)|^2 \, dx \, dy = \int_{-\infty}^{\infty} n(x,y,z) \, dx \, dy = N$$

$$\sigma(z) = \sqrt{\sigma_0^2 + \frac{\epsilon^2}{\sigma_0^2}(z - z_0)^2}$$

Thermal Wave Model (TWM)

R. Fedele and G. Miele, *Nuovo Cim. D* 13, 1527 (1991)

Quantum-like picture: thermal uncertainty and collective effects

- **The thermal spreading introduces an uncertainty in the electron ray positions in the transverse plane at any longitudinal location and in the corresponding electron ray slopes. A paraxial beam emulates the paraxial diffraction of the electromagnetic radiation**

- **The electron optics in thermal regime considers the beams that consist of a large number of charged particles. Such beams are governed by the electromagnetic interactions among the particles i.e. they feature a collective behavior which is affected by the thermal spreading of the particles.**

Quantum-like picture of transverse dynamics: thermal uncertainty and collective effects

$$U\left(x,y,z\right)=U^{ext}\left(x,y,z\right)+U^{coll}\left(\left|\Psi\left(x,y,z\right)\right|^{2}\right)$$

dimensionless potential energy (normalized with respect to $m_0\gamma\beta^2c^2$)

$$i\varepsilon\frac{\partial\Psi}{\partial z}=-\frac{\varepsilon^{2}}{2}\nabla_{\perp}^{2}\Psi+U\left(x,y,z\right)\Psi$$

R. Fedele and G. Miele, *Nuovo Cim. D* 13, 1527 (1991)

R. Fedele and G. Miele, *Phys.Rev.A* 46, 6634 (1992)

R. Fedele and P.K. Shukla, *Phys. Rev. A* 44, 4045 (1992)

R. Fedele, G. Miele, L. Palumbo and V.G. Vaccaro, *Phys. Lett. A* 179, 407 (1993)

Fluid interpretation:

$n\left(x,y,z\right)=$ ***transverse probability density of the beam particles***

$\mathbf{V}\left(x,y,z\right)=\nabla_{\perp}\theta=$ ***transverse current velocity***

Quantum-like description of transverse dynamics in high-energy accelerating machines (configuration space)

- **LINEAR LENS WITH SEXTUPOLAR AND OCTUPOLAR ABERRATIONS**

$$i\epsilon\frac{\partial\Psi}{\partial z}=-\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\Psi+U(x,z)\Psi$$

Optical device:

quadrupole + sextupole + drift space (beam-transport line);

(typically, small sextupole and/or octupole aberrations are included in the quadrupole)

$$U(x,y,z)=\frac{1}{2!}k_1x^2+\frac{1}{3!}k_1x^3+\frac{1}{4!}k_1x^4 \qquad 0\le z\le l$$

$$U(x,y,z)=0 \qquad z>l$$

Quantum-like description of transverse dynamics in high-energy accelerating machines (configuration space)

APPLICATION OF THE TIME-DEPENDENT PERTURBATION THEORY

- AT THE LENS EXIT

$$\Psi(x,l) = \Psi_0(x)\exp\left(-\mathrm{i}\frac{K_1 x^2}{2\epsilon}\right)\left((1-3\mathrm{i}\omega)H_0(x/\sqrt{2}\sigma_0) - \mathrm{i}\frac{3\tau}{\sqrt{2}}H_1(x/\sqrt{2}\sigma_0) - 3\mathrm{i}\omega H_2(x/\sqrt{2}\sigma_0)\right.$$
$$\left. - \mathrm{i}\frac{\tau}{2\sqrt{2}}H_3(x/\sqrt{2}\sigma_0) - \tfrac{1}{4}\mathrm{i}\omega H_4(x/\sqrt{2}\sigma_0)\right),$$

- AFTER THE DRIFT OF LENGTH L

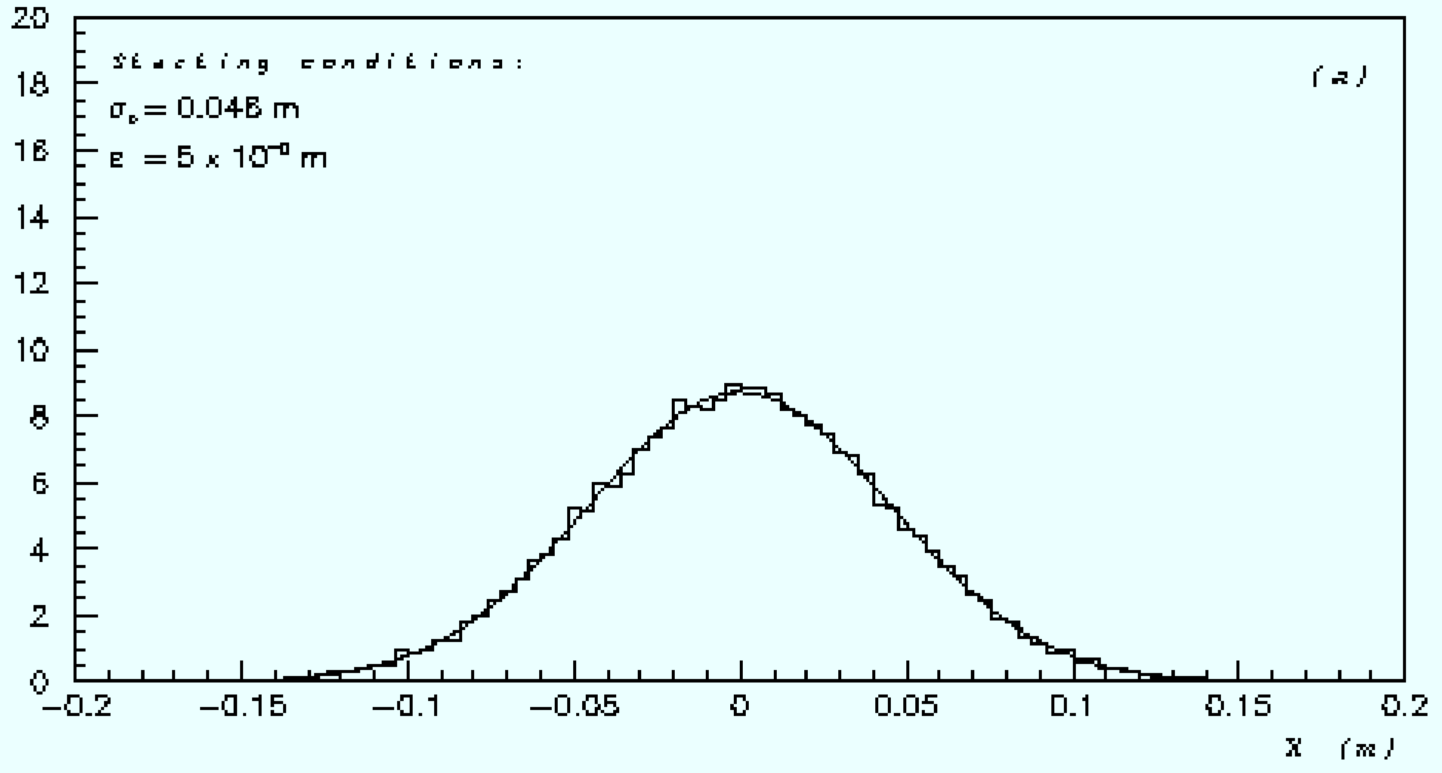
$$\Psi(x,L) = \frac{\exp\left[-x^2/4\sigma^2(L)\right]\exp\left[\mathrm{i}x^2/2\epsilon R(L) + \mathrm{i}\phi(L)\right]}{\left[2\pi\sigma^2(L)(1+15\tau^2+105\omega^2)^2\right]^{1/4}}\left((1-3\mathrm{i}\omega)H_0(x/\sqrt{2}\sigma(L))\right.$$
$$- \mathrm{i}\frac{3\tau}{\sqrt{2}}H_1(x/\sqrt{2}\sigma(L))\,\mathrm{e}^{2\mathrm{i}\phi(L)} - 3\mathrm{i}\omega H_2(x/\sqrt{2}\sigma(L))\,\mathrm{e}^{4\mathrm{i}\phi(L)}$$
$$\left. - \mathrm{i}\frac{\tau}{2\sqrt{2}}H_3(x/\sqrt{2}\sigma(L))\,\mathrm{e}^{6\mathrm{i}\phi(L)} - \tfrac{1}{4}\mathrm{i}\omega H_4(x/\sqrt{2}\sigma(L))\,\mathrm{e}^{8\mathrm{i}\phi(L)}\right),$$

Quantum-like description in configuration space: comparison with tracking code simulations

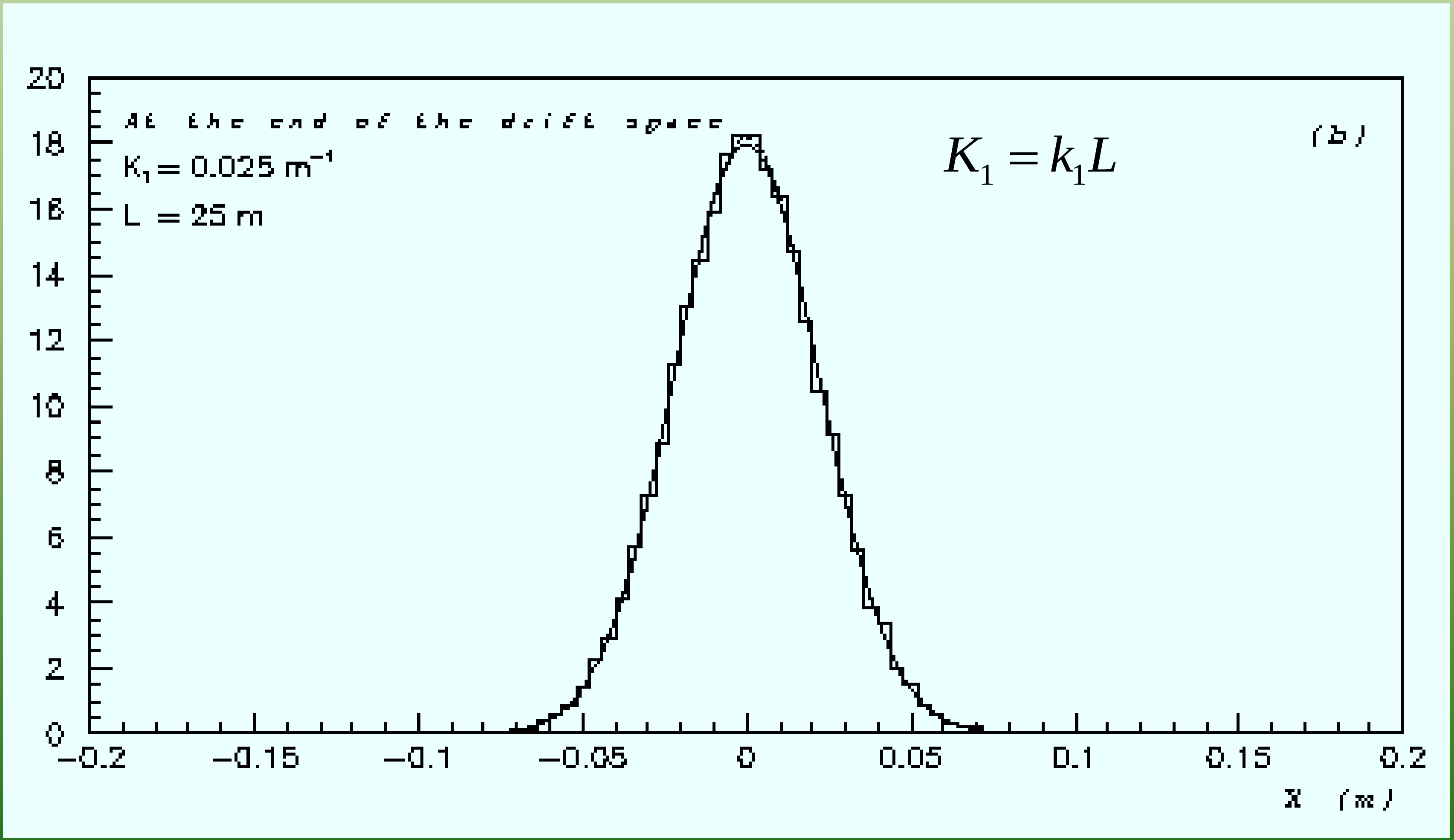
R. Fedele, F. Galluccio, G. Miele, Phys. Lett. A 185 (1994) 93

R. Fedele, F. Galluccio, V.I. Man'ko, G. Miele, Phys. Lett. A 209 (1995) 263.

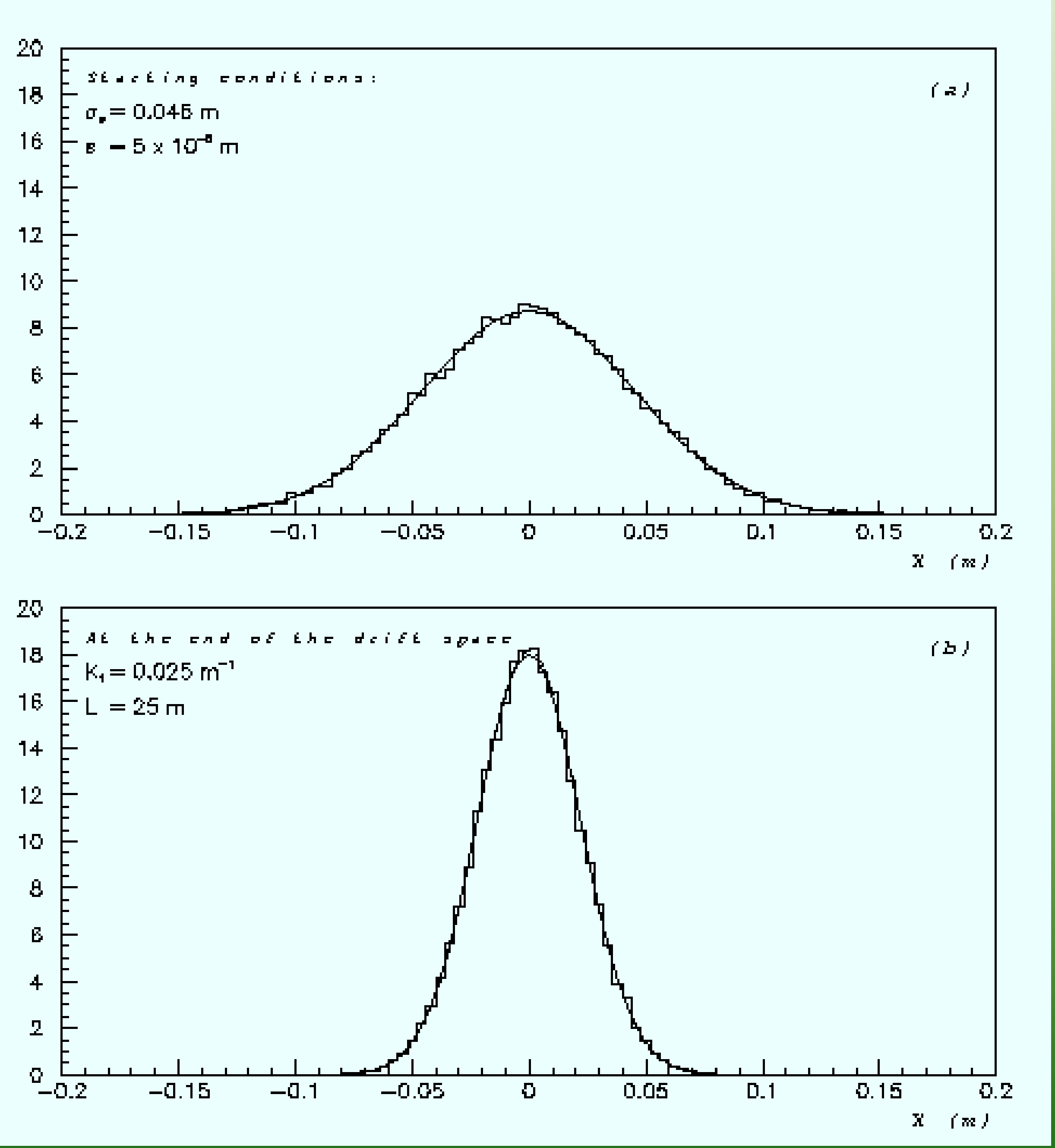
A Gaussian (1+1)-D particle beam with flat envelope



Quantum-like description in configuration space: comparison with tracking code simulations

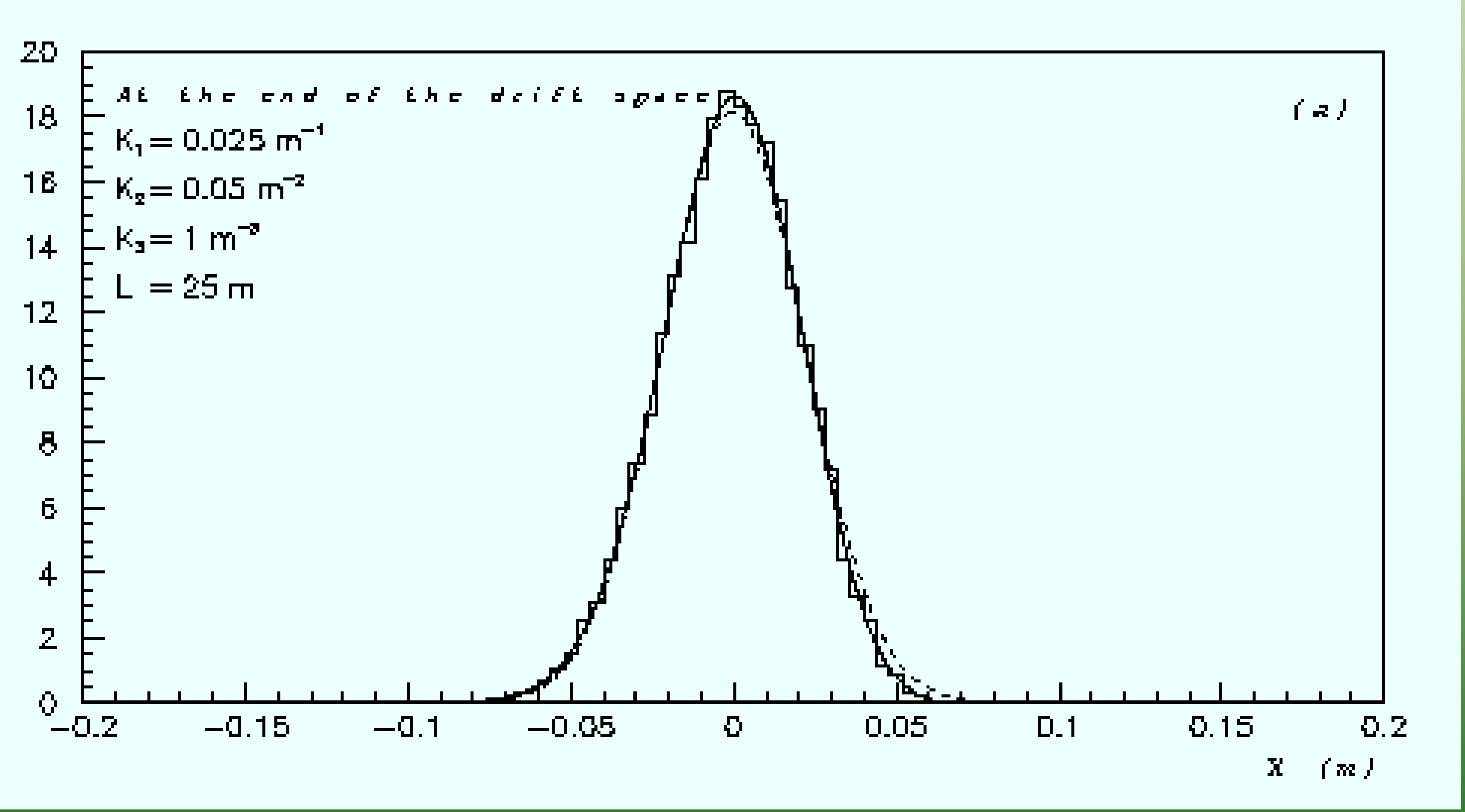


$$K_1 = k_1 L$$



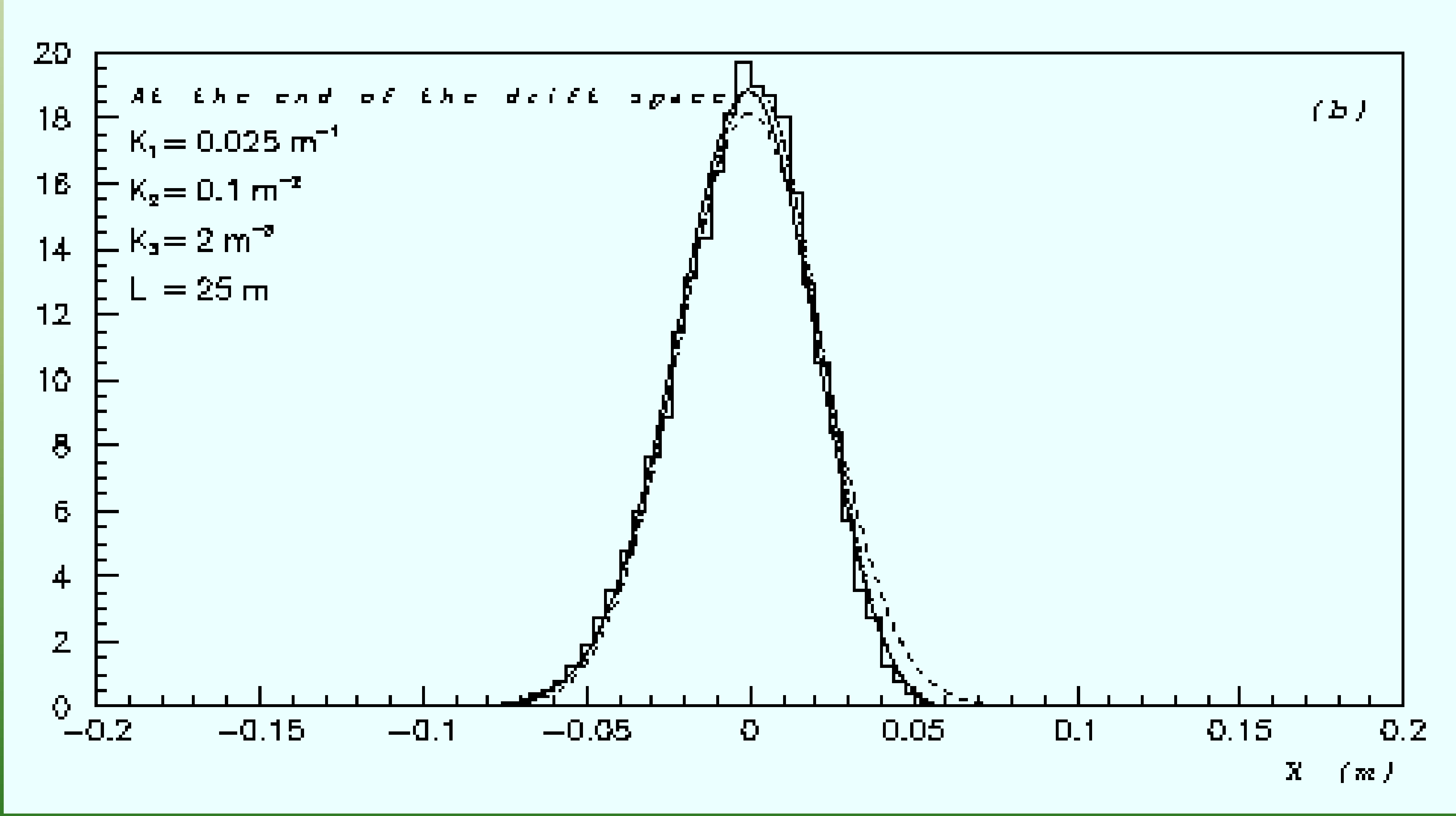
Stacking conditions:
(a)
$\sigma_x = 0.046$ m
$\varepsilon = 5 \times 10^{-6}$ m
X (m)
At the end of the drift space
(b)
$K_1 = 0.025$ m^{-1}
L = 25 m
X (m)

Quantum-like description in configuration space: comparison with tracking code simulations

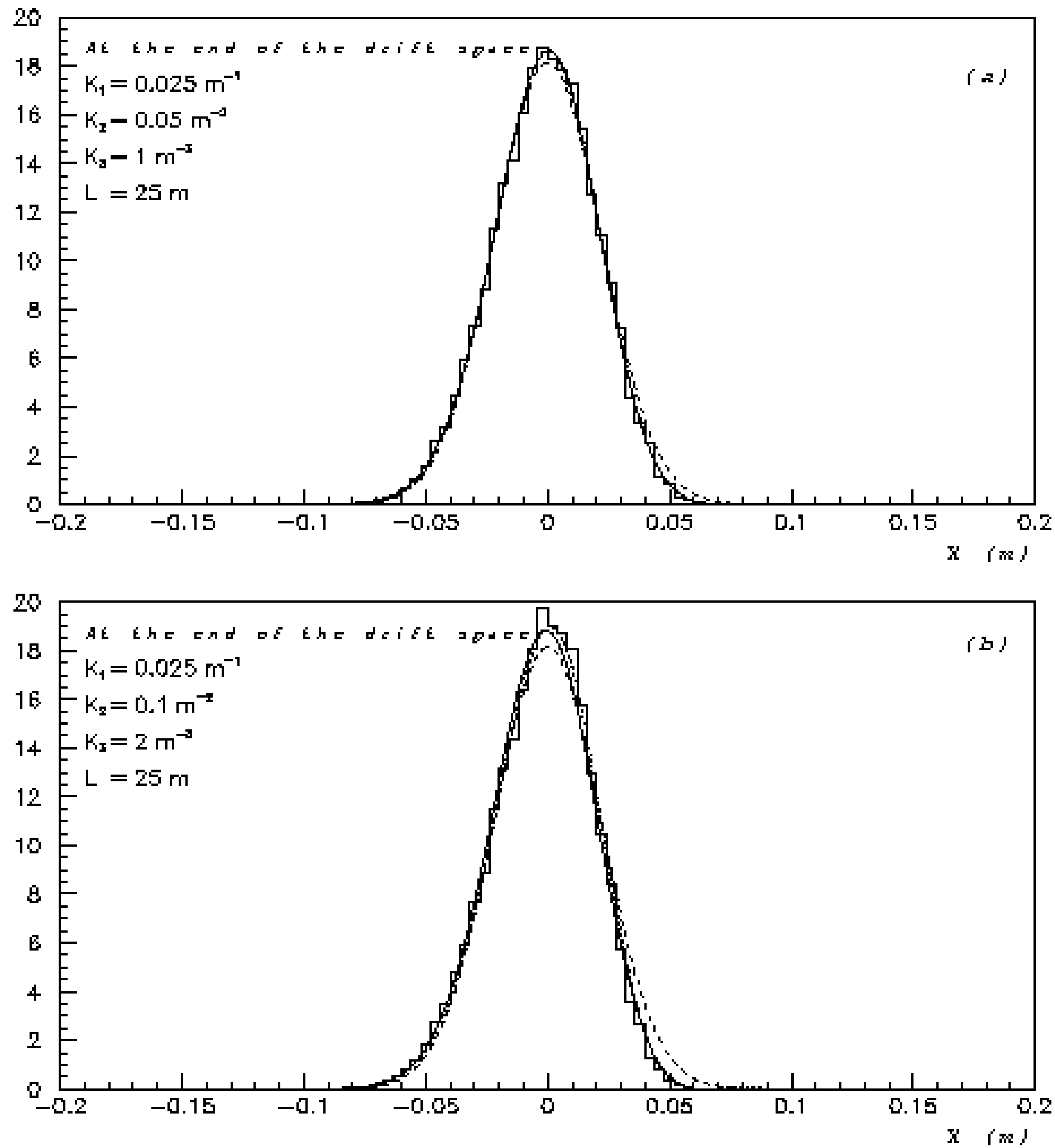


$$K_1 = k_1 L \qquad K_2 = k_2 L \qquad K_3 = k_3 L$$

Quantum-like description in configuration space: comparison with tracking code simulations



$$K_1 = k_1 L \qquad K_2 = k_2 L \qquad K_3 = k_3 L$$



$$K_1 = k_1 L \qquad K_2 = k_2 L \qquad K_3 = k_3 L$$

Quantum-like description in configuration space: comparison with tracking code simulations

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Investigation of the transverse beam dynamics in the thermal wave model with a functional method

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Abstract

We investigated the transverse beam dynamics in a thermal wave model by using a functional method. It can describe the beam optical elements separately with a kernel for a component. The method can be applied to general quadrupole magnets beyond a thin lens approximation as well as drift spaces. We found that the model can successfully describe the PARMILA simulation result through an FODO lattice structure for the Gaussian input beam without space charge effects.

PACS: 29.27.-a; 29.27.Eg

Keywords: Transverse beam dynamics; Thermal wave model; Functional method

Quantum-like description in configuration space: comparison with tracking code simulations

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Transverse beam dynamics including higher order perturbations in the thermal wave model using a functional method

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ABSTRACT

We studied the transverse beam dynamics including sextupole and octupole perturbations in a thermal wave model. A functional integration method was used to calculate the first order perturbation effects. We found that the model successfully explains a PARMILA simulation results for proton beams without space charge effects in a lattice with 10 FODO cells.

Quantum-like description of beam dynamics in phase space

R Fedele, F. Galluccio, V.I. Man'ko and G. Miele, Phys. Lett. A 209,263 (1995)
R. Fedele, V.I. Man'ko, Phys. Rev. E 58, 992 (1998
R Fedele, MA Man'ko, V.I. Man'ko, JOSA A 17, 2506-2512 (2000)
R Fedele, MA Man'ko, Eur. Phys. J. D 27, 263–271 (2003)

The agreement between the TWM predictions and the tracking code simulations is not sufficient if done in the configuration space only. The comparison becomes satisfactory if a similar analysis is carried out in the momentum space or in phase space.

the Wigner quasidistribution

$$W(x,p,z) = \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} \Psi^*\left(x + \tfrac{1}{2}y, z\right) \Psi\left(x - \tfrac{1}{2}y, z\right) \exp\left(\mathrm{i}\frac{py}{\epsilon}\right) \mathrm{d}y.$$

$$\left\{\frac{\partial}{\partial z} + p\frac{\partial}{\partial x} + \frac{\mathrm{i}}{\epsilon}\left[U\left(x + \frac{\mathrm{i}\epsilon}{2}\frac{\partial}{\partial p}\right) - U\left(x - \frac{\mathrm{i}\epsilon}{2}\frac{\partial}{\partial p}\right)\right]\right\} W = 0$$

Quantum-like description in phase space: comparison with tracking code simulations

R Fedele, F. Galluccio, V.I. Man'ko and G. Miele, Phys. Lett. A 209,263 (1995)

***Optical device*:**

quadrupole + sextupole + drift space (beam-transport line);
(typically, small sextupole and/or octupole aberrations are included in the quadrupole)

$$U(x,y,z) = \frac{1}{2!}k_1 x^2 + \frac{1}{3!}k_1 x^3 + \frac{1}{4!}k_1 x^4 \qquad 0 \le z \le l$$

$$U(x,y,z) = 0 \qquad z > l$$

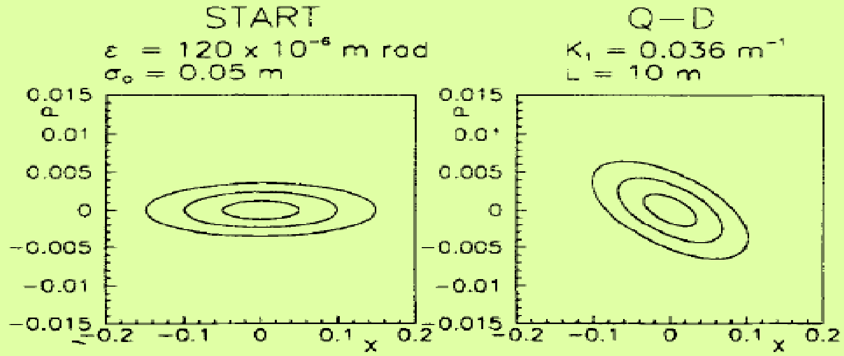
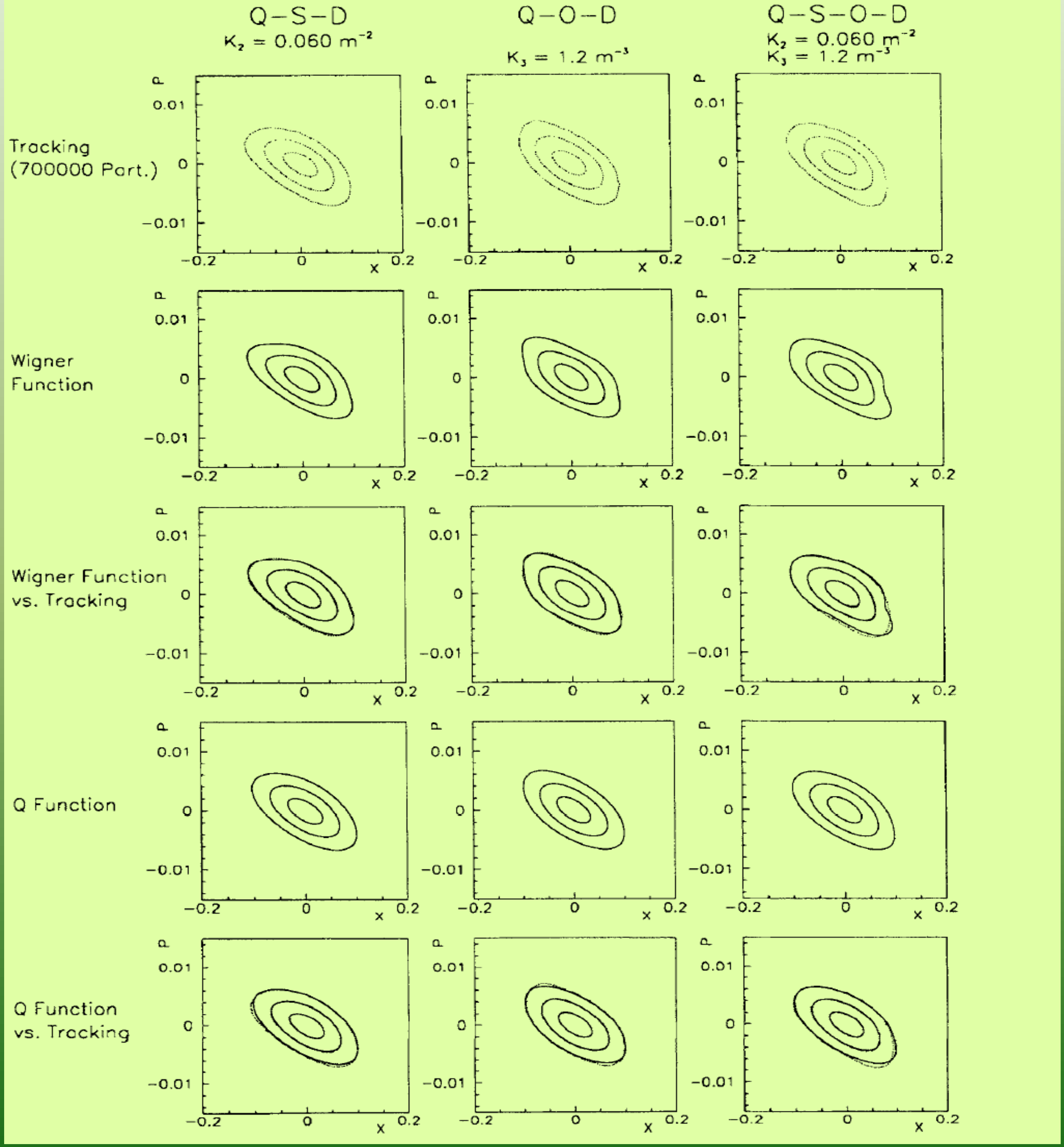
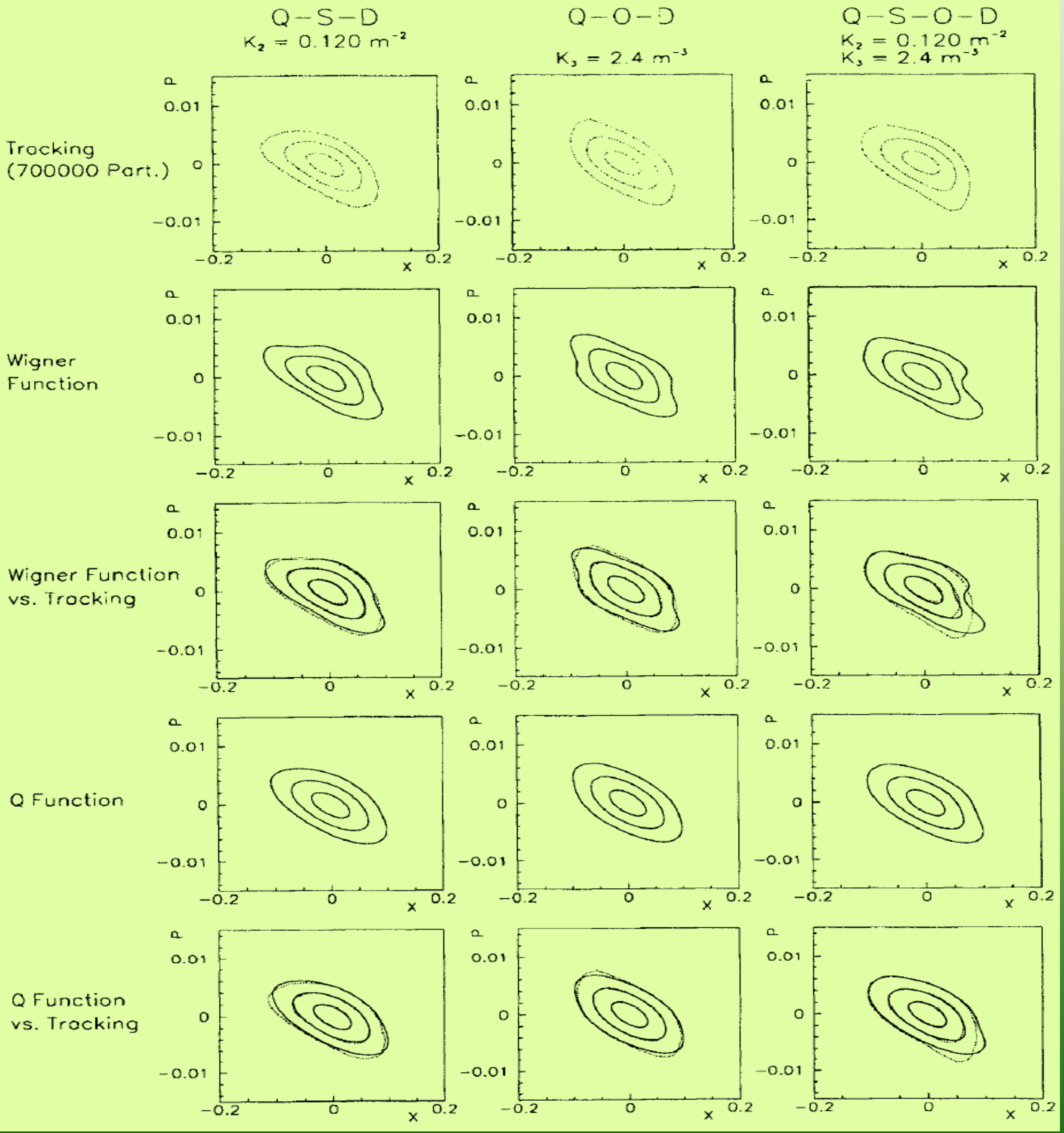


Fig. 1. Phase-space distribution at the starting point (left) and after a quadrupole lens plus drift space (right). Starting from the center of the distributions the isodensity contours correspond to 1σ, 2σ and 3σ, respectively.


Q−S−D
$K_2 = 0.060\ m^{-2}$
Q−O−D
$K_3 = 1.2\ m^{-3}$
Q−S−O−D
$K_2 = 0.060\ m^{-2}$
$K_3 = 1.2\ m^{-3}$
Tracking (700000 Part.)
Wigner Function
Wigner Function vs. Tracking
Q Function
Q Function vs. Tracking
p
0.01
0
−0.01
−0.2
0
x
0.2



Q–S–D
K₂ = 0.120 m⁻²
Q–O–D
K₃ = 2.4 m⁻³
Q–S–O–D
K₂ = 0.120 m⁻²
K₃ = 2.4 m⁻³
Tracking
(700000 Part.)
Wigner
Function
Wigner Function
vs. Tracking
Q Function
Q Function
vs. Tracking
p
x
0.01
0
−0.01
−0.2
0.2

Quantum-like description of longitudinal dynamics in high-energy accelerating machines (configuration space)

$$i\varepsilon_Z \frac{\partial \Psi}{\partial \zeta} = -\frac{\varepsilon_Z^2 \eta}{2} \frac{\partial^2 \Psi}{\partial \xi^2} + U(\xi, \zeta)\Psi$$

$$U(\xi,\zeta) = U^{ext}(\xi,\zeta) + U^{coll}\left(\left|\Psi(\xi,\zeta)\right|^2\right)$$

$$U^{coll}\left(\left|\Psi(\xi,\zeta)\right|^2\right) = \frac{q^2 \beta c}{m\gamma c^2}\left[\rho_0 Z_I^{'}\left(\left|\Psi(\xi,\zeta)\right|^2 - \left|\Psi_0\right|^2\right) + Z_R^{'}\int\left(\left|\Psi(\xi,\zeta)\right|^2 - \left|\Psi_0\right|^2\right)d\xi\right]$$

$$\eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \qquad \left(\frac{\Delta\omega}{\omega} = \eta \frac{\Delta p}{p}\right)$$

1/η plays the role of an effective mass

$$Z = Z_R + iZ_I$$ longitudinal coupling impedance

$$\lambda_1 = \left|\Psi\right|^2 - \left|\Psi_0\right|^2$$ longitudinal bunch density arbitrary perturbation

Quantum-like description of longitudinal dynamics in high-energy accelerating machines (configuration space)

R. Fedele, G. Miele, L. Palumbo and V.G. Vaccaro, *Phys. Lett. A* 179, 407 (1993)

D. Anderson *et al*. ,Phys. Lett. A 258, 244 (1999)

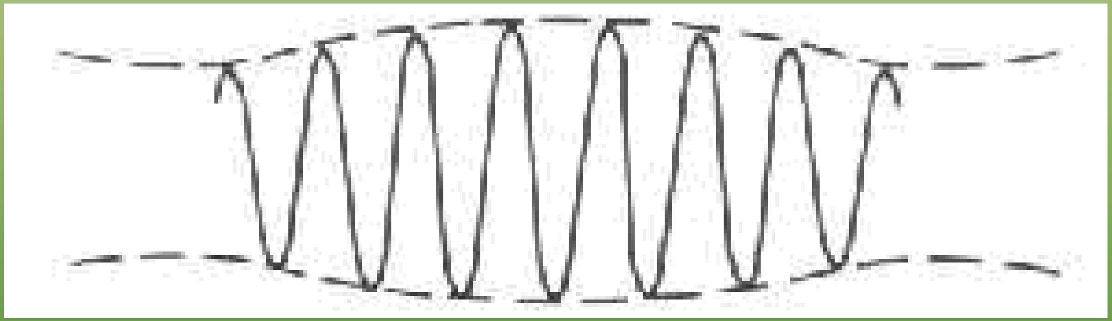
$$i\alpha \frac{\partial \Psi}{\partial \zeta} = -\frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial \xi^2} + \kappa\left(\left|\Psi\right|^2 - \left|\Psi_0\right|^2\right)\Psi + \mu\Psi\int\left(\left|\Psi\right|^2 - \left|\Psi_0\right|^2\right)d\xi$$

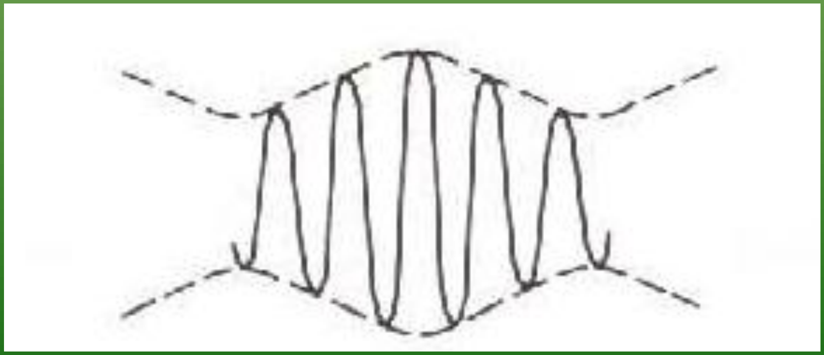
$$\alpha = \varepsilon_z \eta \qquad \kappa \propto Z_I \qquad \mu \propto Z_R$$

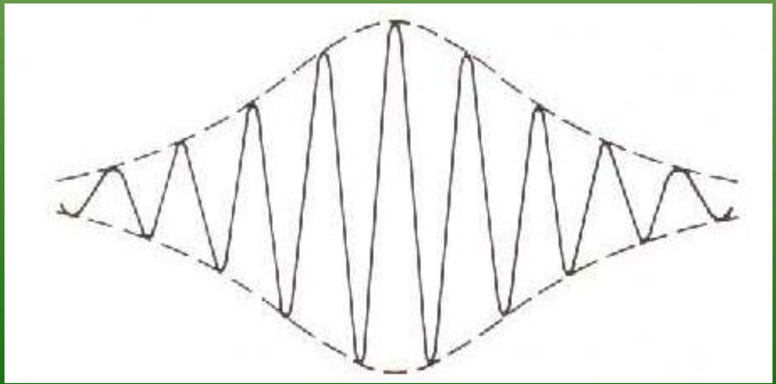
This equation has been used to describe a number of physical problems in high-energy charged particle coasting beams in accelerating machines:

- **synchrotron oscillations with and without radiation damping and quantum excitation effects**
- **soliton structure predictions**
- **coherent instabilities of coasting beams, with the language of the modulational instability, and the stabilizing role played by the Landau damping**
- ***nonlocal* effects of charged-particle beams**

IMPORTANT FEATURE OF NLSE
to govern the evolution of the
MODULATIONAL INSTABILITY (MI)







Purely reactive impedance: $Z_R = 0, Z = iZ_I$

$$\Psi(\xi,\zeta) = \psi(\xi,\zeta)\exp\left(i\kappa|\Psi_0|^2\zeta\right)$$

$$i\alpha\frac{\partial\psi}{\partial\zeta} = -\frac{\alpha^2}{2}\frac{\partial^2\psi}{\partial\xi^2} + \kappa|\psi|^2\psi$$

Envelope solitons: in principle, the cubic NLSE admits bright, dark and grey solitons. In particular, for $\kappa < 0, E < 0$

Bright envelope soliton

$$\psi(\xi,\zeta) = \frac{1}{\sqrt{2\sigma_z}}\operatorname{sech}\left(\frac{\xi - V_0\zeta}{\sigma_z}\right)\exp\left\{\frac{i}{\alpha}\left[V_0\xi - \left(E + \frac{V_0^2}{2}\right)\zeta\right]\right\}$$

$$E = -\frac{\alpha^2}{2\sigma_z^2} = -\frac{\kappa^2}{8\alpha^2}$$

• **If N is the total number of particles of the bunch:**

$$\lambda(\xi,\zeta) = \frac{N}{2\sigma_z}\,\mathrm{sech}^2\left(\frac{\xi - V_0\zeta}{\sigma_z}\right)$$

• **Coherent instability condition for coasting beams (by means of a standard modulational instability analysis): $\eta\, Z_I > 0$, $Z_I = X_L - X_C$, $\eta = 1/\gamma^2 - 1/\gamma_c^2$**

	$Z_I < 0$ (inductive impedance)	**$Z_I > 0$** (capacitive impedance)
$\eta < 0$ (above transition energy)	**instability**	**stability**
$\eta > 0$ (below transition energy)	**stability**	**instability**

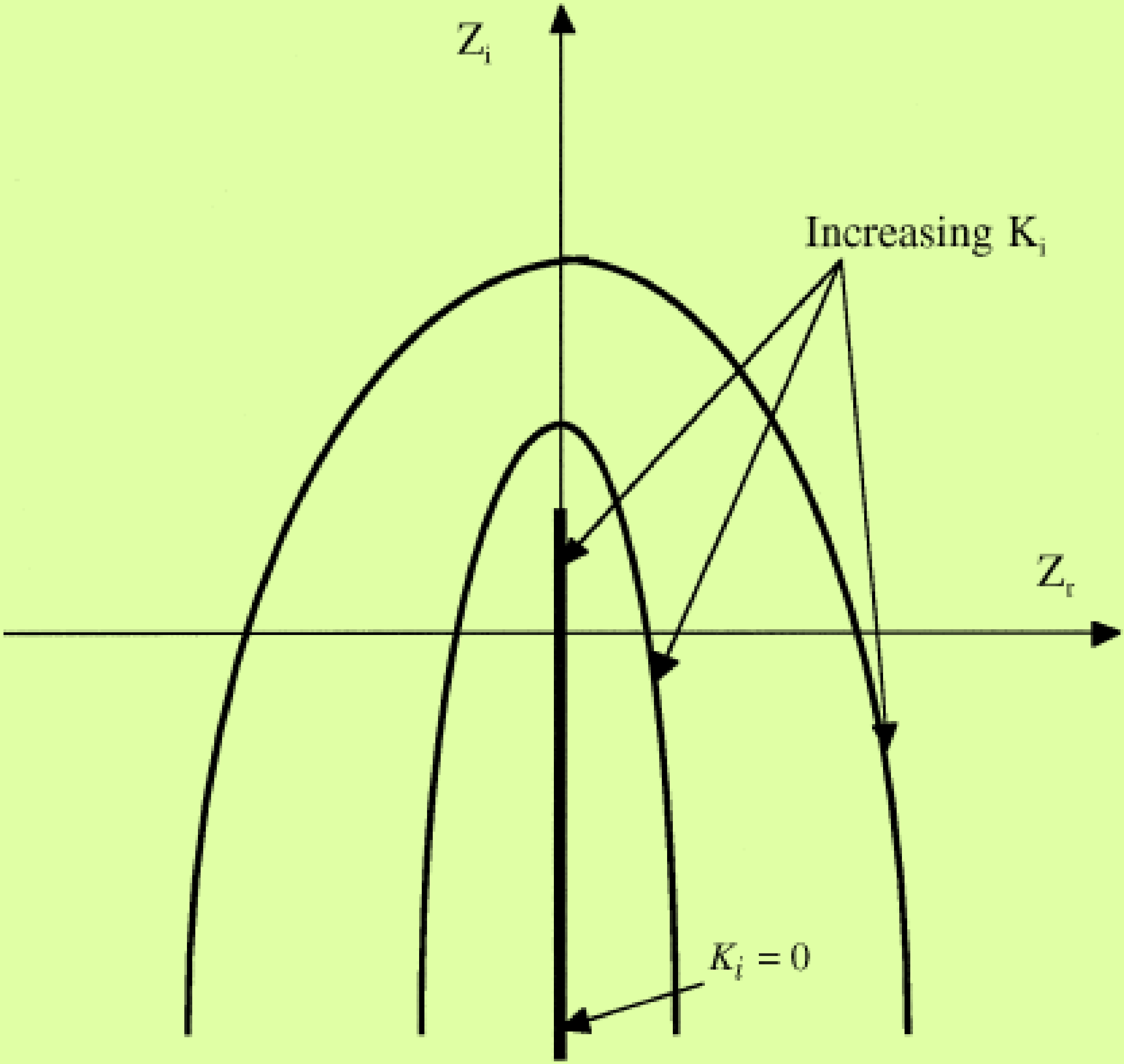


Fig. 1. Qualitative plot of the level curves in the impedance space (Z_r, Z_i) for constant instability growth rate, K_i.

Quantum-like description of longitudinal dynamics in high-energy accelerating machines in phase space

STATISTICAL APPROACH TO MODULATIONAL INSTABILITY

R. Fedele and D. Anderson, J. Opt. B: Quantum Semiclass. Opt. 2 207
R. Fedele, D. Anderson and M. Lisak, Proc. EPAC 2000, CERN 2000
B. Hall *et al.*, Phys. Rev. E 65, 035602(R) (2002)

$$\frac{\partial w}{\partial \zeta} + \mathrm{p}\frac{\partial w}{\partial x} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n} \frac{\partial^{2n+1} U}{\partial \xi^{2n+1}} \frac{\partial^{2n+1} w}{\partial \mathrm{p}^{2n+1}} = 0$$

$$U\left(|\Psi(\xi,\zeta)|^2\right) = \frac{q^2 \beta c}{m\gamma c^2}\left[\rho_0 Z_I'\left(|\Psi(\xi,\zeta)|^2 - |\Psi_0|^2\right) + Z_R' \int \left(|\Psi(\xi,\zeta)|^2 - |\Psi_0|^2\right) d\xi\right]$$

$$|\Psi_0|^2 \equiv \int_{-\infty}^{+\infty} w_0(\mathrm{p})\, d\mathrm{p}$$

$$Z_R \neq 0$$

$$U[w] = \kappa \int_{-\infty}^{+\infty} (w - w_0)\, d\mathrm{p} + \mu \int dx \int_{-\infty}^{+\infty} (w - w_0)\, d\mathrm{p}$$

- Linearization around the stationary state: $w=w_0(p)$, $U[w_0]=0$
- Fourier transform

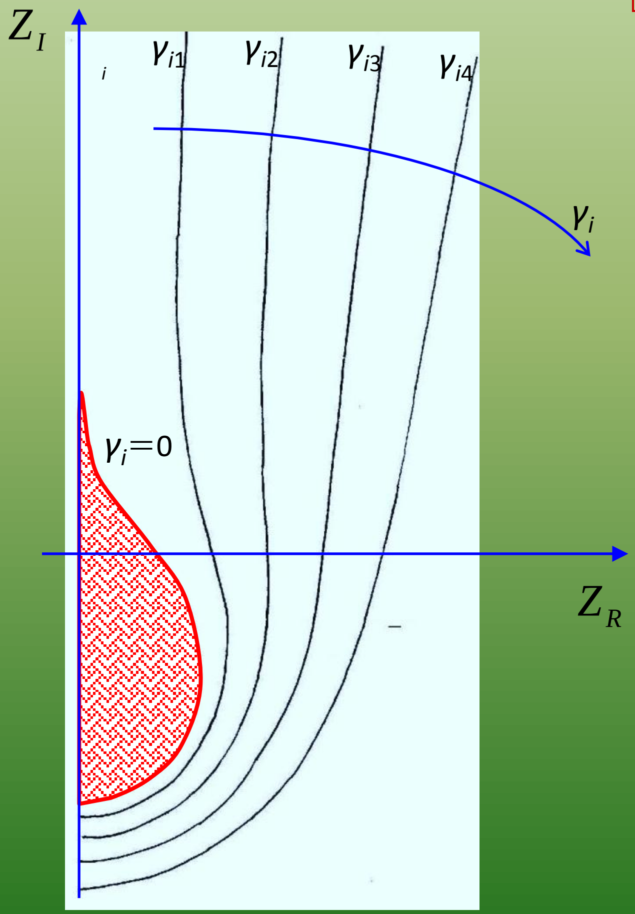
DISPERSION RELATION

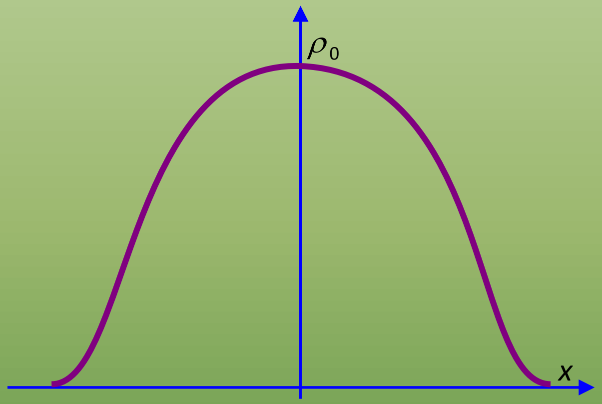
$$1 = i\alpha \frac{Z}{k} \int_{-\infty}^{\infty} \frac{w_0(p+\alpha k/2) - w_0(p-\alpha k/2)}{\alpha k} \frac{dp}{p-\omega/k}$$

Limiting case

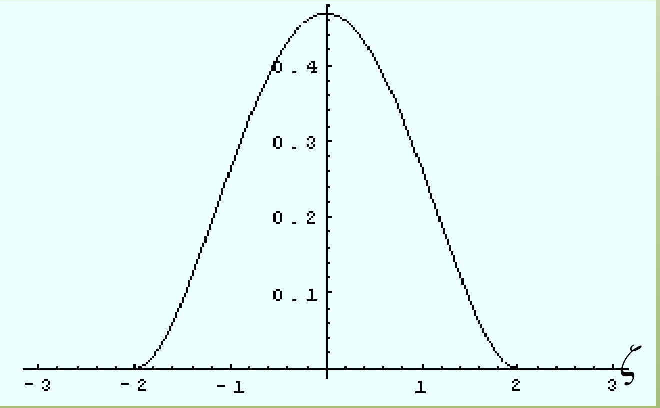
$$\alpha k << 1$$

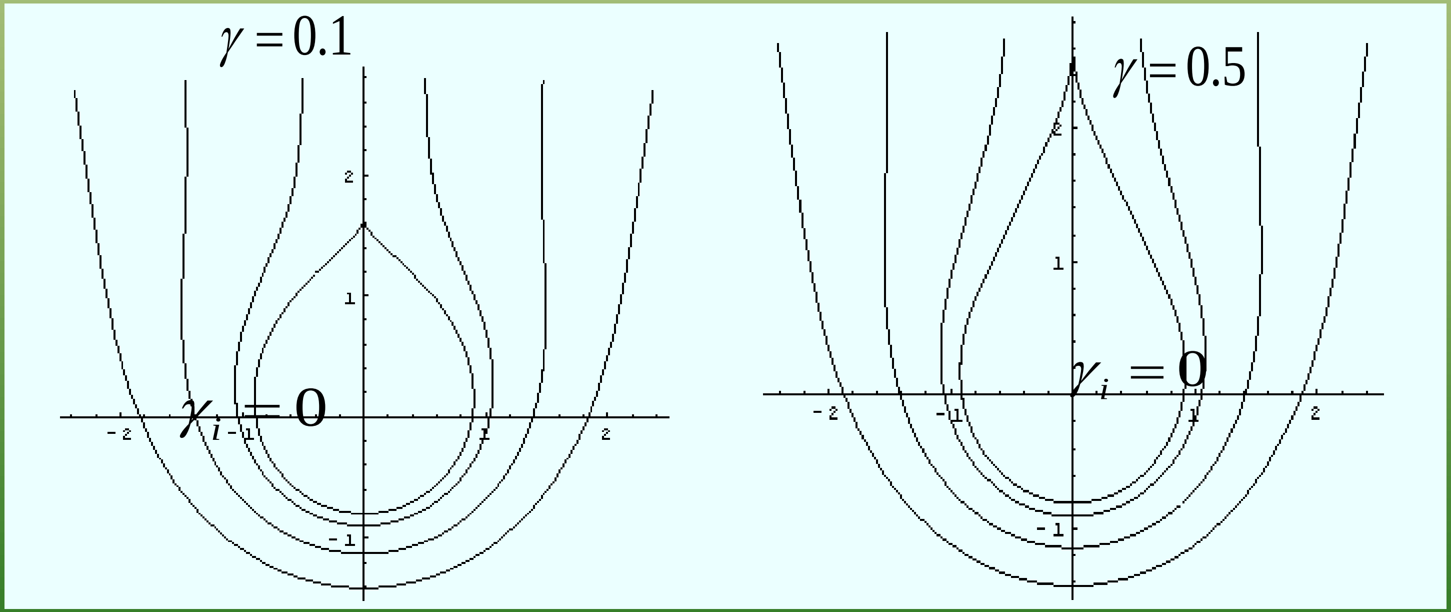
$$1 = i\alpha\left(\frac{Z}{k}\right)\int_{-\infty}^{\infty}\frac{\rho_0'(\mathrm{p})}{\mathrm{p}-\omega/k}d\mathrm{p}$$





arbitrary values of αk





[S. De Nicola, R. Fedele, V.G. Vaccaro, D. Anderson, M. Lisak, Proc. of QABP2000 (P. Chen ed., World Scientific, Singapore, 2002)]

Thank you!