# *Quantum RMS Deviation and Heisenberg's Error-Disturbance Relation*

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@

Wigner 111 - Colourful & Deep

Budapest, 11 November 2013

THE UNIVERSITY *of York*

# Acknowledgements

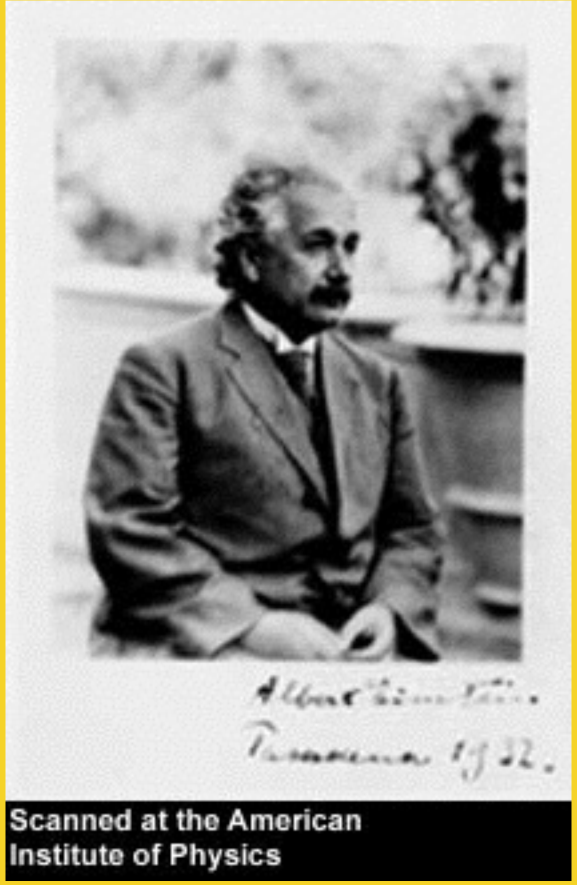
Joint work with Pekka Lahti and Reinhard Werner

- Proof of Heisenberg's Error-Disturbance Relation (PRL 111; arXiv:1306.1565)
- Heisenberg Uncertainty for Qubit Measurements (arXiv:1311.0837)
- Measurement Uncertainty Relations (forthcoming) [proofs, generalisations]
- Noise operators and measures of rms error and disturbance in quantum mechanics (forthcoming) [critique of noise operator based error]
- Measurement uncertainty relations: Reply to critics (in preparation)


BLW (from right to left) in Tianjin, China, August 2012

*“It is the theory that decides what can be observed”*

— Albert Einstein, according to Werner Heisenberg




# *Quantum Uncertainty* - three guises

- *Preparation Uncertainty (PUR)*

$$(\text{Width of } Q_\rho)\,(\text{Width of } P_\rho) \;\geq\; \frac{\hbar}{2}$$ ✓

- *Joint Measurement Error Trade-off Relation (MUR)*

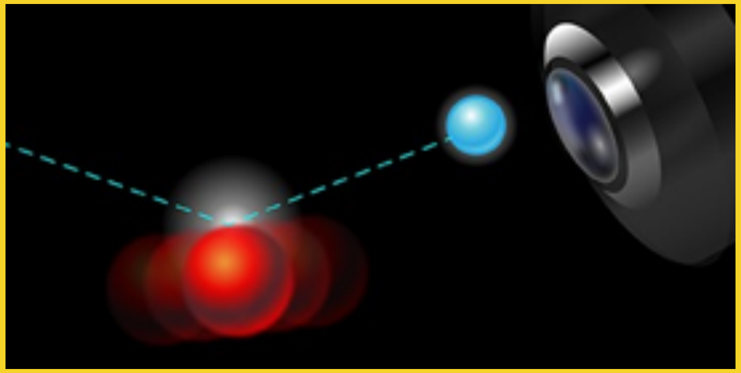
$$(Q - \text{Error})\,(P - \text{error}) \;\geq\; \frac{\hbar}{2}$$ ?

- *Error-Disturbance Trade-off Relation (EDR)*

$$(Q - \text{Error})\,(P - \text{disturbance}) \;\geq\; \frac{\hbar}{2}$$ ✗ (?)

*Heisenberg 1927:*

$$p_1\, q_1 \;\sim\; h$$



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# How (not) to "disprove" Heisenberg's EDR

1. DO: Put words in his mouth (or read his thoughts)

$$\varepsilon_{NO}(A,\rho)\,\varepsilon_{NO}(B,\rho) \not\geq |\langle[A,B]\rangle_\rho|$$

WRONG! *(No-H)*

2. DO: Correct the wrong, creating your own inequality

$$\varepsilon_{NO}(A,\rho)\,\varepsilon_{NO}(B,\rho) + \varepsilon_{NO}(A,\rho)\Delta(E^B_\rho) + \Delta(E^A_\rho)\varepsilon_{NO}(B,\rho) \geq \frac{1}{2}|\langle[A,B]\rangle\rho|$$

(Ozawa 2003-2013) (NO-O)

3. DO: Make an experiment to disprove (No-H) and confirm (NO-O)

4. DON'T: question the meaning of $\varepsilon_{NO}$

nature physics

LETTERS

PUBLISHED ONLINE: 15 JANUARY 2012 | DOI: 10.1038/NPHYS2194

# Experimental demonstration of a universally valid error–disturbance uncertainty relation in spin measurements

Jacqueline Erhart[1], Stephan Sponar[1], Georg Sulyok[1], Gerald Badurek[1], Masanao Ozawa[2] and Yuji Hasegawa[1]*

PRL 109, 100404 (2012) PHYSICAL REVIEW LETTERS week ending 7 SEPTEMBER 2012

# Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

Lee A. Rozema, Ardavan Darabi, Dylan H. Mahler, Alex Hayat, Yasaman Soudagar, and Aephraim M. Steinberg

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(Received 4 July 2012; published 6 September 2012; publisher error corrected 23 October 2012)

While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system, often referred to as "Heisenberg's uncertainty principle," Heisenberg originally formulated his ideas in terms of a relationship between the precision of a *measurement* and the disturbance it must create. Although this latter relationship is not rigorously proven, it is commonly believed (and taught) as an aspect of the broader uncertainty principle. Here, we experimentally observe a violation of Heisenberg's "measurement-disturbance relationship", using weak measurements to characterize a quantum system before and after it interacts with a measurement apparatus. Our experiment implements a 2010 proposal of Lund and Wiseman to confirm a revised measurement-disturbance relationship derived by Ozawa in 2003. Its results have broad implications for the foundations of quantum mechanics and for practical issues in quantum measurement.

# *Press reactions … in an age of uncertainty*

Quantenphysik

## Der große Heisenberg irrte

*17.11.2012* · Werner Heisenberg wollte seine berühmte Unbestimmtheitsbeziehung auch in den Störungen wiedererkennen, die ein Messung verursacht. Diesen Schluss haben kanadische Forscher widerlegt.

Von RAINER SCHARF

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Werner Heisenberg und seine Unschärferelation sind sogar auf einer Briefmarke verewigt

Die von Werner Heisenberg 1927 formulierte Unschärfebeziehung ist trotz ihrer Tiefgründigkeit und Abstraktheit das wohl bekannteste Gesetz der Quantenphysik. Sie besagt vereinfacht, dass man nicht gleichzeitig die Geschwindigkeit und den Ort etwa eines Elektrons mit beliebiger Präzision bestimmen kann. Für die Popularität dieses Gesetzes hat vor allem eine ebenfalls von Heisenberg stammende bildhafte Erläuterung gesorgt, PHYS.ORG

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# Are you certain, Mr. Heisenberg? New measurements deepen understanding of quantum uncertainty

Jan 17, 2012

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7 September 2012 Last updated at 17:24

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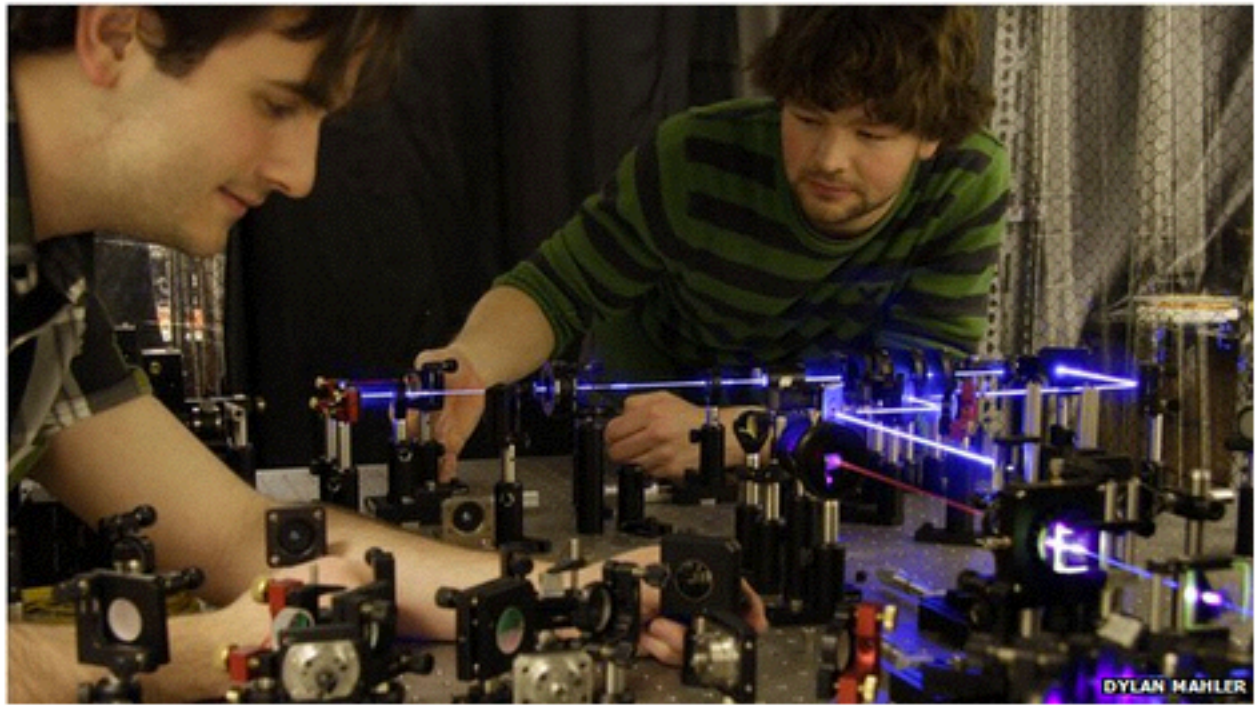
# Heisenberg uncertainty principle stressed in new test

By Jason Palmer

Science and technology reporter, BBC News




The experiment requires preparing pairs of "entangled" photons, the particles from which light is made

**Pioneering experiments have cast doubt on a founding idea of the branch of physics called quantum mechanics.**

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# Common Interpretation of Heisenberg's Uncertainty Principle Is Proved False

A new experiment shows that measuring a quantum system does not necessarily introduce uncertainty

By Geoff Brumfiel

# *Challenging Uncertainty: What It Takes*

*Let QM tell us what limitations there are!*

*(invoke Heisenberg's/Einstein's spirits*
*instead of trying to read their minds)*

*NEEDED:*

- *Language to describe approximate measurements* ✓
- *Notion of joint approximate measurement of incompatible observables* ✓
- *Measures of approximation error and disturbance* ✗
- *These measures need to be operationally significant* ✗
- *MURs and EDRs as consequences of QM — testable!* ✓

✓ *= agree;* ✗ *= disagree with O.*

# *Timeline:*
## *towards a quantum theory of approximate joint measurements*
## *— An Homage to EP Wigner —*

- *1925:* $[Q,P]=i\hbar$ *— Pauli 1926: q-eye & p-eye* $\Longrightarrow$ *going crazy*
- *1927:* $\Delta Q\,\Delta P \;\sim\; h$
  *idea of approximate joint measurement and uncontrollable disturbance (Heisenberg effect) — Pauli: day is dawning in QM*
- *1927-1930: Kennard-Weyl-Robertson-Schrödinger UR (for state preparations)*
- *1931: measurements of 1st & 2nd kind (Landau-Peierls; Pauli)*
- *1932: ``no sharp joint measurement'' / no positive joint probability (Wigner, von Neumann); vN model of approximate position measurement*

# *Timeline:*

- *1940: example of **positive semidefinite phase space probability** representation of quantum states (Husimi)*
- *1940s: theory of selfadjoint dilations of symmetric operators, positive operator valued measures (Naimark)*
- *1950, 1952: WWW, WAY: measurement limitations due to SSR and symmetries Wigner: notion of **unsharp measurement** (POVM implicit)*
- *1960-70s: covariant phase space POVMs as joint approximate measurements of position and momentum (Ludwig, Davies & Lewis, Prugovecki et al) — quantum foundations*
- *1962: noise operator approach for quantum optical theory of linear amplifiers (Haus & Mullen)*
- *1965: model of simultaneous measurement of position and momentum, as an extension of the von Neumann model (Arthurs & Kelly)*

# *Timeline:*

- *1982: Arthurs-Kelly model* ***as realisation of*** *phase space POVMs, Heisenberg's MUR made rigorous for special cases (PB)*

- *1986—… : quantum optical realisations of phase space measurements (homodyne/heterodyne detection)*

- *1990s: growing awareness of joint measurement concept and POVM in quantum optics community*

- *1998: position-momentum joint uncertainty relations for noise-operator based error measure -- model independent (Appleby)*

- *2003–2013: Ozawa's inequality for noise-operator based error measure, and claim of violation of Heisenberg's error-error & error-disturbance relations*

- *2003—… : Critique of noise-operator based error/disturbance measures and rigorous general measurement uncertainty relations — thoroughly ignored (R Werner; PB & P Lahti & collaborators)*

# *Timeline:*

- *2012: Experimental confirmation of Ozawa inequality*
  ***claims of violation of "Heisenberg's relation"*** *(Vienna, Toronto)*
  *several other groups following suit; also some criticisms*

- *2013:*

Physics

spotlighting exceptional research

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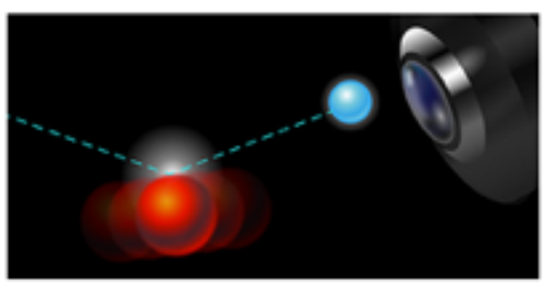
## Synopsis: Rescuing Heisenberg



APS/Alan Stonebraker

Proof of Heisenberg's Error-Disturbance Relation
Paul Busch, Pekka Lahti, and Reinhard F. Werner
Phys. Rev. Lett. **111**, 160405 (2013)
Published October 17, 2013

The popular conception of the Heisenberg uncertainty principle is that measurement is unavoidably invasive. We disturb an object when we observe it, thus introducing error into subsequent measurements. However, recent experiments (see 6 September 2012 Synopsis) claim to have measurement errors below the Heisenberg limit. To address this apparent contradiction, a paper in *Physical Review Letters* reports a new formulation of the uncertainty principle in which measurement disturbance depends on the performance of the measuring device, which is quantified as the maximum possible change in the state of the object.

# Timeline:

# *Wigner's spirit…*

*…. or: his influence on the development of measurement theory*

- *ignoring the "measurement = repeatable measurement" dogma*

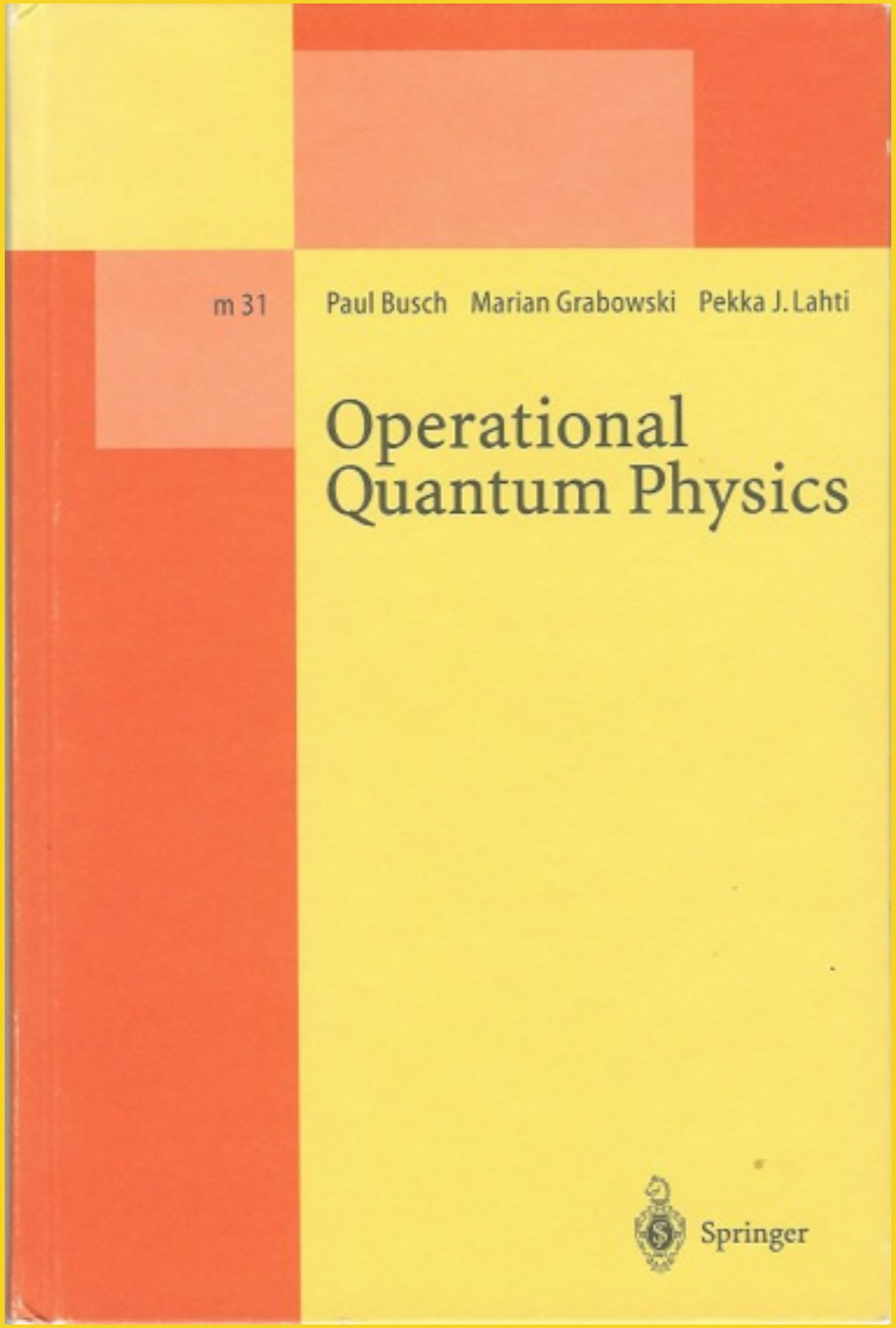
$$\text{“measurement: state} \quad \longrightarrow \quad \text{eigenstate''}$$

- *freedom to work with non-orthodox **unsharp measurement** paradigm*
- *freedom to work with ``impossible'' phase space picture*
- *(to name just a few points)*

## Timeline:

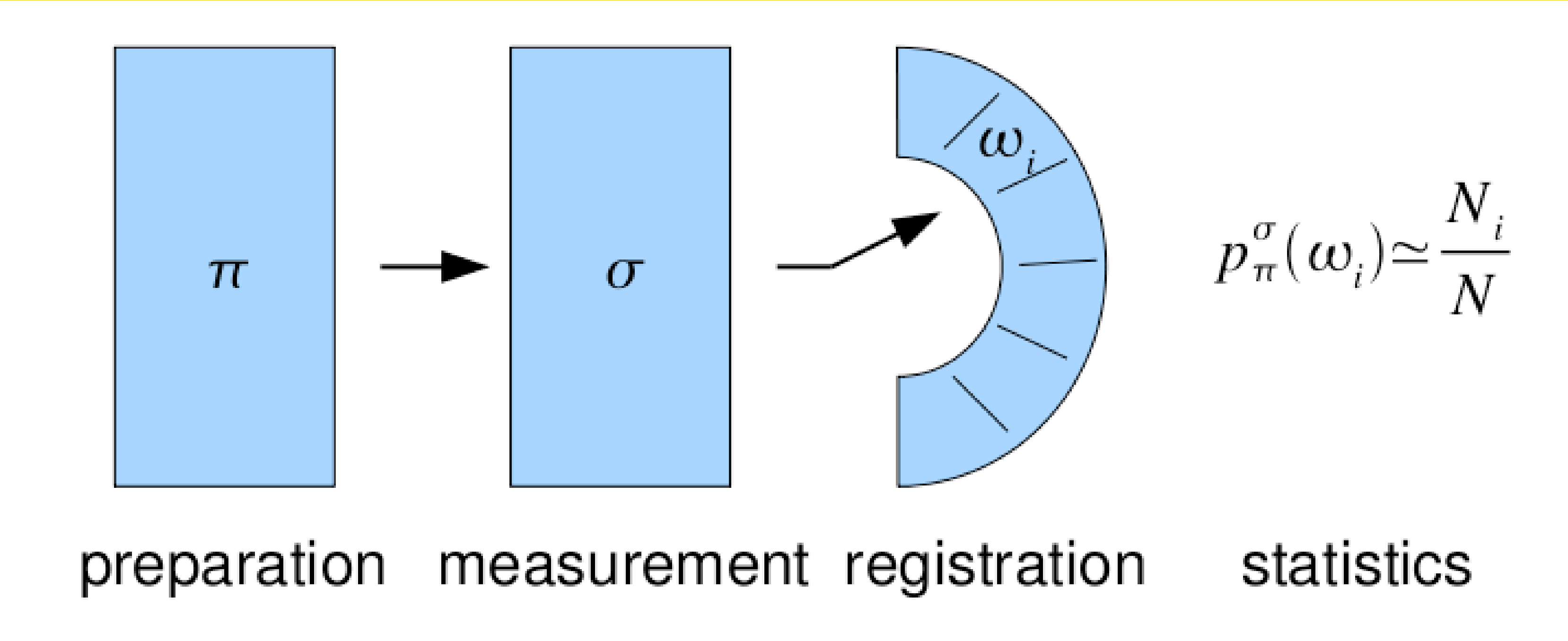
# *Wigner's spirit…and influence*

*In memory of Eugene P. Wigner (1995)*



# *Approximate Measurements: POVMs*

*(positive operator valued measure(ment))*




$$\text{outcome } \omega_i \;\mapsto\; \mathsf{A}_i \quad \text{(positive operator)}, \quad \sum_i \mathsf{A}_i = I$$

$$\text{probability} : \omega_i \mapsto p^{\mathsf{A}}_\rho(\omega_i) = \mathrm{Trace}[\rho \mathsf{A}_i] \equiv \langle \mathsf{A}_i \rangle_\rho \equiv \mathsf{A}_\rho(\omega_i)$$

# *Covariant Phase Space Measurement*

Observables E, F are *jointly measurable* if they are *marginals* of an observable G:

$$\mathsf{E}(X) = \mathsf{G}(X \times \Omega_2), \quad \mathsf{F}(Y) = \mathsf{G}(\Omega_1 \times Y)$$

Does *not* require commutativity*!*

$$\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto G(Z) = \frac{1}{2\pi\hbar} \int_Z W(q,p)^* m W(q,p)\, dq dp$$

$m$ = a positive operator of trace 1 (**NOT(!)** a '*state*')

$$W(q,p) = \exp\left(\frac{i}{\hbar}(Pq - Qp)\right)$$

# *Covariant Phase Space Measurement*

Covariance:

$$W(q,p)^* G(Z) W(q,p) = G(Z + (q,p))$$

Marginals:

$$G_m(X \times \mathbb{R}) = \mathsf{Q}_\mu(X) \qquad \mu(X) = \mathrm{Trace}[\Pi m \Pi \mathsf{Q}(X)]$$

$$G_m(\mathbb{R} \times Y) = \mathsf{P}_\nu(Y) \qquad \nu(Y) = \mathrm{Trace}[\Pi m \Pi \mathsf{P}(Y)]$$

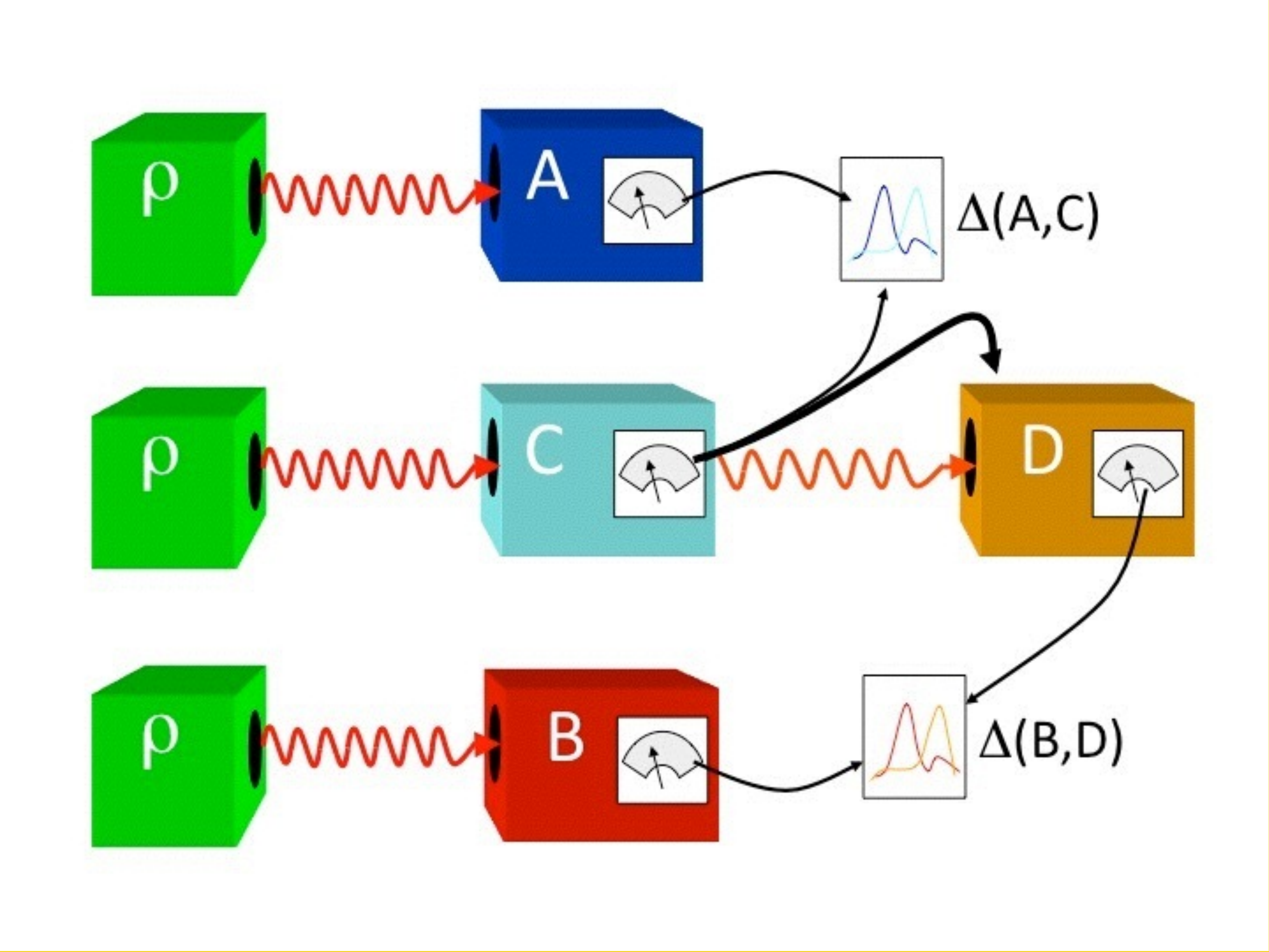
$$\Pi \;=\; \text{parity operation}$$

*Smeared* position and momentum observables

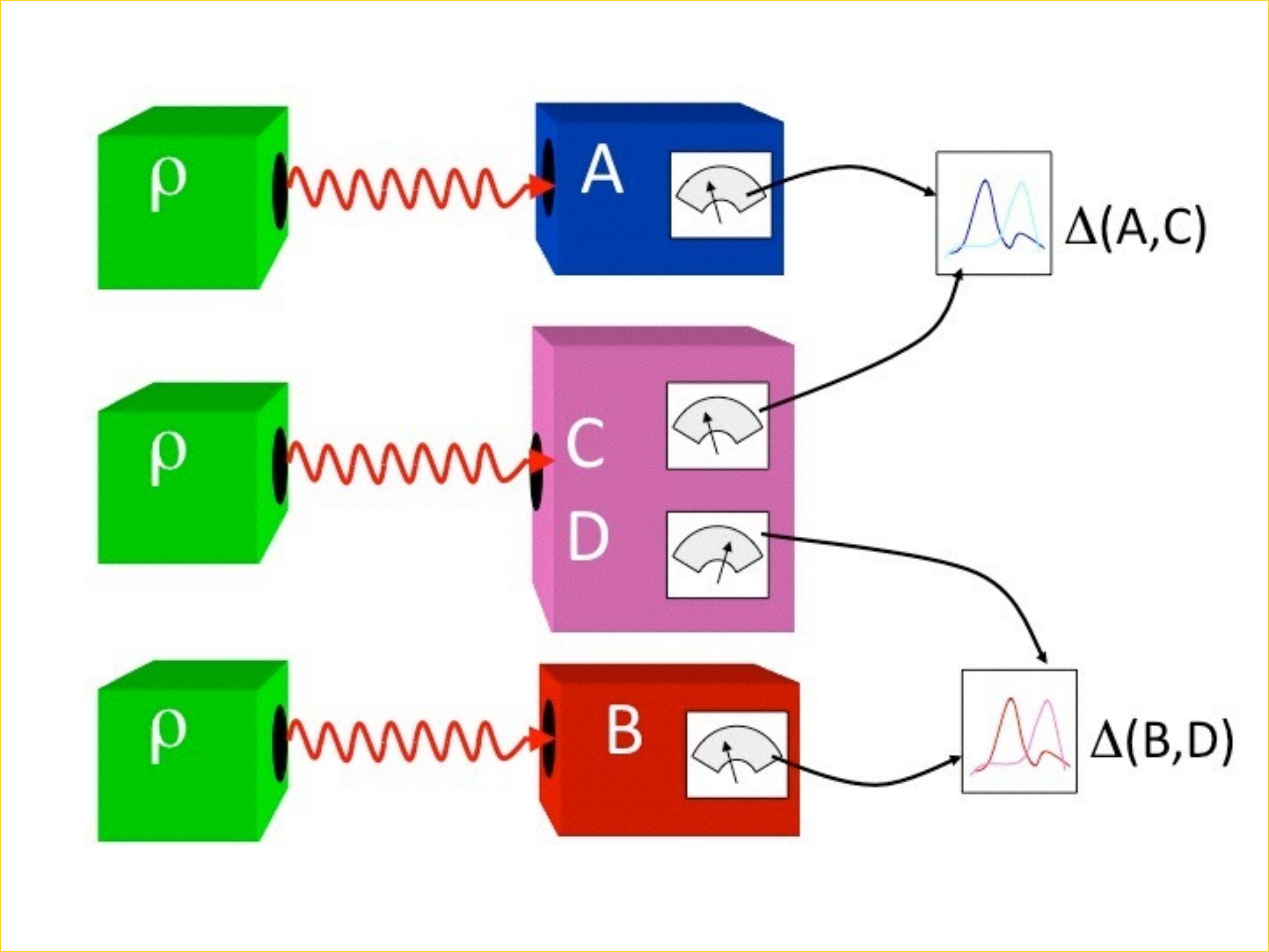
$$\Delta(\mu)\Delta(\nu) = \Delta(Q,m)\,\Delta(P,m) \;\geq\; \frac{\hbar}{2}$$ Cov-MUR

Consequence of PUR

# *Error & Disturbance: operational definition*



## *Error & Disturbance:*
## *an instance of **joint measurement***



# *Defining **measurement error** (I): how **not** to do it*

$$\varepsilon^{O}(Z, A; w \otimes a) := \langle (U^* I \otimes ZU - A \otimes I)^2 \rangle_{w \otimes a}^{1/2} \qquad \times$$

$$\equiv \varepsilon_{NO}(A, \rho) \qquad (w = \rho, \quad a = \sigma)$$

Motivation:

- similar to classical Gauss RMS error;
- similar to *noise operator* approach for linear amplifiers

BUT:

$Z_{out} = U^* I \otimes ZU$ and $A_{in} = A \otimes I$ may not commute!

Hence: $\varepsilon_{NO}(A, \rho)$ is NOT determined by distributions of $Z_{out}$ and $A_{in}$

# Flaws of $\varepsilon_{NO}(A,\rho)$

- Can be ***zero*** where input and output distributions differ
- Can be ***arbitrarily large*** where input and output distributions are ***identical***
- Is adequate for *linear models* or *unbiased* measurements

***Must be used JUDICIOUSLY***

***NOT suitable for "universally valid" MURs***

# *Defining* ***measurement error*** *(II): Wasserstein 2-distance*

$$D^\gamma(\mu,\nu) := \left(\int (x-y)^2\, d\gamma(x,y)\right)^{\frac{1}{2}}$$

$$D(\mu,\nu) := \inf_\gamma D_2^\gamma(\mu,\nu)$$

$$\Delta(\mathsf{A},\mathsf{B}) := \sup_\rho D(\mathsf{A}_\rho,\mathsf{B}_\rho)$$

- $\gamma$ is any *coupling* of the probability measures $\mu,\ \nu$
- operational: direct comparison of probability distributions $\mathsf{A}_\rho,\ \mathsf{B}_\rho$
- ***quantum generalisation of Gauss' RMS error***

# Compare with $\varepsilon_{NO}(A, \rho)$

$$\varepsilon_{NO}(A, \mathsf{E}; \rho)^2 = \mathrm{tr}\left[\rho(\mathsf{E}[x] - A)^2\right] + \mathrm{tr}\Big[\rho\big(\underbrace{\mathsf{E}[x^2] - \mathsf{E}[x]^2}_{\text{intrinsic noise}}\big)\Big] \quad \times$$

$$= \iint (x' - x)^2 \underbrace{\mathrm{Re}\,\mathrm{tr}\left[\rho \mathsf{E}(dx')\mathsf{E}^A(dx)\right]}_{\textit{bi-measure}}$$

- Here E is the actually measured (estimator) observable
- non-positive bi-measure when E and A do not commute
- when commutative:

$$\varepsilon_{NO}(A, \mathsf{E}; \rho) \;\geq\; \Delta(\mathsf{E}_\rho, \mathsf{E}^A_\rho)$$

# *Error measures: state dependent or not?*

- State-dependent: $D(A_\rho, C_\rho), \quad \varepsilon_{NO}(A, C; \rho)$

  trade-off inequality gives necessary error for fixed disturbance threshold — and vice versa

- State-independent: $\Delta(A, C), \quad \varepsilon_{NO}(A, C)$

  trade-off inequality gives guarantee for maximal error/disturbance
  figure of merit characterising measuring device

# *Error measures: state dependent or not?*

- Violation of (No-H) is ***unsurprising — to be expected***

$$\varepsilon_{NO}(A,\rho)\,\varepsilon_{NO}(B,\rho) \;\geq\; |\langle[A,B]\rangle_\rho| \qquad \text{WRONG!}$$

(No-H)

- Similarly for Wasserstein:

$$D(A_\rho, C_\rho)\, D(B_\rho, D_\rho) \;\geq\; 0$$

- accurate A measurement (C=A) with constant output channel $\rho \;\rightarrow\; \rho_0$ has

$$D(A_\rho, C_\rho) \;=\; 0 \;=\; D(B_\rho, D_\rho) \quad \text{if} \quad \rho = \rho_0$$

**Hence, violation of (No-H) is nothing exciting…**

# *Measurement Uncertainty Relations (I): Position and Momentum*

THEOREM

For every observable G on phase space with

$$\Delta(\mathsf{G}_1, \mathsf{Q}) < \infty, \quad \Delta(\mathsf{G}_2, \mathsf{P}) < \infty$$

there is a covariant $\mathsf{G}_{\mathbf{m}}$ such that

$$\Delta(\mathsf{G}_1, \mathsf{Q}) \geq \Delta_2(\mathsf{G}_{\mathbf{m},1}, \mathsf{Q}), \qquad \Delta(\mathsf{G}_2, \mathsf{P}) \geq \Delta_2(\mathsf{G}_{\mathbf{m},2}, \mathsf{P})$$

and therefore

$$\Delta(\mathsf{G}_1, \mathsf{Q})\,\Delta(\mathsf{G}_2, \mathsf{P}) \;\geq\; \frac{\hbar}{2}$$

- PUR $\Longrightarrow$ MUR
- (proof uses “invariant mean”; technically involved)

# *Measurement Uncertainty Relations (II): Qubit Observables*

Sharp +/- 1 valued qubit observables A, B approximated by C, D:

$$A : \pm 1 \mapsto A_\pm = \frac{1}{2}(I \pm \mathbf{a}\cdot\boldsymbol{\sigma})$$

$$C : +1 \mapsto C_+ = \frac{1}{2}(c_0 I + \mathbf{c}\cdot\boldsymbol{\sigma}), \quad -1 \mapsto C_- = I - C_+$$

$$B : \pm 1 \mapsto B_\pm = \frac{1}{2}(I + \mathbf{b}\cdot\boldsymbol{\sigma})$$

$$D : +1 \mapsto D_+ = \frac{1}{2}(d_0 I + \mathbf{d}\cdot\boldsymbol{\sigma}), \quad -1 \mapsto D_- = I - D_+$$

- Optimal approximations found among C, D with $c_0 = d_0 = 1$
- Such C, D are *covariant* under shifts $\tau : \pm 1 \ \mapsto \ \mp 1$

$$U_\tau = \mathbf{u}\cdot\boldsymbol{\sigma} : \quad C_\pm \mapsto C_\mp, D_\pm \mapsto D_\mp, \quad \mathbf{u} \perp \mathbf{c},\ \mathbf{d}$$

# *Measurement Uncertainty Relations (II): Qubit Observables*

$$\Delta(A,C)^2 = 2|1-c_0| + 2\|\mathbf{a}-\mathbf{c}\| \geq 2\|\mathbf{a}-\mathbf{c}\|$$
$$\Delta(B,D)^2 = 2|1-d_0| + 2\|\mathbf{b}-\mathbf{d}\| \geq 2\|\mathbf{b}-\mathbf{d}\|$$

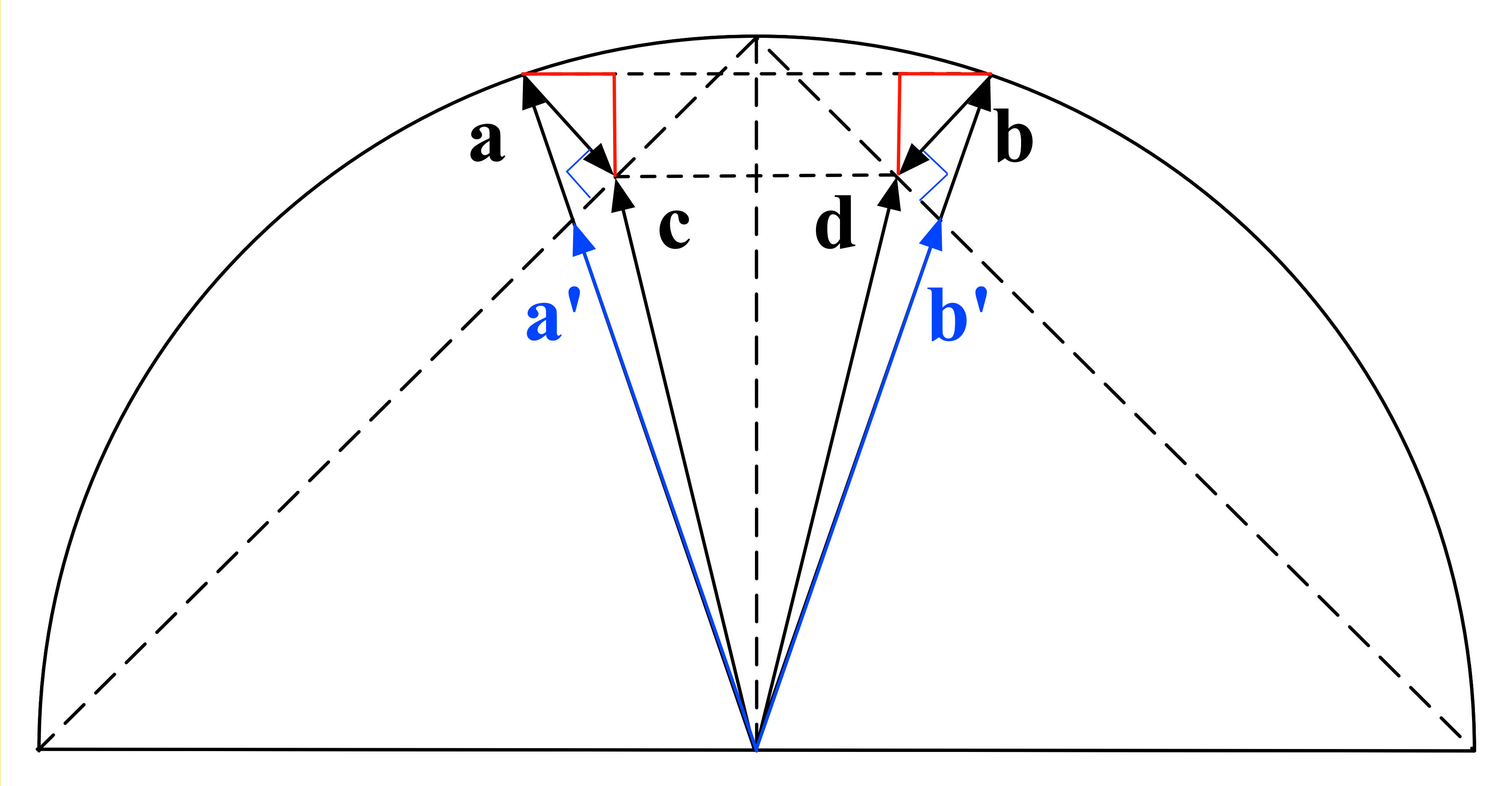
$$\begin{aligned}\Delta(C,A)^2 + \Delta(D,B)^2 &\geq \sqrt{2}\left[\|\mathbf{a}-\mathbf{b}\| + \|\mathbf{a}+\mathbf{b}\| - 2\right] \\ &= \mathrm{incompatibility}(A,B)\end{aligned}$$

- ***Tight*** lower bound, required by the constraint that C, D must be ***compatible***:

$$\|\mathbf{c}-\mathbf{d}\| + \|\mathbf{c}+\mathbf{d}\| \leq 2$$

# *Measurement Uncertainty Relations (II): Proof of Qubit MUR*

1. Reduction to covariant case
2. Diagram:




PUR $\Longrightarrow$ MUR

# *Measurement Uncertainty Relations (III): Uncertainty Relations: additive?!*

- Bound for error *product* is trivial: =0
- ***PUR*** for uncertainty sums?

$$\Delta(A,\rho) + \Delta(B,\rho) \;\geq\; \|\mathbf{a}\times\mathbf{b}\| \;=\; 2\|[A_+, B_+]\|$$

$$\Delta(A,\rho)^2 + \Delta(B,\rho)^2 \geq 1 - |\mathbf{a}\cdot\mathbf{b}| = 1 - \sqrt{1 - \|\mathbf{a}\times\mathbf{b}\|^2}$$

$$= 1 - \sqrt{1 - 4\|[A_+, B_+]\|^2}$$

# *Measurement Uncertainty Relations (III): Uncertainty Relations: additive?!*

- Can have additive relations for position and momentum

$$\frac{4\hbar^2}{x_0^2}\Delta(Q,\rho)^2 + x_0^2\Delta(P,\rho)^2 \geq 2\hbar^2$$

$$\Longleftrightarrow$$

$$\frac{2\hbar}{x_0}\Delta(Q,\rho) + x_0\Delta(P,\rho) \geq 2\hbar$$

$$\Longleftrightarrow$$

$$\Delta(Q,\rho)\,\Delta(P.\rho) \;\geq\; \frac{\hbar}{2}$$

- Finding minimiser of quadratic error sum is equivalent to finding minimiser of Harmonic Oscillator Hamiltonian

# *Measurement Uncertainty Relations (IV): Alternative error measures*

- Wasserstein 2-distance $\longrightarrow$ Wasserstein $\alpha$-distance

$$D_\alpha(\mu,\nu) := \inf_\gamma \left[ \iint |x-y|^\alpha \, d\gamma(x,y) \right]^{1/\alpha}$$

(BWL 2013; R Werner, 2004: $\alpha$= 1)

- Error-bar-width & calibration errors
(PB, DB Pearson, 2007)

- Entropic error measures:
bounds for combined in-out correlations (calibration)
(Buscemi, Hall, Ozawa, Wilde, arXiv:1310.6603

# *Ozawa vs Wasserstein: Qubit case*

$$\Delta(A,C)^2 = 2|1-c_0| + 2\|\mathbf{a}-\mathbf{c}\| \geq 2\|\mathbf{a}-\mathbf{c}\|$$

$$\begin{aligned}\varepsilon_{NO}(A,C;\rho)^2 &= (1-\|\mathbf{c}\|)^2 + \|\mathbf{b}-\mathbf{d}\|^2 \\ &= U(C)^2 + \frac{1}{4}\Delta(A,C)^4\end{aligned}$$

$$\begin{aligned}\varepsilon_{NO}(C,A;\rho) + \varepsilon_{NO}(D,B;\rho) &\geq \frac{1}{\sqrt{2}}\left[\|\mathbf{a}-\mathbf{b}\| + \|\mathbf{a}+\mathbf{b}\| - 2\right] \\ &= \frac{1}{2}\mathrm{incompatibility}(A,B)\end{aligned}$$

Irony:

- State Independent!
- ***Satisfies Heisenberg-type MUR***

# *CONCLUSION*

- Error/error & error/disturbance trade-off relations for position/momentum and qubit observables
- in terms of ***quantum RMS error***
- ***fully in Heisenberg's spirit:***

  **(error of A) (error of B) $\geq$ incompatibility(A,B)**

- Ozawa's error overestimates quantum RMS error (in qubit case); satisfies MUR
- ***…and No End to Uncertainty***