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Poisson Geometry of Difference Lax Operators, and Difference Galois Theory, or Quantum groups from Poisson brackets anomalies

Michael Semenov-Tian-Shansky

Institut Mathématique de Bourgogne, Dijon, France

and St.Petersburg Department of Steklov Mathematical Institute

Introduction

- Groups in classical and quantum mechanics.
  Why classical symmetry groups are not quantized?
- Well-known answer: this is because classical symmetry groups acting on a phase space preserve Poisson brackets.
- Anomalous Poisson brackets: this is when the group action fails to preserve them.
- Anomalous Poisson bracket relations arise in practice in connection with dynamical symmetry groups.
- The notion of dynamical symmetry group is a bit vague; it's a group which is not directly related to the geometric symmetry of the problem. An interesting example related to the Poisson brackets anomaly arises in the theory of the celebrated Korteweg–de Vries equation.

KdV equation and its phase space

- KdV equation is $u_t = 6uu_x + u_{xxx}$. Its phase space may be identified with the space of Schroedinger operators on the line or on the circle. The algebra of observables consists of local functionals of the potential u.
- The phase space carries a natural Poisson structure related to the famous Virasoro algebra. For this reason the KdV phase space is also related to models of 2d-gravity.
- Is there a natural Poisson structure on the space of *wave functions* of a Schroedinger operator?
- This question is in fact of practical interest in application to a family of KdV-like equation.

Schwarzian Derivative

Let us recall a nice geometric way to restore the potential from the wave functions.

- For a given u the space of solutions of the Schroedinger equation

$$-\psi'' - u\psi = 0$$

  is 2-dimensional; any two solutions ϕ, ψ have constant wronskian $W = \phi\psi' - \phi'\psi$.

- Set $\eta(x) = \phi(x)/\psi(x)$. The potential u may be restored from η by the formula

$$u = \tfrac{1}{2}S(\eta), \quad \text{where} \quad S(\eta) = \frac{\eta'''}{\eta'} - \tfrac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

  is the Schwarzian derivative.

Projective group

- Change of basis amounts to linear transformation $(\phi, \psi) \mapsto (\phi, \psi)g,\ g \in SL(2)$ and to fractional linear transformation of $\eta = \phi/\psi$
- $S(u)$ is projective invariant: if $\tilde{\eta} = \frac{a\eta+c}{b\eta+d}$, then $S(\tilde{\eta}) = S(u)$.

A tower of KdV-like equations. The KdV equation is included into an interesting family of “KdV-like equations” related to projective group and its subgroups. Atop the tower of these “KdV-like equations” is the “Schwarz–KdV equation”

$$\eta_t = S(\eta)\eta_x, \quad S(\eta) = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

Tower of KdV-like equations

- Claim:
  - if η is its solution, so is $\frac{a\eta+c}{b\eta+d}$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$.
  - Original KdV holds for $u = S(\eta)$ which is $PSL(2)$-invariant.
  - Other KdV-like equations hold for the invariants of various subgroups of $PSL(2)$.
- More formally:
  - We identify observables of KdV with local densities which are rational functions of u and its derivatives; in a similar way, observables for Schwarz–KdV are rational functions of η and its derivatives (in finite number)
  - The group $G = SL(2)$ acts on this algebra of observables and commutes with derivation ∂_x.

Point of view of differential Galois theory

- We define the differential field $\mathbb{C}\langle\psi_1,\psi_2\rangle$ as a free algebra of rational functions in an infinite set of variables $\psi_1,\psi_2,\psi_1',\psi_2',\psi_1'',\psi_2'',\ldots$ with a formal derivation ∂ such that $\partial\psi_i^{(n)}=\psi_i^{(n+1)}$.
- A *differential automorphism* is an automorphism of $\mathbb{C}\langle\psi_1,\psi_2\rangle$ (as an algebra) which commutes with ∂. All differential automorphisms are induced by linear transformations $(\psi_1,\psi_2)\mapsto(\psi_1,\psi_2)\cdot g,\ g\in GL(2,\mathbb{C})$.
- Let (W) be the differential ideal in $\mathbb{C}\langle\psi_1,\psi_2\rangle$ generated by $\psi_1\psi_2'-\psi_1'\psi_2-1$. Automorphisms which preserve W belong to $G=SL(2)$.
- The differential subfield of G-invariants coincides with $\mathbb{C}\langle u\rangle$.

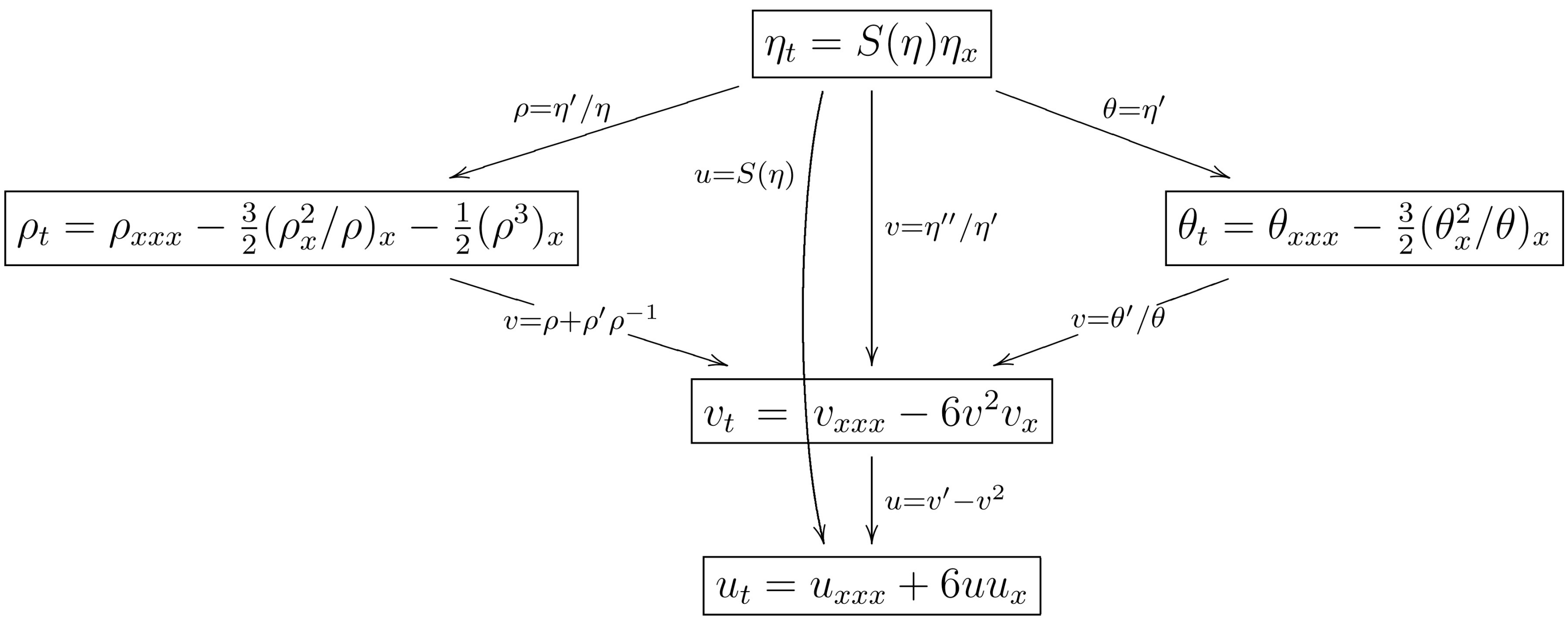
Intermediate differential fields

Let $Z = \{\pm 1\}$ be the center of G and N, A, $B = AN$ its standard subgroups (nilpotent, split Cartan & Borel). The subfields of invariants are freely generated differential algebras:

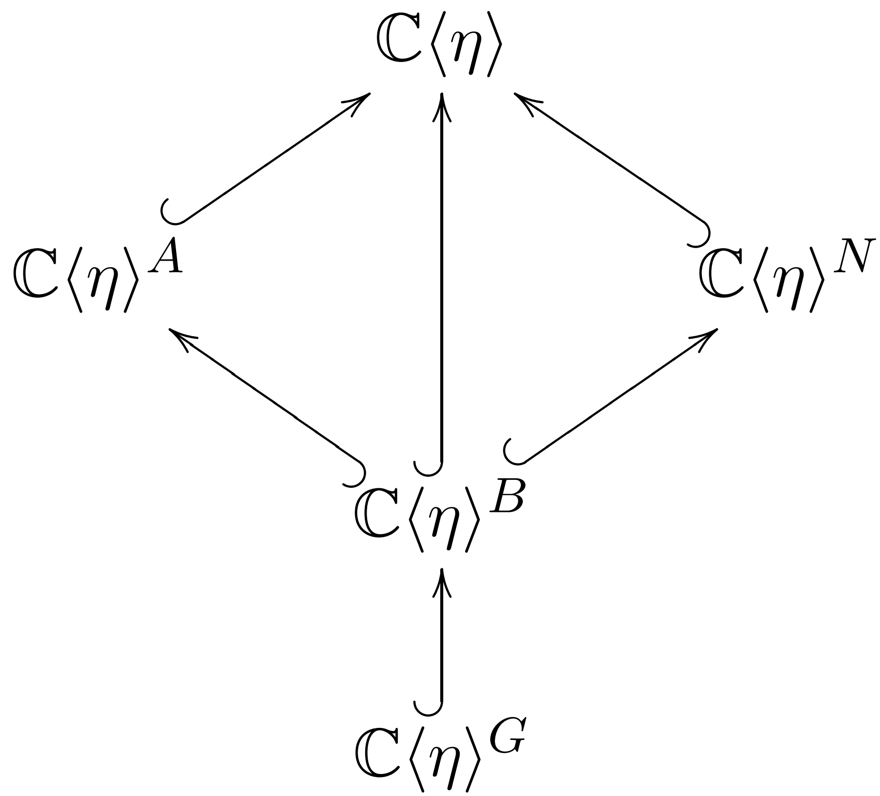
- $\mathbb{C}\langle\phi, \psi\rangle^Z = \mathbb{C}\langle\eta\rangle, \quad \eta = \phi/\psi$,
- $\mathbb{C}\langle\eta\rangle^A = \mathbb{C}\langle\rho\rangle, \quad \rho = \eta'/\eta$,
- $\mathbb{C}\langle\eta\rangle^N = \mathbb{C}\langle\theta\rangle, \quad \theta = \eta'$,
- $\mathbb{C}\langle\eta\rangle^B = \mathbb{C}\langle v\rangle, \quad v = \eta''/\eta'$,
- $\mathbb{C}\langle\eta\rangle^G = \mathbb{C}\langle u\rangle, \quad u = S(\eta)$.

Tower of KdV-like flows

Tower of compatible integrable KdV-like equations associated with subgroups of $PSL(2)$:



Extension Tower



Inclusions in this diagram match with “differential substitutions” listed above and with the tower of KdV-like equations

Differential Galois theory

Question: Put this picture into the Hamiltonian framework. As well known, the space of 2nd order differential operators carries a natural Poisson structure (incidentally, this is the Poisson–Virasoro algebra).
Question: Extend this Poisson structure to the space of wave functions in such a way that all arrows in the diagram above become Poisson maps.

- The answer to this question is non-trivial: All arrows go in wrong direction! (Poisson structure cannot be pulled back!) Still, the lift is possible (and even almost unique); however, the resulting brackets are anomalous: they are not projective invariant. Instead, a new axiom holds true: the projective group becomes a Poisson Lie group (and eventually a *quantum group*); all brackets are Poisson covariant.

Basic Poisson bracket relations

In our paper with Ian Marshall we explored this question for the KdV case. The resulting Poisson brackets are listed below:

- Basic Poisson bracket relations for differential Galois invariants:
  - $\{\eta(x), \eta(y)\} = \eta(x)^2 - \eta(y)^2 - \mathrm{sign}(x-y)\left(\eta(x) - \eta(y)\right)^2$.
  - For $\theta = \eta'$ we have
  $$\{\theta(x), \theta(y)\} = 2\,\mathrm{sign}(x-y)\theta(x)\theta(y).$$
  - For $v = \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$ we have
  $$\{v(x), v(y)\} = \frac{1}{2}\delta'(x-y).$$

Basic Poisson bracket relations

- For $u = \frac{1}{2}v' - v^2 = S(\eta)$ we have

$$\{u(x), u(y)\} = \frac{1}{2}\delta'''(x-y) + \delta'(x-y)\big[u(x) + u(y)\big]. \quad (1)$$

The Poisson bracket for u is the *Poisson–Virasoro algebra*. In these formulas $\mathrm{sign}(x-y)$ is the distribution kernel of the operator ∂_x^{-1} and $\delta'(x-y)$ is the distribution kernel of ∂_x. Poisson brackets for η are *not projective invariant*: there are correction terms which mean that this action is Poisson, with the group $SL(2)$ itself carrying the so called standard (Sklyanin) Poisson bracket.

Reformulation

In a sense, on has to *guess* the Poisson brackets for η. The situation becomes more transparent if we pass to 1st order matrix differential operators and then to difference operators, which are the main subject of the present talk.

- As well known, 2nd order differential operators may be written as 1st order 2×2 matrix differential operators. One can start with arbitrary 1st order matrix differential operators; to get the correct Poisson bracket relations for 2nd order operators on needs a reduction procedure *(Drinfeld–Sokolov theory)*.
- Again, the choice of a Poisson structure in the space of wave functions is nontrivial.
- We shall put the theory on a lattice replacing differential operators with difference operators. (This also leads to an interesting deformation of Virasoro algebra.)

Abstract difference operators

To set up a general framework for the difference case let us assume that $\mathbb{G}$ is a Lie group equipped with an automorphism τ Let $G = \mathbb{G}^\tau$ be the group of "quasi-constants", $G = \{g \in \mathbb{G}; g^\tau = g\}$.
The "auxiliary linear problem" reads:

$$\psi^\tau \psi^{-1} = L.$$

There is a natural action of $\mathbb{G}$ on itself by left multiplication which induces gauge transformations for L:

$$g \colon \psi \mapsto g \cdot \psi, L \mapsto g^\tau L g^{-1}.$$

The quasi-constants act by right multiplications, $\psi \mapsto \psi h$ and leave L invariant.

Natural realizations

There are several natural realizations of this scheme:

- $\mathbb{G}$ consists of functions on a lattice $\mathbb{Z}$ with values in a matrix group G and τ is a shift operator; it is also possible to introduce multi-dimensional lattices with several commuting shift automorphisms.
- $\mathbb{G}$ consists of functions on the line with $g^{\tau}(x) = g(x+1)$.
- $\mathbb{G}$ consists of functions which are meromorphic in $\mathbb{C}^*$ and τ acts by $g^{\tau}(z) = g(qz), q \neq 1$.

In the first case, the group of quasi-constants consists of genuine constant functions on the lattice with values in G, in the second case it consists of G-valued periodic functions, in the 3d case it consists of G-valued functions on the elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Poisson framework

- There is a natural Poisson structure on the space of 1st order differential operators on the line (Schwinger bracket) and the gauge action is Hamiltonian.
- In the difference case the gauge action is not Hamiltonian; the gauge group is a Poisson Lie group; the Poisson structure on the gauge group is fixed by the choice of a classical r-matrix.
- Once r is chosen, there is a unique Poisson structure on the space of 1st order difference operators which is Poisson covariant with respect to the gauge action. This is the “discrete current algebra” discovered by Alexeev, Faddeev, Volkov and myself (both in Poisson and in Quantum group setting)

Choice of a Poisson structure

We shall define the Poisson structure directly on the space of wave functions.

- Consistency check:
  - It should be Poisson covariant with respect to left translations (gauge action) and also with respect to right translations.
  - It should yield the correct Poisson structure on the space of potentials $L = \psi^\tau \psi^{-1}$.

Definition. For $f \in \mathbf{Fun}(\mathbb{G})$ we denote by ∇_f, ∇'_f its left and right gradients defined by

$$\langle \nabla_f(\psi), \xi \rangle = \frac{d}{dt}\bigg|_{t=0} f(e^{t\xi}\psi), \quad \langle \nabla'_f(\psi), \xi \rangle = \frac{d}{dt}\bigg|_{t=0} f(\psi e^{t\xi}).$$

Choice of a Poisson structure – 2

We put

$$\{f_1, f_2\} = \langle l(\nabla_{f_1}), \nabla_{f_2}\rangle + \langle r(\nabla'_{f_1}), \nabla'_{f_2}\rangle.$$

Here l and r are two (a priori, different) classical r-matrices. In tensor form this formula may be written as

$$\{\psi_1, \psi_2\} = l_{12}\psi_1\psi_2 + \psi_1\psi_2 r_{12}, \quad \text{where} \quad \psi_1 = \psi \otimes I, \psi_2 = I \otimes \psi.$$

This may be regarded as an abstract version of the Exchange algebra introduced in the early 1990's (notably, by Babelon). The main question, of course, is to restrict and explain the choice of l and r.

Contribution of l

Easy observations.

- For a given l the gauge action becomes Poisson if the gauge group carries the Sklyanin bracket associated with l.
- Left and right bracket are almost independent, but they are linked via the Jacoby/Yang–Baxter identity.

A simple computation. We look at the contribution of l to the Poisson brackets of potentials. Suppose that $f(\psi) = F(\psi^\tau \psi^{-1})$. We denote by X_F, X'_F left and right gradients of F. Then

$$\{f_1, f_2\}^l(\psi) = \langle l(X_1), X_2\rangle + \langle l(X'_1), X'_2\rangle - \langle l \circ \tau^{-1}(X_1), X'_2\rangle - \langle \tau \circ l(X'_1), X_2\rangle$$

Contribution of l

In tensor notation, this yields for $L = \psi^\tau \psi^{-1}$:

$$\{L_1, L_2\}^l = lL_1L_2 + L_1L_2l - L_1l^\tau L_2 - L_2l^{\tau^{-1}}L_1.$$

This formula resembles the "discrete current algebra", but it lacks some crucial terms.
The rescue comes through the choice of r which appears to be very rigid.

The role of r

Key observation. The mapping $\psi \mapsto \psi^\tau \psi^{-1}$ is Poisson if and only if :

$r = r_0 + \frac{\tau + I}{\tau - I}$, where r_0 acts in the subspace of quasi-constants.

Explanation. Left gradient of $f(\psi) = F(\psi^\tau \psi^{-1})$ depends only on left and right gradients of F (regarded as a function of $L = \psi^\tau \psi^{-1}$). By contrast, its right gradient depends on ψ. After some calculations, one gets explicitly:

$$\{f_1, f_2\}^r(\psi) = \langle (r - \tau \cdot r + r - r \cdot \tau^{-1}) \mathrm{Ad}\psi^{-1} X_1', \mathrm{Ad}\psi^{-1} X_2' \rangle.$$

Our mapping is Poisson if and only if $\mathrm{Ad}\psi^{-1}$ cancels.

The role of r

Remarkably, this cancellation is achieved by a simple and unique choice $r = \frac{\tau+I}{\tau-I}$ if we assume that $\tau - I$ is *invertible*. Morever, we have:

Proposition. Assume that $\tau - I$ is invertible. Then $r = \frac{\tau+I}{\tau-I}$ satisfies the modified classical Yang–Baxter identity

$$[rX, rY] - r([rX, Y] + [X, rY]) + [X, Y] = 0;$$

It is skew iff τ is orthogonal.
With our choice of r, the contribution of the right bracket finally becomes

$$\{f_1, f_2\}^r(\psi) = \langle X_1, \tau X_2' \rangle - \langle \tau X_1', \ X_2 \rangle.$$

The role of r

Easy observation.

- The above formula precisely provides the missing terms to convert the left bracket for L's into the correct lattice algebra.
- If l also satisfies the modified Yang–Baxter identity, the full bracket satisfies Jacoby.

The formula we derived fixes the choice of r up to the subspace of quasi-constants.
One more crucial step:

- r is a singular integral operator; one has to define its regularization.

Regularization of r

- **Example**: τ is a shift operator on the line, $\tau f(x) = f(x+1)$. Then

$$r f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{cotan}(k/2) \hat{f}(k) e^{ikx}\, dk.$$

Choose a finite dimensional r-matrix r_0 acting in the space of values.
We set ${}^0r_{\pm} = \frac{1}{2}(r_0 \pm t)$, so that

$${}^0r_{+} + {}^0r_{-} = r_0, \quad {}^0r_{+} - {}^0r_{-} = t,$$

Regularization of r

- The correct regularization of r is given by

$$(rf)(x) =^0 r_+ \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{f}(k) \mathrm{cotan}\,(k + i0) e^{ikx}\, dk \right) + {}^0 r_- \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{f}(k) \mathrm{cotan}\,(k - i0) e^{ikx}\, dk \right), \quad (2)$$

Regularization of r

Set

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n) e^{2\pi i n x}.$$

F is a quasiconstant lying in the kernel of τ. Using the Sokhotsky and Poisson formulas. we can rewrite (2) in equivalent form:

$$(rf)(x) = r_0(F(x)) + \text{v.p.}\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{cotan}(k/2)\hat{f}(k)e^{ikx}\,dk, \quad (3)$$

where r_0 is acting pointwise in the subspace of quasiconstants.
For q-difference operators the kernel of r is expressed through theta functions.

q-deformed Drinfeld–Sokolov theory

Reduction to the case of higher order scalar difference is non-trivial. It allows to fix both l and r_0 completely. Unfortunately, it is already impossible to give explicit formulas, due to the lack of time.

Thank you for your attention!