Constraining the correlations in multi-qubit systems

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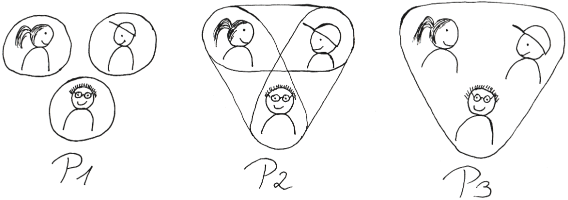
The Bloch representation

Consider the state $|\text{GHZ}\rangle \propto |000\rangle + |111\rangle$, expand in terms of Pauli operators:

$$\rho_{\text{GHZ}} \propto \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \underbrace{Z\otimes Z\otimes\mathbb{1} + Z\otimes\mathbb{1}\otimes Z + \mathbb{1}\otimes Z\otimes Z}_{P_2} + \underbrace{X\otimes X\otimes X - X$$

Question: Are there any relations between P_1, P_2 and P_3?



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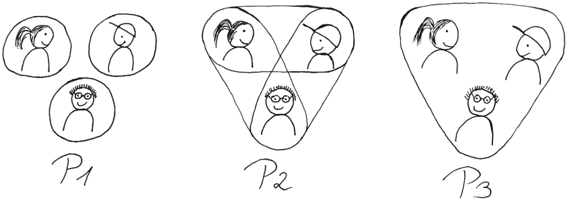

Bloch representation cont'd

- Define the *weight* (wt) of a Pauli tensor product as number of particles it acts on nontrivially, e.g.

$$\mathrm{wt}(\mathbb{1}Z\mathbb{1}) = 1, \qquad \mathrm{wt}(XY\mathbb{1}) = 2, \qquad \mathrm{wt}(ZYY) = 3$$

- Group terms by the weight ($P_k =$ all terms of weight k):

$$\rho = \frac{1}{2^n}(\mathbb{1}^{\otimes n} + \sum_{k=1}^{n} P_k).$$

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Marginal problem

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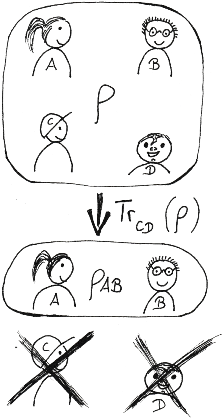
Marginals

Definition

Let $S \subset \{1, \ldots n\}$, define the $|S|$-body marginal of parties S as

$$\rho_S := \mathrm{Tr}_{\{1,\ldots,n\}\setminus S}(\rho)$$

Note: A k-body marginal is a function of $P_1, \ldots, P_k$ of the original state only.



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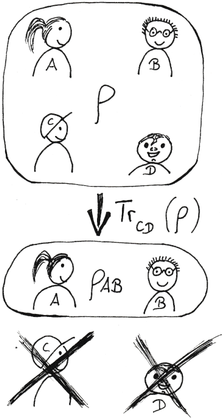

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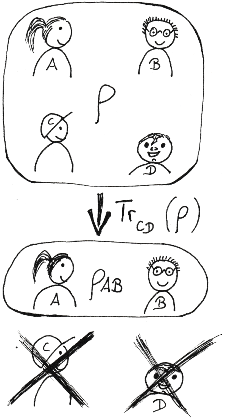

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UDP and UDA

Definition

A pure state $|\psi\rangle$ is called k-**UDP** (uniquely determined among pure states) if there is no other *pure* state $|\phi\rangle$ with the same k-body marginals.

$$|\psi\rangle\langle\psi| \longrightarrow \{\rho_k\} \xleftarrow{?} |\phi\rangle\langle\phi|$$

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Two-body systems

- Consider a two-particle pure state in Schmidt decomposition

$$|\psi\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j}\, |j\rangle_{\mathsf{A}} \otimes |j\rangle_{\mathsf{B}}\,.$$

- Then the reduced states are given by

$$\rho_{\mathsf{A/B}} = \sum \lambda_j\, |j\rangle\langle j|_{\mathsf{A/B}}\,.$$

But there are many compatible states:

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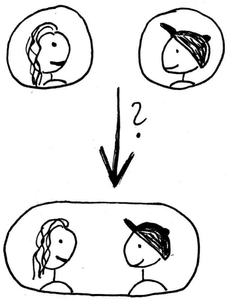

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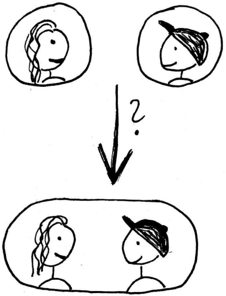
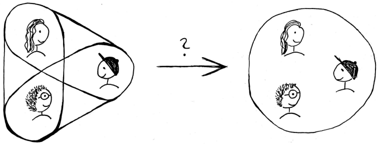


n-body systems

Theorem (Linden, Popescu, Wootters (2002))

Almost all three-qubit pure states are 2-UDA.



Theorem (Jones et al. (2005))

Almost all n-qudit pure states are UDA by $\lfloor \frac{n}{2} \rfloor$ of their $(\lceil \frac{n}{2} \rceil + 1)$-body marginals.

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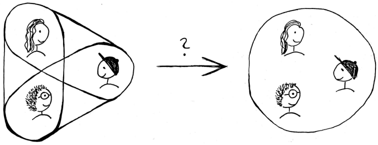

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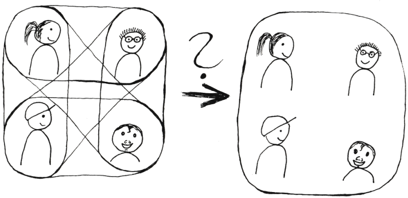
Four-body systems

- Jones: almost all 4-particle states are determined by two of their 3-body marginals.
- How about having just the 2-body marginals?



Main result I

Theorem

Almost all 4-qudit pure states are 2-UDP by certain sets of three of the six 2-body marginals.

Proof works for certain sets of three two-body marginals:

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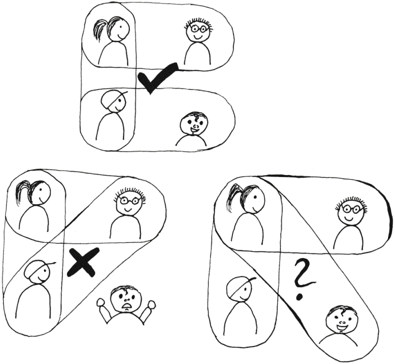

n-particle states

Corollary

For $n \geq 4$, almost all n-qudit pure states are $(n-2)$-UDP by certain sets of three of the $(n-2)$-body marginals.

TL;DR: (Parts of) $P_1, \ldots, P_{n-2}$ determine P_{n-1} and P_n in pure states.

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For $n \geq 4$, almost all n-qudit pure states are $(n-2)$-UDP by certain sets of three of the $(n-2)$-body marginals.

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Even and odd correlations

PRA 97, 060101 (2018)

Even and odd correlations

- Recall: Write ρ as (P_k = all terms of weight k):

$$\rho = \frac{1}{2^n}(\mathbb{1}^{\otimes n} + \sum_{k=1}^{n} P_k).$$

- It will be particularly useful to group terms with an even and an odd number of Paulis:

$$P_e := \sum_{i \text{ even}} P_i, \quad P_o := \sum_{i \text{ odd}} P_i,$$

- From Pauli commutation relations:[1]

$$\{P_e, P_e'\} = \text{only even Paulis}, \quad \{P_o, P_o'\} = \text{only even Paulis},$$
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[1] F. Huber, O. Gühne, J. Siewert, PRL **118**, 200502 (2017)

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Bloch representation for pure states

- For pure states, write $\rho = \frac{1}{2^n}(\mathbb{1} + P_e + P_o)$ and expand $\rho^2 = \rho$.
- Sort in even and odd components:

$$\begin{aligned} 0 &= (2^n - 1)\mathbb{1} + (2^n - 2)P_e - (P_e^2 + P_o^2), \\ 0 &= (2^n - 2)P_o - \{P_e, P_o\}. \end{aligned}$$

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State inversion

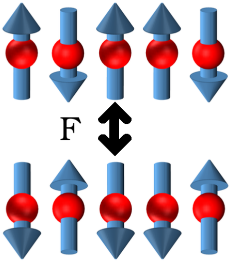
- Define universal state inversion (spin flipped state):

$$\tilde{\rho} := Y^{\otimes n} \bar{\rho} Y^{\otimes n}.$$

- In terms of correlations:

$$\tilde{\rho} = \frac{1}{2^n}(\mathbb{1} + P_e - P_o).$$

- Key observation: If n is odd, $\rho\tilde{\rho} = 0$.



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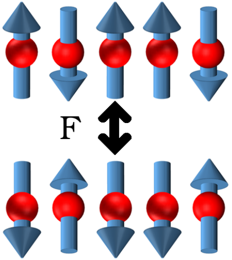

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Main result II

Expand $\rho\tilde{\rho} = 0$ in terms of even and odd components:

Theorem

For pure states of n parties, where n is odd holds:

$$\begin{aligned}(\mathbb{1} + P_e)^2 &= P_o^2, \\ [P_e, P_o] &= 0.\end{aligned}$$

Use additional equations from $\rho^2 = \rho$:

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For pure states of n parties, where n is odd, the even correlations are uniquely determined by the odd correlations:

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Example:

$$\rho_{\mathsf{GHZ}} \propto \mathbb{1}\mathbb{1}\mathbb{1} + \underbrace{ZZ\mathbb{1} + Z\mathbb{1}Z + \mathbb{1}ZZ}_{P_2} + \underbrace{XXX - XYY - YXY - YYX}_{P_3}$$

Check:

$$(XXX - XYY - YXY - YYX)^2 = 4(\mathbb{1}\mathbb{1}\mathbb{1} + ZZ\mathbb{1} + Z\mathbb{1}Z + \mathbb{1}ZZ)$$

Previous results and extensions

- Linden, Popescu, Wootters (2002): P_1 and P_2 determine P_3 (almost always).
- Here: P_1 and P_3 determine P_2.

Corollary

A pure state of n parties, where n is odd, is UDA by its odd correlations (i.e., there is no other mixed or pure state with the same P_o).

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The case of even n

If n is even, the state inversion does not map to orthogonal states. Define the n-concurrence $C_n := |\langle\psi|\tilde{\psi}\rangle|$, then

Theorem

For pure states of n parties, where n is even and $C_n > 0$, the odd correlations are uniquely determined up to the sign by the even correlations.

Summary:

	n even and $0 < C_n < 1$	n odd or $C_n = 0$
P_o given	One-dimensional solution space for P_e	P_e is uniquely determined (even UDA)
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Structure of ground states of Hamiltonians, robustness under white noise, entanglement structure and unitary time evolution under odd Hamiltonians.

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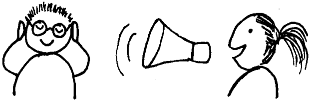

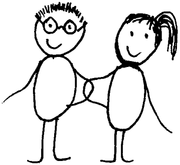

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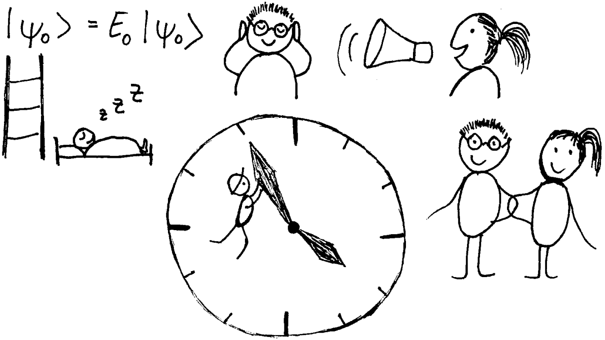

Odd Hamiltonians

Corollary

Let H_o be a Hamiltonian that contains odd body interactions only. Then the n-concurrence C_n of a state ρ is constant under unitary time evolution w.r.t. H_o.

Corollary

Let n be even. Starting with a state $|\psi(0)\rangle$ with $C_n(|\psi(0)\rangle) = 0$ and using an odd Hamiltonian H_o, then the fidelity of $|\psi(t)\rangle$ with $|GHZ\rangle$ never exceeds 50%.

- Can be used to check for the presence of even terms in Hamiltonians.

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Summary/Outlook

- Bloch decomposition is a powerful tool to gain insight into correlations in pure quantum states.
- Often, subsets of correlations determine the rest (4-qubit states det. by P_1 and P_2, ...)
- Natural decomposition in even and odd correlations (odd determine even correlations for $n =$odd states).
- What about states beyond qubits?
- What about inversions on subsets of particles?

Thank you for your attention!

N. Wyderka, F. Huber, O. Gühne
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*PRA **96**, 010102 (2017)*

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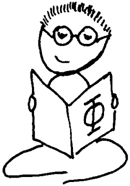
Constraints on correlations in multiqubit systems.
*PRA **97**, 060101 (2018)*



Appendix

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Backup Slides (Noise robustness)

- Let $\rho = \frac{1}{2^n}(\mathbb{1} + P_{\mathsf{e}} + P_{\mathsf{o}})$ be a pure state with $C_n = \alpha$. Then for the eigenvalues holds

$$n \text{ even: } \sigma(P_{\mathsf{e}}) = (2^{n-1}(1+\alpha) - 1, 2^{n-1}(1-\alpha) - 1, -1, -1, \dots)$$
$$n \text{ odd: } \sigma(P_{\mathsf{o}}) = (2^{n-1}, -2^{n-1}, 0, 0, \dots)$$

- Add white noise: $\rho_p = p\rho + \frac{(1-p)}{2^n}\mathbb{1} = \frac{1}{2^n}(\mathbb{1} + pP_{\mathsf{e}} + pP_{\mathsf{o}})$:

$$n \text{ even: } p(P_e) = (\lambda_1 + \lambda_2)/(2^n - 2)$$
$$n \text{ odd: } p(P_{\mathsf{o}}) = (\lambda_1 - \lambda_2)/2^n$$

Backup Slides (Fidelity test)

- Start with $|\psi_0\rangle$ with $C_n(|\psi_0\rangle) = 0$. Then

$$|\psi_t\rangle = e^{-iH_\circ t}|\psi_0\rangle = \sqrt{F}|\text{GHZ}\rangle + \sqrt{1-F}|\chi\rangle$$

with $\langle\chi|\text{GHZ}\rangle = 0$. Then

$$C_n(|\psi_t\rangle) = |\langle\tilde{\psi}_t|\psi_t\rangle| = |F + (1-F)\langle\tilde{\chi}|\chi\rangle| \geq F - (1-F).$$

Thus, if $F > 50\%$, $C_n \neq 0$ and even terms must have been present in $H_\circ$.