Multipartite Entanglement and Combinatorial Designs

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Dénes Petz, 8.04.1953 – 6.02.2018

In Memory of Professor Dénes Petz, Editor of OSID in 1992 – 2018

Composed systems & **entangled states**

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Schmidt decomposition & Entanglement measures

Any pure state from $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$|\psi\rangle = \sum_{ij} G_{ij} |i\rangle \otimes |j\rangle = \sum_i \sqrt{\lambda_i} |i'\rangle \otimes |i''\rangle$, where $|\psi|^2 = \mathrm{Tr}\, GG^\dagger = 1$.

The partial trace, $\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi| = GG^\dagger$, has spectrum given by the **Schmidt vector** $\{\lambda_i\}$ = squared **singular values** of G.

Entanglement entropy of $|\psi\rangle$ is equal to **von Neumann entropy** of the reduced state σ

$$E(|\psi\rangle) := -\mathrm{Tr}\, \sigma \ln \sigma = S(\lambda).$$

Maximally entangled bi–partite quantum states

Bipartite systems $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}_d \otimes \mathcal{H}_d$

generalized Bell state (for two qu*d*its),

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

distinguished by the fact that all **singular values** are equal, $\lambda_i = 1/\sqrt{d}$,
hence the reduced state is **maximally mixed**,
$\rho_A = \mathrm{Tr}_B |\psi_d^+\rangle\langle\psi_d^+| = \mathbb{1}_d/d$.
This property holds for all locally equivalent states, $(U_A \otimes U_B)|\psi_d^+\rangle$.

Observations:

A) State $|\psi\rangle$ is **maximally entangled** if $\rho_A = GG^\dagger = \mathbb{1}_d/d$,
which is the case if the matrix $U = G/\sqrt{d}$ of size d is **unitary**,
(and all its **singular values** are equal to 1).

B) For a **bi–partite** state the **singular values** of G characterize
entanglement of the state $|\psi\rangle = \sum_{i,j} G_{ij} |i,j\rangle$.

Multipartite pure quantum states: $3 \gg 2$

States on N parties are determined by a **tensor** with N indices
e.g. for $N = 3$: $|\Psi_{ABC}\rangle = \sum_{i,j,k} T_{i,j,k} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C$.

Mathematical problem: in general for a **tensor** T_{ijk} there is no (unique) **Singular Value Decomposition** and it is not simple to find the **tensor rank** or **tensor norms** (nuclear, spectral).

Open question: Which state of N subsystems with d–levels each is the **most entangled** ?

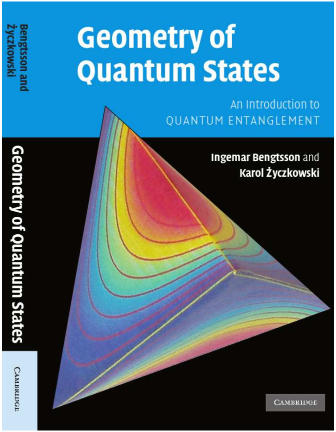
example for three qubits, $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}_2^{\otimes 3}$

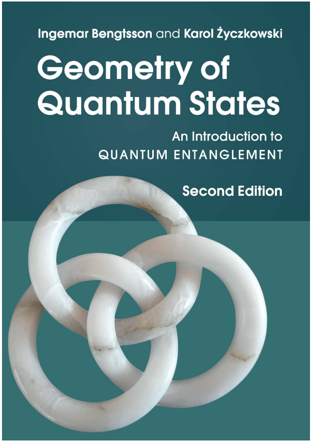
GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$ has a similar property:
all three one-partite reductions are **maximally mixed**
$\rho_A = Tr_{BC}|GHZ\rangle\langle GHZ| = \mathbb{1}_2 = \rho_B = Tr_{AC}|GHZ\rangle\langle GHZ|$.

(what is **not** the case e.g. for $|W\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle)$

Geometry of Quantum States is discussed in a book

published by Cambridge University Press in **2006**,





II edition (with new chapters on multipartite entanglement & discrete structures in the Hilbert space),

August 2017

Genuinely multipartite entangled states

k-uniform states of N qu$d$its

Definition. State $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ is called ***k*-uniform**
if for all possible splittings of the system into k and $N-k$ parts the reduced states are maximally mixed (**Scott 2001**),
(also called **MM**-states (maximally multipartite entangled)
Facchi et al. (2008,2010), **Arnaud & Cerf** (2012)

Applications: quantum error correction codes, teleportation, etc...

Example: 1–uniform states of N qu$d$its

Observation. A generalized, N–qu$d$it **GHZ** state,

$$|GHZ_N^d\rangle := \frac{1}{\sqrt{d}}\big[|1,1,...,1\rangle + |2,2,....,2\rangle + \cdots + |d,d,...,d\rangle\big]$$

is 1–**uniform** (but not 2–uniform!)

Examples of k–uniform states

Observation: k–uniform states may exist if $N \geq 2k$ (**Scott 2001**) (traced out ancilla of size $(N-k)$ cannot be smaller than the principal k–partite system).

Hence there are no 2-uniform states of 3 **qubits**.

However, there exist no 2-uniform state of 4 qubits either!

Higuchi & Sudbery (2000) - **frustration** like in spin systems –
Facchi, Florio, Marzolino, Parisi, Pascazio (2010) –
it is not possible to satisfy simultaneously so many constraints...

2-uniform state of 5 and 6 qubits

$$|\Phi_5\rangle = |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle + \\ + |10000\rangle + |00101\rangle - |00011\rangle - |10110\rangle,$$

related to 5–qubit error correction code by **Laflamme et al.** (1996)

$$|\Phi_6\rangle = |111111\rangle + |101010\rangle + |001100\rangle + |011001\rangle + \\ + |110000\rangle + |100101\rangle + |000011\rangle + |010110\rangle.$$

Combinatorial Designs

$\Longrightarrow$ An introduction to *"Quantum Combinatorics"*

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

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b) - in every row and column there is only a **single** card of each rank

$$\begin{array}{cccc} A\spadesuit & K\diamondsuit & Q\heartsuit & J\clubsuit \\ K\heartsuit & A\clubsuit & J\spadesuit & Q\diamondsuit \\ Q\clubsuit & J\heartsuit & A\diamondsuit & K\spadesuit \\ J\diamondsuit & Q\spadesuit & K\clubsuit & A\heartsuit \end{array}$$

Two **mutually orthogonal Latin squares** of size $N = 4$

Graeco–Latin square !

Mutually orthogonal Latin Squares (MOLS)

♣) $N = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)

♡) $N = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(N - 1)$ MOLS.

♠) $N = 6$. Only a **single** Latin Square exists (No OLS!).

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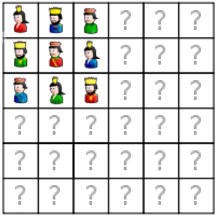
♡) $N = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(N - 1)$ MOLS.
♠) $N = 6$. Only a **single** Latin Square exists (No OLS!).

Euler's problem: **36** officers of six different ranks from six different units come for a **military parade**. Arrange them in a square such that in each row / each column all uniforms are different.



No solution exists ! (conjectured by **Euler**), proof by:
Gaston Terry "Le Probléme de 36 Officiers". *Compte Rendu* **(1901)**.

Mutually orthogonal Latin Squares (MOLS)

An apparent solution of the $N = 6$ **Euler's** problem of **36 officers**.

Wawel castle in Cracow

Orthogonal Arrays

Combinatorial arrangements introduced by **Rao** in 1946 used in statistics and design of experiments, OA($r, N, d,$ **k**)

0	0
1	1

0	0	0
0	1	1
1	0	1
1	1	0

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

Orthogonal arrays OA(2,2,2,**1**), OA(4,3,2,**2**) and OA(8,4,2,**3**):

in each **column** each **symbol** occurres the **same number** of times.

Definition of an **Orthogonal Array**

An array A of size $r \times N$ with entries taken from a d–element set S is called **Orthogonal array** $\mathrm{OA}(r, N, d, k)$ with r runs, N factors, d levels, **strength** k and index λ **if** every $r \times k$ subarray of A contains each $k-$tuple of symbols from S exactly λ times as a row.

Example a) **Two qubit, 1–uniform state**

Orthogonal array

$$OA(2,2,2,1) = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

leads to the **Bell state** $|\Psi_2^+\rangle = |01\rangle + |10\rangle$, which is 1–uniform

Example b) **Three–qubit, 1–uniform state**

Orthogonal array

$$OA(4,3,2,2) = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

leads to a 1–**uniform state**: $|\Phi_3\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle$.

Orthogonal Arrays & k-uniform states

A link between them

	orthogonal arrays	multipartite quantum state $|\Phi\rangle$
r	Runs	Number of terms in the state
N	Factors	Number of qudits
d	Levels	dimension d of the subsystem
k	Strength	class of entanglement (k–uniform)

holds

provided an **orthogonal array** OA(r, N, d, k)
satisfies additional constraints !

(this relation is NOT one–to–one)

Goyeneche, K.Ż. (2014)

k–uniform states and Orthogonal Arrays I

Consider a **pure state** $|\Phi\rangle$ of N qudits,

$$|\Phi\rangle = \sum_{s_1,\ldots,s_N} a_{s_1,\ldots,s_N} |s_1,\ldots,s_N\rangle,$$

where $a_{s_1,\ldots,s_N} \in \mathbb{C}$, $s_1,\ldots,s_N \in S$ and $S = \{0,\ldots,d-1\}$.
Vectors $\{|s_1,\ldots,s_N\rangle\}$ form an orthonormal basis.

Density matrix ρ reads

$$\rho_{AB} = |\Phi\rangle\langle\Phi| = \sum_{\substack{s_1,\ldots,s_N\\ s'_1,\ldots,s'_N}} a_{s_1,\ldots,s_N} a^*_{s'_1,\ldots,s'_N} \, |s_1,\ldots,s_N\rangle\langle s'_1,\ldots,s'_N|.$$

We split the system into **two** parts $\mathcal{S}_A$ and $\mathcal{S}_B$ containing N_A and N_B qudits, $N_A + N_B = N$, remove N_B subsystems to obtain **reduced state**
$\rho_A = \mathrm{Tr}_B(\rho_{AB})$
$= \sum_{\substack{s_1\ldots s_N\\ s'_1\ldots s'_N}} a_{s_1\ldots s_N} a^*_{s'_1\ldots s'_N} \langle s'_{N_A+1},\ldots,s'_N | s_{N_A+1}\ldots s_N\rangle \, |s_1\ldots s_{N_A}\rangle\langle s'_1\ldots s'_{N_A}|.$

$k-$uniform states and Orthogonal Arrays II

A simple, **special case**: coefficients $a_{s_1,\ldots,s_N}$ are *zero* or *one*. Then

$|\Phi\rangle = |s_1^1, s_2^1, \ldots, s_N^1\rangle + |s_1^2, s_2^2, \ldots, s_N^2\rangle + \cdots + |s_1^r, s_2^r, \ldots, s_N^r\rangle,$

upper index i on s denotes the $i-th$ term in $|\Phi\rangle$. These coefficients can be arranged in an **array**

$$A = \begin{array}{cccc} s_1^1 & s_2^1 & \ldots & s_N^1 \\ s_1^2 & s_2^2 & \ldots & s_N^2 \\ \vdots & \vdots & \ldots & \vdots \\ s_1^r & s_2^r & \ldots & s_N^r \end{array} .$$

i). If A forms an **orthogonal array** for any partition the diagonal elements of the reduced state ρ_A are equal.

ii). If the sequence of N_B symbols appearing in every row of removed columns **is not repeated** along the r rows (**irredundant OA**), the reduced density matrix ρ_A becomes diagonal.

Hadamard matrices & Orthogonal Arrays

A Hadamard matrix $H_8 = H_2^{\otimes 3}$ of order $N = 8$ implies OA(8,7,2,2)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

This **Orthogonal Array** of **strength** $k = 2$ allows us to construct a 2–**uniform state** of 7 qubits:

$$|\Phi_7\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle.$$

– the **simplex** state $|\Phi_7\rangle$.

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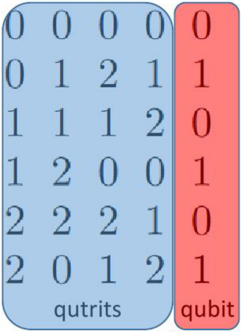
– the **simplex** state $|\Phi_7\rangle$.

No 3-**uniform** states of 7 qubits: **Huber, Gühne, Siewert (2017)**

Heterogeneous systems (e.g. qubits & qutrits)

generalized OA (with mixed alphabet) allow us to construct highly entangled **heterogeneous** states

Example: **four qutrits** and one **qubit**



$$|\Psi_{3^4,2^1}\rangle = |00000\rangle + |01211\rangle + |11120\rangle + |12001\rangle + |22210\rangle + |20121\rangle.$$

Goyeneche, Bielawski, K.Ż (2016)

Absolutely maximally entangled state (AME)

Homogeneous systems (subsystems of the same kind)

Definition. A k–uniform state of N qu*d*its is called **absolutely maximally entangled AME(N,d)** if $k = [N/2]$

Examples:

a) **Bell state** - 1-uniform state of 2 qubits = AME(2,2)

b) **GHZ state** - 1-uniform state of 3 qubits = AME(3,2)

x) **none** - no 2-uniform state of 4 qubits
Higuchi & Sudbery (2000)

c) 2-uniform state $|\Psi_3^4\rangle$ of 4 qutrits, AME(4,3)

d) 3-uniform state $|\Psi_4^6\rangle$ of 6 ququarts, AME(6,4)

e) no **3-uniform** states of 7 qubits
Huber, Gühne, Siewert (2017)

Higher dimensions: AME(4,3) state of four qutrits

From OA(9,4,3,2) we get a **2–uniform** state of **4 qutrits**:

$$
\begin{aligned}
|\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\
& |1011\rangle + |1120\rangle + |1202\rangle + \\
& |2022\rangle + |2101\rangle + |2210\rangle .
\end{aligned}
$$

This state is also encoded in a pair of orthogonal Latin squares of size 3,

$$
\begin{array}{|c|c|c|}
\hline
0\alpha & 1\beta & 2\gamma \\ \hline
1\gamma & 2\alpha & 0\beta \\ \hline
2\beta & 0\gamma & 1\alpha \\ \hline
\end{array}
\quad = \quad
\begin{array}{|c|c|c|}
\hline
A\spadesuit & K\clubsuit & Q\diamondsuit \\ \hline
K\diamondsuit & Q\spadesuit & A\clubsuit \\ \hline
Q\clubsuit & A\diamondsuit & K\spadesuit \\ \hline
\end{array} .
$$

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0α	1β	2γ		$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$	
1γ	2α	0β	$=$	$K\diamondsuit$	$Q\spadesuit$	$A\clubsuit$	.
2β	0γ	1α		$Q\clubsuit$	$A\diamondsuit$	$K\spadesuit$	

Corresponding **Quantum Code**:

$$
\begin{aligned}
|0\rangle &\to |\tilde{0}\rangle := |000\rangle + |112\rangle + |221\rangle \\
|1\rangle &\to |\tilde{1}\rangle := |011\rangle + |120\rangle + |202\rangle \\
|2\rangle &\to |\tilde{2}\rangle := |022\rangle + |101\rangle + |210\rangle
\end{aligned}
$$

Why do we care about AME states?

Since they can be used for various purposes
(e.g. **Quantum codes**, **teleportation**,...)

Resources needed for **quantum teleportation**:

a) **2-qubit Bell state** allows one to teleport **1 qubit** from A to B

b) **2-qudit generalized Bell state** allows one to teleport **1 qudit**

c) **3-qubit GHZ state** allows one to teleport **1 qubit** between any users

d) **4-qutrit GHZ state** allows one to teleport **1 qutrit**
between any two out of four users

f) **4-qutrit state AME(4,3)** allows one to teleport **2 qutrits** between
any pair chosen from four users to the other pair!
- say from the pair (A & C) to (B & D)

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relations between **AME states** and **multiunitary matrices,**
perfect tensors and holographic codes

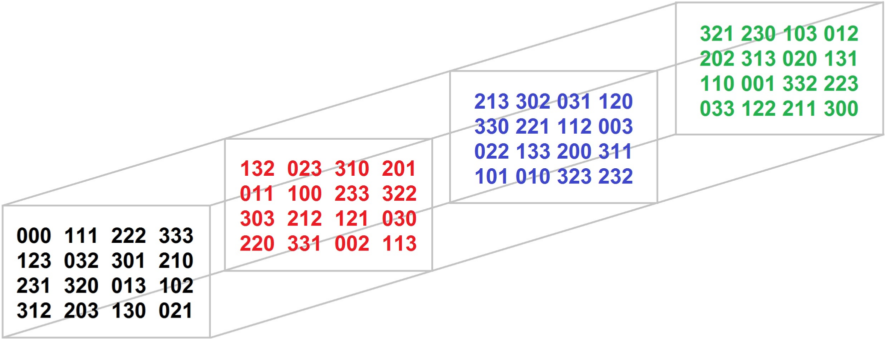
State AME(6,4) of six ququarts:

3–uniform state of **6 ququarts**: read from

three **Mutually orthogonal Latin cubes**

$$|\Psi_4^6\rangle =$$

$$|000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle +$$
$$|012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle + |022013\rangle + |023102\rangle +$$
$$|030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle +$$
$$|102310\rangle + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle +$$
$$|120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle + |130220\rangle + |131331\rangle +$$
$$|132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle +$$
$$|210330\rangle + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle +$$
$$|222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle + |232323\rangle + |233232\rangle +$$
$$|300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle +$$
$$|312020\rangle + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle +$$
$$|330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle.$$



State $|\Psi_4^6\rangle$ of **six** **ququarts** can be generated by three mutually orthogonal **Latin cubes of order four!**

(three address quarts + three cube quarts = 6 quarts in $4^3 = 64$ terms)

Absolutely maximally entangled state (AME) II

Key issue For what number N of qu*d*its the state **AME(N,d)** exist?

How to construct them??

AME(5,2) [**five qubits**] and AME(6,2) [**six qubits**] do exist

but

they contain terms with negative signs $\Rightarrow$ cannot be obtained with OA

new construction needed...

"every good notion can be quantized"

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The new notion of

Quantum Latin Square (QLS) by **Musto & Vicary** (2016)

(square array of N^2 quantum states from $\mathcal{H}_N$:

every column and every row forms a basis)

inspired us to introduce

Mutually Orthogonal Quantum Latin Squares (MOQLS)

and related

Quantum Orthogonal Array (QOA)

Superpositions, entangled states and "quantum designs"

Quantum orthogonal Latin square

Example of order $N = 4$ by **Vicary, Musto (2016)**

$$\begin{vmatrix} |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |2\rangle & |1\rangle & |0\rangle \\ |\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\ |\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle \end{vmatrix}$$

where $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ denote **Bell states**, while
$|\xi_+\rangle = \frac{1}{\sqrt{5}}(i\,|0\rangle + 2\,|3\rangle)$ $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2\,|0\rangle + i\,|3\rangle)$ other **entangled** states.
Four states in each row & column form an **orthogonal basis** in $\mathcal{H}_4$

Standard **combinatorics**: **discrete** set of symbols, $1, 2, \dots, N$,
$+$ **permutation** group
generalized ("Quantum") **combinatorics**: **continuous** family
of states $|\psi\rangle \in \mathcal{H}_N$ $+$ **unitary** group $U(N)$.

Quantum Orthogonal arrays and AME states

Quantum orthogonal array: (entangled strategies $\rightarrow$ **quantum games**)

$$QOA(4, 3+2, 2, 2) = \left(\begin{array}{ccc|c} |0\rangle & |0\rangle & |1\rangle & |\phi^+\rangle \\ |0\rangle & |1\rangle & |0\rangle & |\phi^-\rangle \\ |1\rangle & |0\rangle & |0\rangle & |\psi^+\rangle \\ |1\rangle & |1\rangle & |1\rangle & |\psi^-\rangle \end{array} \right).$$

constructed out of the classical OA(4,3,2,2) and the quantum **Bell basis**

yields the **five qubit AME** state:

$$AME(5,2) = OA(4,3,2,4) \cup \{|\psi_j\rangle\}_{j=1}^4 =$$
$$= |001\rangle \otimes |\phi^+\rangle + |010\rangle \otimes |\phi^-\rangle + |100\rangle \otimes |\psi^+\rangle + |111\rangle \otimes |\psi^-\rangle.$$

Orthogonal Quantum Latin Squares

"every good notion can be quantized"

Definition. A table of N^2 bipartite states $|\phi_{i,j}\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$

$$QOLS = \begin{pmatrix} |\phi_{11}\rangle & |\phi_{12}\rangle & \dots & |\phi_{1N}\rangle \\ |\phi_{21}\rangle & |\phi_{22}\rangle & \dots & |\phi_{2N}\rangle \\ \dots & \dots & \dots & \dots \\ |\phi_{N1}\rangle & |\phi_{N2}\rangle & \dots & |\phi_{NN}\rangle \end{pmatrix}$$

forms a pair of two **Orthogonal Quantum Latin Squares**
if the 4-partite state: $|\Psi_4\rangle := \sum_{i=1}^{N} \sum_{j=1}^{N} |i,j\rangle \otimes |\phi_{ij}\rangle$
is 2–uniform, so it forms the state $|AME(4,N)\rangle$.
This implies that the states are orthogonal, $\langle\phi_{ij}|\phi_{kl}\rangle = \delta_{ik}\delta_{jl}$.

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This implies that the states are orthogonal, $\langle \phi_{ij}|\phi_{kl}\rangle = \delta_{ik}\delta_{jl}$.

However, the **Bell square**: $\begin{matrix} |\phi^+\rangle & |\psi^-\rangle \\ |\psi^+\rangle & |\phi^-\rangle \end{matrix}$

does not form a OQLS. Furtheremore, there are **no** two OQLS(2),
(as there are **no absolutely maximally entangled** states of 4 qubits!)

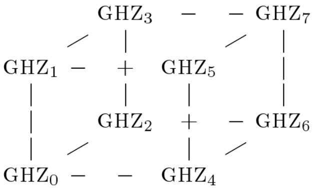
Mutually Orthogonal Quantum Latin Cubes

"every good notion can be quantized"

Definition. A cube of N^3 states $|\phi_{ijk}\rangle \in \mathcal{H}_N^{\otimes 3}$ forms a **Mutually Orthogonal Latin Cube** if the 6-party superposition $|\Psi_6\rangle \ := \ \sum_{i,j,k=1}^{N} |i,j,j\rangle \otimes |\phi_{ijk}\rangle$ is 3–uniform (so it forms the state $|AME(6,N)\rangle$).

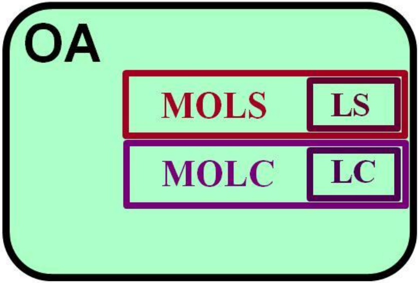
Example. **Cube** of 8 states forming three-qubit GHZ basis:

$0\ 0\ 0\ |\mathrm{GHZ}_0\rangle$
$0\ 0\ 1\ |\mathrm{GHZ}_1\rangle$
$0\ 1\ 0\ |\mathrm{GHZ}_2\rangle$
$0\ 1\ 1\ |\mathrm{GHZ}_3\rangle$
$1\ 0\ 0\ |\mathrm{GHZ}_4\rangle$
$1\ 0\ 1\ |\mathrm{GHZ}_5\rangle$
$1\ 1\ 0\ |\mathrm{GHZ}_6\rangle$
$1\ 1\ 1\ |\mathrm{GHZ}_7\rangle$



leads to QOA(8,3+**3**,2,3) and six-qubit AME state of **Borras**
$$|AME(6,2)\rangle = \sum_{x=0}^{7} |x\rangle \otimes |GHZ_x\rangle.$$
(analogy to state $|\Psi(f)\rangle = \sum_x |x\rangle \otimes |f(x)\rangle$ used in the Shor algorithm!)

Classical **combinatorial designs**...
include: Orthogonal Arrays, Latin Squares, Latin Cubes



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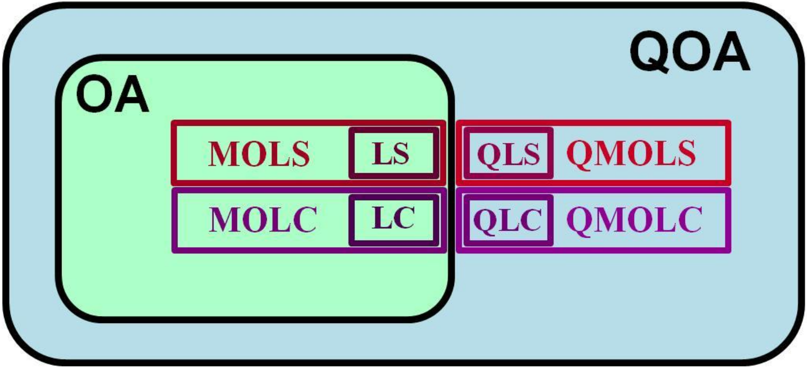


More general **quantum combinatorial designs**

include: **Quantum** Orthogonal Arrays, **Quantum** Latin Squares and Cubes

Goyeneche, Raissi, Di Martino, K.Ż. Phys. Rev. A (2018)

k–uniform states and k–unitary matrices

Consider a **2–uniform** state of four parties A, B, C, D with d levels each,

$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} \Gamma_{ijlm}|i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:

$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,
$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$

Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor $d\Gamma_{ijkl}$ are **unitary** for three reorderings:

a) $\mu, \nu = ij, lm$, b) $\mu, \nu = im, jl$, c) $\mu, \nu = il, jm$.

Such a tensor Γ is called **perfect**.

Corresponding **unitary matrix** U of order d^2 is called **two–unitary** if reordered matrices U^{R_1} and U^{R_2} remain **unitary.**

Unitary matrix U of order d^k with analogous property is called **k–unitary**

Exemplary multiunitary matrices

Two–unitary permutation matrix of size $9 = 3^2$
associated to 2 **MOLS(3)** and 2–**uniform** state $|\Psi_3^4\rangle$ of 4 qutrits

$$U = U_{\substack{ij \\ ml}} = \left(\begin{array}{ccc|ccc|ccc} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{array}\right) \in U(9)$$

Furthermore, also two reordered matrices
(by partial transposition and reshuffling) remain **unitary**:

$$U^{T_1} = U_{\substack{il \\ mj}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in U(9)$$

$$U^{R} = U_{\substack{im \\ jl}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \in U(9)$$

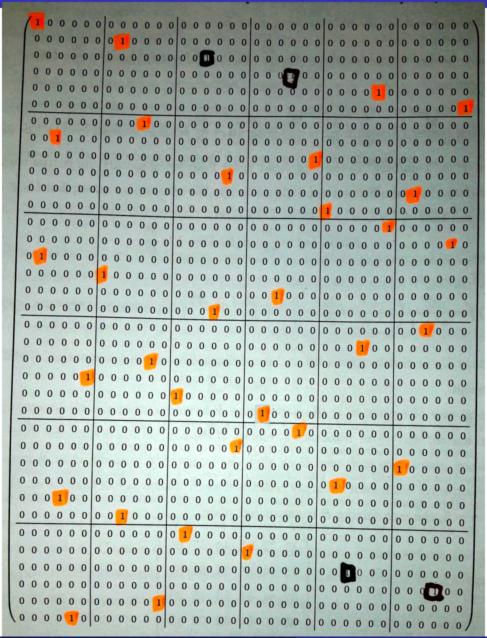
9-sudoku & two orthogonal Latin squares (3)

8	1	6	**2**	4	9	**5**	7	3
3	5	7	**6**	8	1	**9**	2	4
4	9	2	**7**	3	5	**1**	6	8
7	3	5	1	6	8	4	9	2
2	4	9	5	7	3	8	1	6
6	8	1	9	2	4	3	5	7
9	2	4	3	5	7	6	8	1
1	6	8	4	9	2	7	3	5
5	7	3	8	1	6	2	4	9

special sudoku matrix:

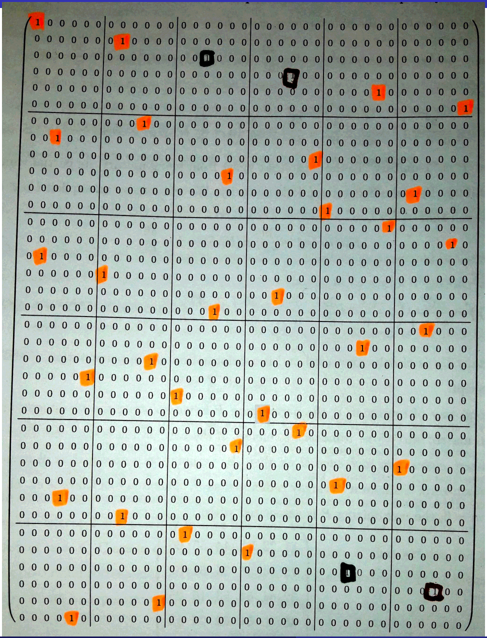
- each symbol appears only once in each row, each column and each box
- each **location** of a given symbol in each box is different !

36-sudoku & two orthogonal Latin squares (6)



What goes **wrong** here?

36-sudoku & two orthogonal Latin squares (6)



What goes **wrong** here?

two pairs of boxes contain **1** in the same **locations** !

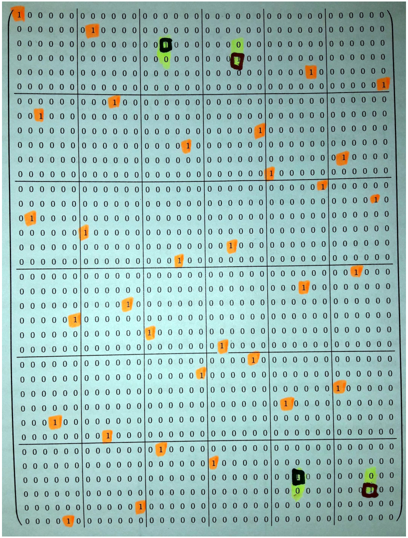
Euler was right:
there are no two OLS(6)

In search for 36 entangled officers of Euler

Two OLS(6) would correspond to a 2-unitary **permutation** matrix P_{36} such that its partial transpose $P_{36}^{T_2}$ and reshufled matrix P_{36}^{R} are **unitary**.

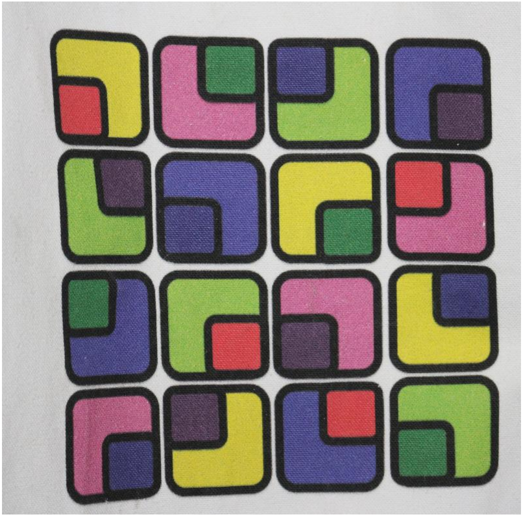
Such matrix does not exists (**Euler**) but one can look for two **quantum** OLS(6):
a **unitary** U_{36} such that its partial transpose $U_{36}^{T_2}$ and reshufled matrix U_{36}^{R} are **unitary**.

the best solution found, $U_{36} =$
is still **not** perfect...



see also poster by **W. Bruzda**

A quick quiz



What **quantum state** can be associated with this design ?

Hints

$A\spadesuit$	$K\diamondsuit$	$Q\heartsuit$	$J\clubsuit$
$K\heartsuit$	$A\clubsuit$	$J\spadesuit$	$Q\diamondsuit$
$Q\clubsuit$	$J\heartsuit$	$A\diamondsuit$	$K\spadesuit$
$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Two **mutually orthogonal** **Latin squares** of size $N = 4$

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Two **mutually orthogonal Latin squares** of size $N = 4$

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$J\diamondsuit$	$Q\spadesuit$	$K\clubsuit$	$A\heartsuit$

Three mutually orthogonal Latin squares of size $N = 4$

The answer

Bag shows **three mutually orthogonal Latin squares** of size $N = 4$ with three attributes A, B, C of each of $4^2 = 16$ squares.
Appending two indices, $i, j = 0, 1, 2, 3$ we obtain a 16×5 table,
$A_{00}, B_{00}, C_{00}, 0, 0$
$A_{01}, B_{01}, C_{01}, 0, 1$
....
$A_{33}, B_{33}, C_{33}, 3, 3$.
It forms an **orthogonal array OA(16,5,4,2)**
leading to the 2–**uniform** state of **5 ququarts**,

$$
\begin{aligned}
|\Psi_4^5\rangle = \quad & |00000\rangle + |12301\rangle + |23102\rangle + |31203\rangle \\
& |13210\rangle + |01111\rangle + |30312\rangle + |22013\rangle + \\
& |21320\rangle + |33021\rangle + |02222\rangle + |10123\rangle + \\
& |32130\rangle + |20231\rangle + |11032\rangle + |03333\rangle
\end{aligned}
$$

related to the **Reed–Solomon code** of length 5.

Concluding Remarks I

1. Basing on **Orthogonal Arrays (OA)** we constructed several **strongly entangled multipartite** quantum pure states
2. Generalized **OA** with **mixed** alphabets allow us to extend the construction for **heterogeneous** systems: e.g **qubits** and **qutrits**.
3. We introduced the notion of **Mutually Orthogonal Quantum Latin Squares** (MOQLS), and **Mutually Orthogonal Quantum Latin Cubes** (MOQLC), which allow us to identify several Absolutely Maximally Entangled states (AME)
4. MOQLS and MOQLC form special cases of **Quantum Orthogonal Arrays** (QOA), which generalize the combinatorial notion of **orthogonal arrays** and lead to a vast garden of highly **entangled multipartite states**.

Open Questions

1. For what number N of subsystems with d levels each an **Absolutely Maximally Entangled** state $|AME(N,d)\rangle$ exists?
2. Are all $|AME(N,d)\rangle$ states related to **Quantum Orthogonal Arrays** ?
3. Are there two **Orthogonal Quantum Latin Squares** for $N=6$,

   AME(4,6) $=$ **36 entangled** officers of **Euler**?

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   numerical results $\Longrightarrow$ **possibly not** (but the question is **open!**)

4. Find further applications of **Absolutely Maximally Entangled** states for quantum error correction codes, quantum protocols, quantum computing and multiuser quantum games.

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4. Find further applications of **Absolutely Maximally Entangled** states for quantum error correction codes, quantum protocols, quantum computing and multiuser quantum games.

5. A speculation whether **Quantum Combinatorics**
will evolve someday into a mature research field for its own?

Kraków -
just on the other side of the mountains...