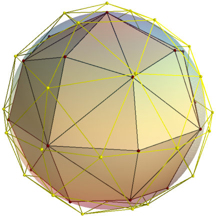
The geometry of EPR correlations

Otfried Gühne

Ana C.S. Costa, H. Chau Nguyen, H. Viet Nguyen, Roope Uola



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Theoretical Quantum Optics

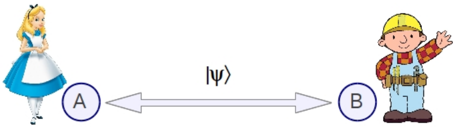
Overview

1. Motivation: Entanglement and steering
2. Steering criteria from entropic uncertainty relations
3. Two-qubit steering: Geometric approach
4. Two-qubit steering: Practical calculation
5. Conclusion

Steering

Bell inequalities

Alice and Bob share a state $|\psi\rangle$.



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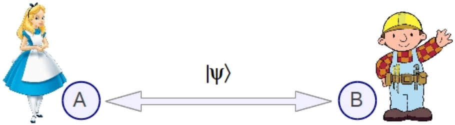

Bell inequalities

- Alice and Bob measure A_x and B_y, obtain the results a, b.
- Question: Can the probabilities be written as

$$P(a, b|x, y) \stackrel{?}{=} \sum_{\lambda} p_{\lambda} p(a|x, \lambda) p(b|y, \lambda)$$

Entanglement

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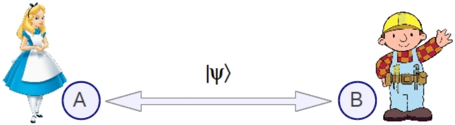

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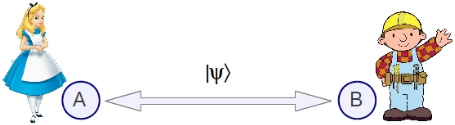

Entanglement vs. Separability

- A separable state can be written as

$$\varrho = \sum_\lambda p_\lambda \varrho_\lambda^A \otimes \varrho_\lambda^B$$

- Otherwise, the state is entangled.
- For the probabilities, this means that

$$P(a, b|x, y) = \sum_\lambda p_\lambda \mathrm{tr}(E_{a|x} \varrho_\lambda^A) \mathrm{tr}(E_{b|y} \varrho_\lambda^B)$$

Entanglement criterion

Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X
- For a product space one can also consider the partial transposition. If $X = A \otimes B$:

$$X^{T_B} = A \otimes B^T$$

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If a state is separable, then its partial transposition has no negative eigenvalues ("the state is PPT" or $\varrho^{T_B} \geq 0$).

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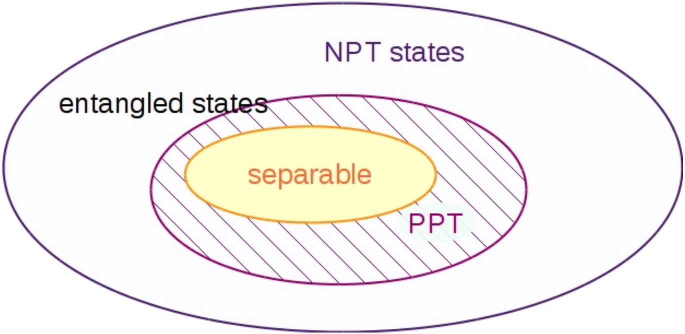
Peres & Horodecki

For small dimensions (2×2 or 2×3): ϱ is PPT $\Leftrightarrow$ ϱ is separable.

A. Peres, PRL 77, 1413 (1996), Horodecki$^{\otimes 3}$, PLA 223, 1 (1996)

Geometry



What happens in between?

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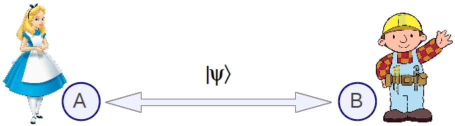

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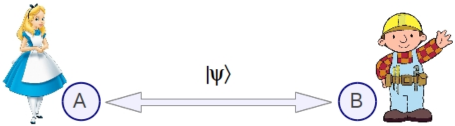


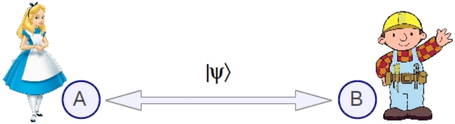
Mixed Scenario

- Consider probabilities of the form

$$P(a,b|x,y) = \sum_\lambda p_\lambda p(a|x,\lambda)\, Tr(E_{b|y}\varrho_\lambda^B) = Tr(E_{b|y}\varrho_{a|x})$$

- What is the physical meaning of such probabilities?

Steering scenario



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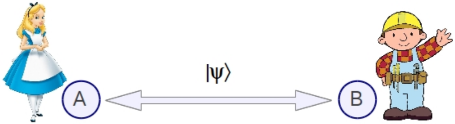

- Alice makes measurements and claims that she can steer Bob's state with that. Bob does not believe it.
- Bob has conditional states $\varrho_{a|x}$ depending on A_x and result a.

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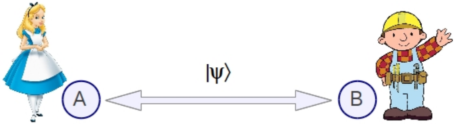

- Alice makes measurements and claims that she can steer Bob's state with that. Bob does not believe it.
- Bob has conditional states $\varrho_{a|x}$ depending on A_x and result a.
- If they are of the form

$$\varrho_{a|x} = \sum_\lambda p_\lambda p(a|x,\lambda)\sigma_\lambda^B = p(a|x)\sum_\lambda p(\lambda|a,x)\sigma_\lambda^B$$

  then Bob is not convinced: Alice's results give only information about existing hidden states σ_λ^B
- Otherwise: Bob has to believe in a spooky action at a distance.

E. Schrödinger, Proc. Camb. Phil. Soc. 31, 555 (1935), H. M. Wiseman et al., PRL 98, 140402 (2007).

Consequences

- Steering is entanglement verification with one untrusted party.
- Typical question: Given an ensemble $\{\varrho_{a|x}\}$ of conditional states, is it steerable? Or is there a local hidden state model?
- Inclusion: Violation of a BI $\Rightarrow$ Steerability $\Rightarrow$ Entanglement

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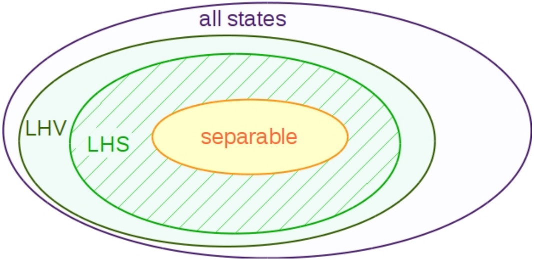

Steering helps

Peres Conjecture

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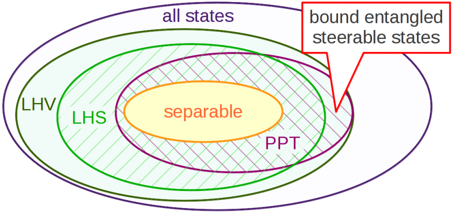
  T. Moroder et al., PRL 113, 050404 (2014).



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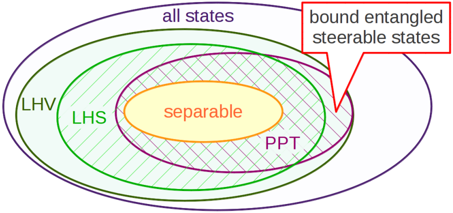

- These PPT entangled states violate a Bell inequality.
  T. Vertesi, N. Brunner, Nature Comm. 5, 5297 (2014).

Menu

Steering criteria

- For entanglement, positive maps provide a systematic way to derive entanglement criteria.
- Is there a systematic method to derive steering criteria?

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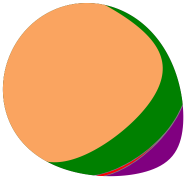
Two qubits

- The border of separability for two qubits can be computed with the PPT criterion



- How to compute the border of steerable states for two qubits?

Entropic Uncertainty Relations & Steering

Photo: A. Jaffe

Entropies

Von Neumann et al.

- Entropy for a distribution $\mathcal{P} = \{p_k\}$

$$S = -\sum_k p_k \ln(p_k)$$

- Relative entropy $D(\mathcal{P}\|\mathcal{Q}) = \sum_k p_k \ln(p_k/q_k)$ and conditional entropy

$$S(B|A) = S(A,B) - S(A)$$

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$$S(B|A) = S(A,B) - S(A)$$

- Tsallis entropy

$$S_q = -\sum_k p_k^q \ln_q(p_k)$$

with $\ln_q(x) = (x^{1-q} - 1)/(1-q)$

EUR

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Let $S(X)$ denote the entropy of the probability distribution of a measurement X. Then, e.g.,

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First criteria for steering

Consider the measurements $X = \sigma_x^A \otimes \sigma_x^B$ and $Z = \sigma_z^A \otimes \sigma_z^B$. Then, for unsteerable states

$$S(\sigma_x^B | \sigma_x^A) + S(\sigma_z^B | \sigma_z^A) \geq \ln(2)$$

This has a nice interpretation, but can it be generalized?

J. Schneeloch et al., PRA 87, 062103 (2013).

General criteria

Theorem

Any entropic uncertainty relation (arbitrary measurements and nearly arbitrary entropy) can be converted into a steering inequality.

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Any entropic uncertainty relation (arbitrary measurements and nearly arbitrary entropy) can be converted into a steering inequality.

Example: Tsallis

Let B_k be some observables on Bob's space with an EUR

$$\sum_k S_q(B_k) \geq C_q$$

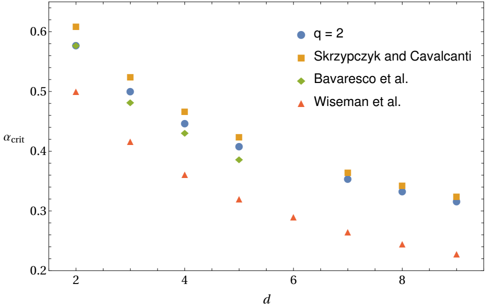
Then

$$\sum_k \left[S_q(B_k|A_k) + (1-q)\Gamma(A_k, B_k) \right] \geq C_q$$

with $\Gamma = f(p_{\alpha\beta})$ being a correction term.

Application

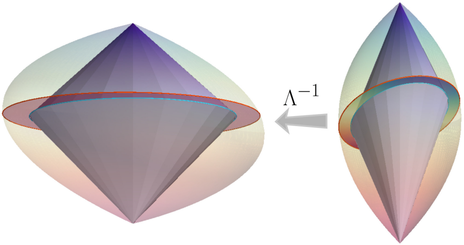
Isotropic states and $q = 2$



$$\varrho = \alpha|\Phi^+\rangle\langle\Phi^+| + (1-\alpha)\frac{\mathbb{1}}{d^2}$$

Other examples: One-way steerable states

Steering: Geometric approach



The map Λ

Reminder: the task

Consider the the conditional states $\varrho_{a|x}$. Can they be written as

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The map Λ

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Consider the the conditional states $\varrho_{a|x}$. Can they be written as

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The map

Define for a state ϱ_{AB} the map:

$$\Lambda(X_A) = Tr_A(\varrho_{AB} X_A \otimes \mathbb{1}_B)$$

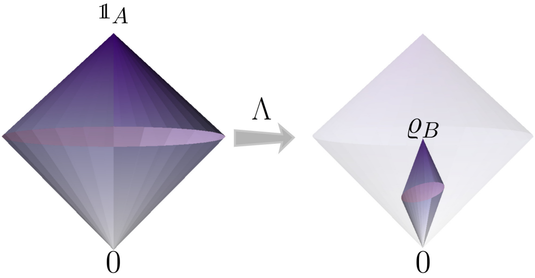
For a measurement effect $E_{a|x}$ one has:

$$\varrho_{a|x} = Tr_A(\varrho_{AB} E_{a|x} \otimes \mathbb{1}_B) = \Lambda(E_{a|x})$$

so Λ describes the steering outcomes.

Geometry of the map Λ

- The effects $0 \leq E_{a|x} \leq \mathbb{1}$ form a 4D double cone.
- Λ maps it to another double cone:



- Wlog: Λ is invertible.

The capacity $\mathcal{K}$

- For projective measurements with $E_{+|x} + E_{-|x} = \mathbb{1}_A$ on a qubit we have to solve:

$$\varrho_{\pm|x} = \Lambda(E_{\pm|x}) \stackrel{?}{=} \int d\mu(\sigma) G_{\pm|s}(\sigma)\sigma,$$

with $G_{+|x} + G_{-|x} = 1$ and $\varrho_B = \Lambda(\mathbb{1}_A) = \int d\mu(\sigma)\sigma$.

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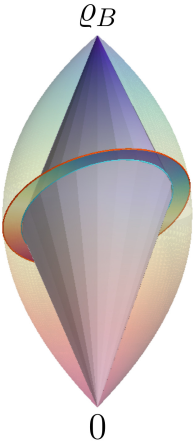
  with $G_{+|x} + G_{-|x} = 1$ and $\varrho_B = \Lambda(\mathbb{1}_A) = \int d\mu(\sigma)\sigma$.
- The set of all reachable $\varrho_{\pm|x}$ is the capacity of μ

$$\mathcal{K}(\mu) = \Big\{ K = \int d\mu(\sigma) g(\sigma)\sigma : 0 \leq g(\sigma) \leq 1 \Big\}.$$

- For a given μ the set $\mathcal{K}(\mu)$ is convex and contains 0_B and ϱ_B.

Geometry



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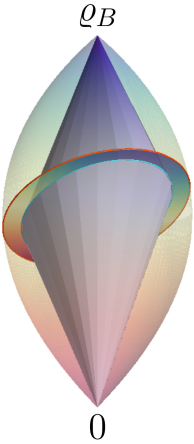

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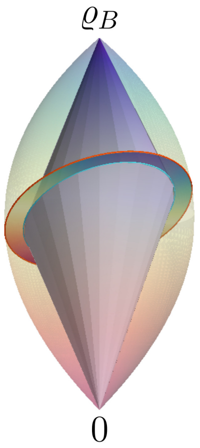

- We want to check the relation between the image of Λ and $\mathcal{K}(\mu)$
- This is simplified be geometry: The equator suffices!
- We need to compute the principal radius $r(\varrho_{AB}, \mu)$ in the appropriate norm!
- If $r(\varrho_{AB}, \mu) \geq 1$ then ϱ_{AB} is not steerable.

Geometry

- For qubits, this boils down to a simple optimization problem:

$$r(\varrho_{AB}, \mu) = \min_C \frac{1}{\sqrt{2}\| Tr_B[\bar{\varrho}(\mathbb{1}_A \otimes C)]\|} \int d\mu(\sigma)| Tr_B(C\sigma)|,$$

where $\bar{\varrho} = \varrho_{AB} - (\mathbb{1}_A \otimes \varrho_B)/2$, $\|X\| = \sqrt{Tr(X^\dagger X)}$, and C denotes an observable on Bob's space.

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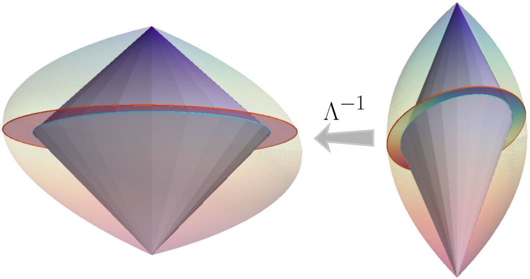
- Proof idea: Characterize $\mathcal{K}(\mu)$ by linear inequalities, apply Λ^{-1}, and project onto equator.



Criterion

The critical radius

- The critical radius is obtained via optimization over all μ:

$$R(\varrho_{AB}) = \max_{\mu} r(\varrho_{AB}, \mu).$$

- Subtle points under the carpet: $R(\varrho_{AB}) \stackrel{?}{=} \infty$, $R(\varrho_{AB})$ not continuous, μ^* exists, ...

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Main result for two qubits & projective measurements

$$\varrho_{AB} \text{ is steerable} \Leftrightarrow R(\varrho_{AB}) < 1$$

Remaining task: Compute the critical radius ...

Scaling property of the critical radius

- States mixed with separable noise $\varrho_\alpha^{\mathrm{n}} = \alpha\varrho_{AB} + (1-\alpha)\mathbb{1}_A \otimes \varrho_B/2$ obey:

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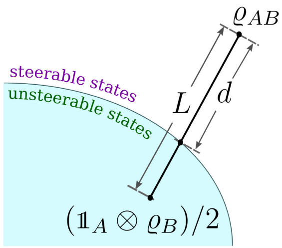
- Consequently, the border of unsteerable states can be computed:



Symmetry of the critical radius

- Given a state ϱ_{AB}, consider the family of states

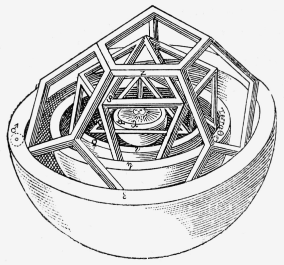
$$\tilde{\varrho} = \frac{1}{\mathcal{N}}(U_A \otimes V_B)\varrho_{AB}(U_A^\dagger \otimes V_B^\dagger),$$

- Then, the critical radius stays the same

$$R(\varrho_{AB}) = R(\tilde{\varrho})$$

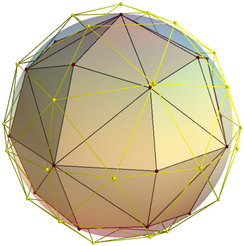
- For steerability, this was known.
  R. Gallego et al., PRX 2015, M. Tulio Quintino et al., PRA 2015, R. Uola et al., PRL 2014.
- Under these transformations, any state can be brought into a canonical form.

Calculation of the critical radius



Calculation of the critical radius

- Idea: Approximate probability distributions on the sphere by distributions on the vertices of an inner and outer polytope. This gives upper and lower bounds.



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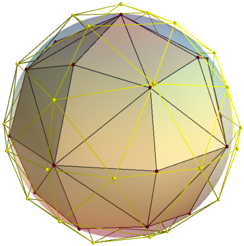

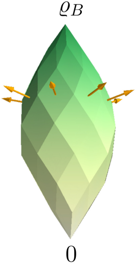

- Even better: For an inner polytope with inscribed radius r_{in}:

$$R_{\mathrm{in}} \leq R(\varrho_{AB}) \leq R_{\mathrm{in}}/r_{\mathrm{in}}$$

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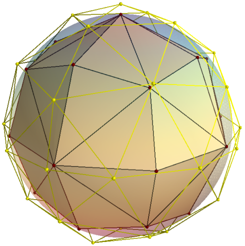

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- For a polytope with N vertices, $\mathcal{K}(\mu)$ has $O(N^3)$ facets.
- For the principal radius, there are only $O(N^3)$ possible C to check.

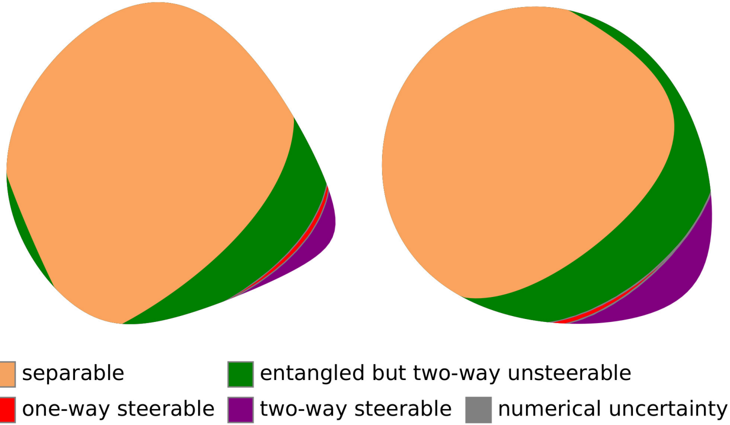
Steering for Babies

For a truncated icosahedron with $N = 60$:

$$r_{\text{in}} = \sqrt{\frac{3}{109}(17 + 6\sqrt{5})} \approx 0.915$$

$\Rightarrow 60 \times 59 \times 58/6 = 34.220$ linear programs with 60 variables each determine the critical radius with 9% error for *any* state.

Result I: Random cross-sections



252 vertices, 40 min

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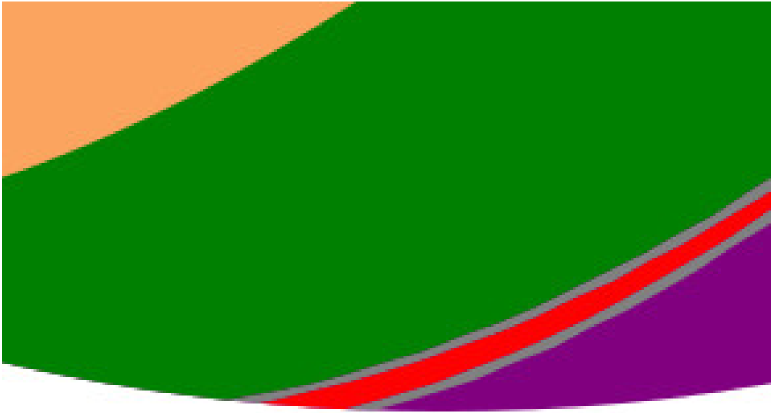

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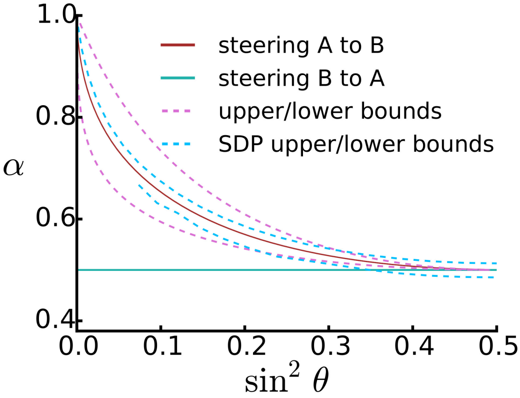
Result II: One-way steerable states



$\varrho_{AB} = \alpha|\theta\rangle\langle\theta| + (1-\alpha)\varrho_A \otimes \frac{\mathbb{1}}{2}$ with $|\theta\rangle = \cos(\theta/2)|00\rangle + \sin(\theta/2)|11\rangle$

J. Bowles et al., PRA 2016, M. Fillettaz et al., arXiv:1804.07576.

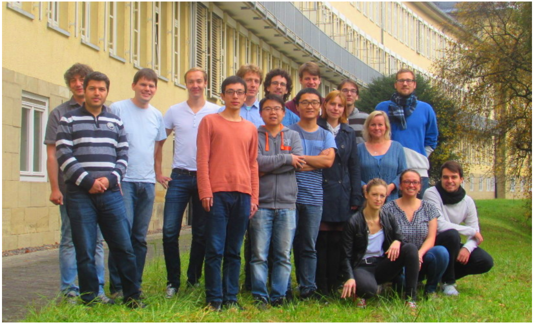
Conclusion

- Steering is an interesting problem.
- Entropic uncertainty relations offer a systematic way to derive steering criteria.
- For two-qubits and projective measurements the problem can be solved with a geometric approach.
- Many open problems remain: higher dimensions, POVMs, ...
- Other applications: Simulatability of measurements, joint measureability, entropic uncertainty relations, ...

Literature

- A. C.S. Costa, R. Uola, and O. Gühne, arXiv:1710.04541
- H.C. Nguyen, H.-V. Nguyen, and O. Gühne, arXiv:1808.09349

Acknowledgements



DFG

FQXi FOUNDATIONAL QUESTIONS INSTITUTE

DAAD

F U FAUNDATION

FWF Der Wissenschaftsfonds.

SUOMEN KULTTUURIRAHASTO

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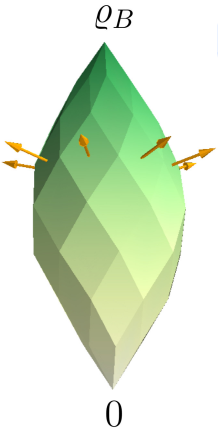
- The critical radius can be defined for any operator. It is invariant under partial transposition.
  $\Rightarrow$ Steering $\neq$ Entanglement

Calculation of the critical radius



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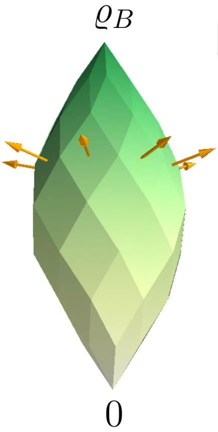

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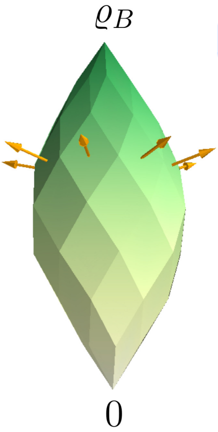

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- For any μ, the capacity $\mathcal{K}(\mu)$ is a polytope.
- For a polytope with N vertices, $\mathcal{K}$ has $O(N^3)$ facets.
- This implies that for the pricipal radius, there are only $O(N^3)$ possible C to check.
- These C are independent of μ

All in all:
$O(N^3)$ linear programs with $O(N)$ variables.

PVMs vs POVMS

- The critical radius can also be defined for higher-dimensional systems and POVMs.
- Scaling properties still hold, but no simple evaluation.
- Numerical evidence for qubits: Already the principal radius for PVMs and POVMs is the same for any μ.