Disproving hidden variable models with spin magnitude conservation

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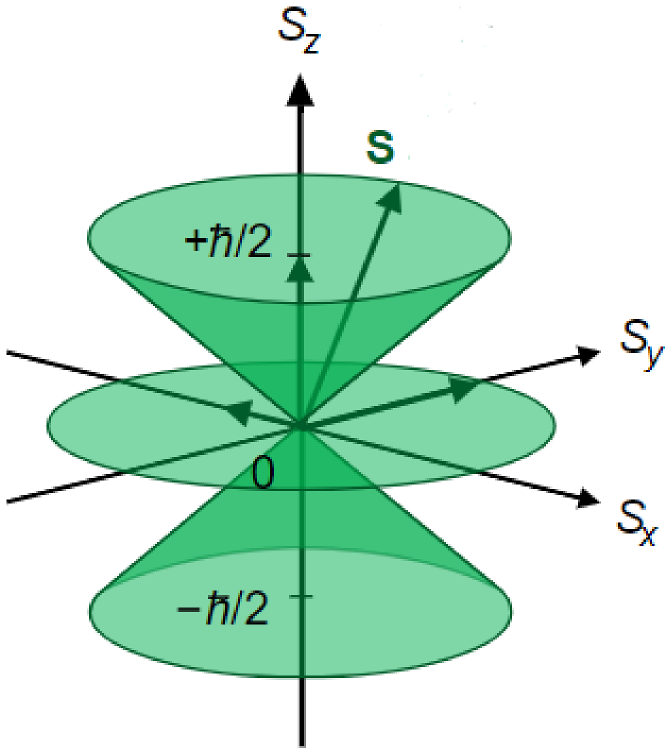

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Our problem

We argue that a physical HV model should be
not only local, realistic and noncontextual,
but should also obey generally accepted physical principles.

K. Nagata, W. Laskowski, M. Wieśniak, M. Żukowski
Rotational Invariance as an Additional Constraint on Local Realism
Phys. Rev. Lett. 93, 230403 (2004).

C. F. Wildfeuer, J. P. Dowling
Strong violations of Bell-type inequalities for Werner-like states
Phys. Rev. A 78, 032113 (2008)

I. J. Pitowsky
New Bell inequalities for the singlet state
Math. Phys. 49, 012101 (2008).

rotational
invariance

symmetry

In case of spin systems the model should conserve angular
momentum, i.e., the length of spin vectors should be fixed

Some basic facts ...

Consider a spin ***s*** particle whose spin state is represented by a vector or a density matrix in a ***(2s + 1)***-dimensional Hilbert space.

The number ***s*** is either an <u>integer</u> or a <u>half-integer.</u>

The average values of spin coordinates can be determined with a help of the three spin operators ***Sx***, ***Sy***, and ***Sz.***

For s=1

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

The generalization of these matrices for arbitrary spin s is

$$(S_x)_{ab} = \frac{\hbar}{2} \left(\delta_{a,b+1} + \delta_{a+1,b}\right) \sqrt{(s+1)(a+b-1) - ab}$$

$$(S_y)_{ab} = \frac{i\hbar}{2} \left(\delta_{a,b+1} - \delta_{a+1,b}\right) \sqrt{(s+1)(a+b-1) - ab}$$

$$(S_z)_{ab} = \hbar(s+1-a)\delta_{a,b} = \hbar(s+1-b)\delta_{a,b}$$

where indices a, b are integer numbers such that

$1 \le a \le 2s+1$ and

$1 \le b \le 2s+1$

Some basic facts ...

Consider a spin ***s*** particle whose spin state is represented by a vector or a density matrix in a ***(2s + 1)***-dimensional Hilbert space.

The number ***s*** is either an <u>integer</u> or a <u>half-integer.</u>

The average values of spin coordinates can be determined with a help of the three spin operators ***Sx***, ***Sy***, and ***Sz***.

The spectrum of each operator is: ***s, s – 1, …. , -s***.

These eigenvalues correspond to possible spin projections onto a given axis.

Some basic facts ...

$$\boxed{\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = s(s+1)\mathbb{I}}$$

The length of the spin vector is $\sqrt{s(s+1)}$
and that within quantum formalism this property is conserved – it does not depend on a spin state.

In addition, the relation for the sum of squares does not depend on the choice of directions x, y, and z. What matters, is that the directions are mutually orthogonal.

HV models

Imagine that there is a HV theory that assigns a well defined value to each spin operator.

Each spin operator is preassigned one of ***2s+1*** eigenvalues:

$$(\hat{S}_x, \hat{S}_y, \hat{S}_z) \rightarrow (v(\hat{S}_x), v(\hat{S}_y), v(\hat{S}_z)) \xrightarrow{\text{for example}} (s-1, s-1, s-1)$$

<u>Kochen-Specker theorem:</u>

For $s \geq 1$ such assignment is not possible for some properly chosen sets of directions, but it is in principle possible to do it for three mutually orthogonal directions *x*, *y* and *z* for any s.

The main problem

It is possible to deterministically assign eigenvalues to three spin operators for mutually orthogonal axes.

$$\vec{s} = \left(v(\hat{S}_x), v(\hat{S}_y), v(\hat{S}_z)\right)$$

However, is it possible to satisfy:

$$|s|^2 = v(\hat{S}_x)^2 + v(\hat{S}_y)^2 + v(\hat{S}_z)^2 = s(s+1)$$

?

Example – spin 1/2

$\vec{s} \in \{-\frac{1}{2}, \frac{1}{2}\}$

The HV model leads to:

$$\vec{s} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$$

Hence:

$$|\vec{s}|^2 = \left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{2}\right)^2 + \left(\pm\frac{1}{2}\right)^2 = \frac{3}{4} \stackrel{\checkmark}{=} s(s+1)$$

Example – spin 1 $\vec{s} \in \{-1,0,1\}$

For spin-1: $|\vec{s}|^2 = s(s+1) = 2$

The HV model leads to:

$\vec{s}$	$\|\vec{s}\|^2$
$(\pm 1, \pm 1, \pm 1)$	3 ✗
$(0, \pm 1, \pm 1)$ $(\pm 1, 0, \pm 1)$ $(\pm 1, \pm 1, 0)$	2 ✓
$(0, \pm 1, 0)$ $(\pm 1, 0, 0)$ $(0, 0, \pm 1)$	1 ✗
$(0,0,0)$	0 ✗

Example – spin-3/2 $\vec{s} \in \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$

For spin-3/2: $|s|^2 = s(s+1) = \frac{15}{4}$

The HV model leads to:

$\vec{s}$	$\lvert\vec{s}\rvert^2$
$\left(\pm\frac{3}{2}, \pm\frac{3}{2}, \pm\frac{3}{2}\right)$	27/4 ✗
$\left(\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{2}\right)$ $\left(\pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{3}{2}\right)$ $\left(\pm\frac{3}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}\right)$	19/4 ✗
$\left(\pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ $\left(\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}\right)$ $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{3}{2}\right)$	11/4 ✗
$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$	3/4 ✗

Magnitude conservation for arbitrary s

$$v(\hat{S}_x)^2 + v(\hat{S}_y)^2 + v(\hat{S}_z)^2 = s(s+1) \quad (*)$$

Legendre's three-square theorem (1798)

A non−negative integer n can be written as a sum of three squares of integers iff n is not of the form $4^a(8b+7)$.

Half-integer case

(*) has a solution if

$$2s = 1 \bmod 4$$

No solution for: $s = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$

Integer case

(*) has a solution if s cannot be written as either of the forms:

$$\begin{aligned} s &= 4(8i+3), \\ s &= 16(8i+7)\times 4^j, \\ s &= 4(8i+5)-1, \\ s &= 16(8i+1)\times 4^j - 1, \end{aligned}$$

No solution for: s = 12, 15, 19, 44, 51, ...

Numerology

$$S_x^2 + S_y^2 + S_z^2 = s(s+1) \quad \Big| \cdot 4$$

$$\underbrace{(2S_x)^2}_{} + \underbrace{(2S_y)^2}_{} + \underbrace{(2S_z)^2}_{} = \underbrace{2s(2s+2)}_{\equiv \text{ RHS}}$$

odd integers: $1, 3, 5, 7, \ldots\ldots$

RHS is odd

$4^a(8b+7)$ is odd for $a = 0$

RHS can be written as a sum of $3 \,\square$ of integers iff it is not 7 mod 8.

I

$2s = 1 \bmod 4$

$4s = 2 \bmod 8$

$\text{RHS} = (2s)^2 + 4s = \underline{3 \bmod 8}$

II

$2s = 3 \bmod 4$

$4s = 6 \bmod 8$

$\text{RHS} = 7 \bmod 8$

Numerology

(odd integer)^2 is odd

(even integer)^2 is even

$$(2k+1)^2 + (2l+1)^2 + (2m+1)^2$$

$$= 4k^2 + 4k + 1 + 4l^2 + 4l + 1 + 4m^2 + 4m + 1$$

$$= 4(\underbrace{\underbrace{k^2+k}_{(k+1)k \ \text{even}} + \underbrace{l^2+l}_{\text{even}} + \underbrace{m^2+m}_{\text{even}}}_{2n}) + 3$$

$$= 8n + 3 \longrightarrow 3 \bmod 8.$$

Numerology

$s(s+1)$ is even $\Rightarrow$ $4^a(8b+7)$ can not be even

$\Rightarrow a \neq 0.$

s even

$s = 4^a q$ $\quad$ q – positive integer

$s(s+1) = 4^a\underbrace{(4^a q^2 + q)}_{8b+7} \Rightarrow q$ must be odd

$4^a q^2 + q = 7 \bmod 8$

From previous proof: $q^2 = 1 \bmod 8 \Rightarrow 4^a + q = 7 \bmod 8$

If $a = 1$ $\quad q = 3 \bmod 8 \longrightarrow 4(8i+3)$

If $a > 1$ $\quad q = 7 \bmod 8 \longrightarrow 16(8i+7)\cdot 4^j$

s odd

$s + 1 = 4^a q$

$s(s+1) = (4^a q - 1)\,4^a q = 4^a(4^a q^2 - q)$

$\Downarrow\ 4^a(8b+7)$

$4^a - q = 7 \bmod 8$

If $a = 1$ $\quad q = 5 \bmod 8 \longrightarrow 4(8i+5) - 1$

If $a > 1$ $\quad q = 1 \bmod 8 \longrightarrow 16(8i+1)\cdot 4^j - 1$

Magnitude conservation for arbitrary s

No solution for: $s = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$

No solution for: s = 12, 15, 19, 44, 51, …

no HV model for any state

a phenomenon that is analogous to the state-independent contextuality

---

Even if there is a model, only a limited number of possible spin projections can be measured if the spin magnitude is to be conserved.

For $s = 2$, $\quad s(s+1) = 6$

The only HV assignments allowed by our model are of the form $(\pm 2, \pm 1, \pm 1)$

The remaining projections can never be measured according to the model, so if they are measured, the model is contradicted.

a projection of the spin onto some axis could be zero

$$|s = 2, s_z = 0\rangle$$

it automatically refutes the HV model

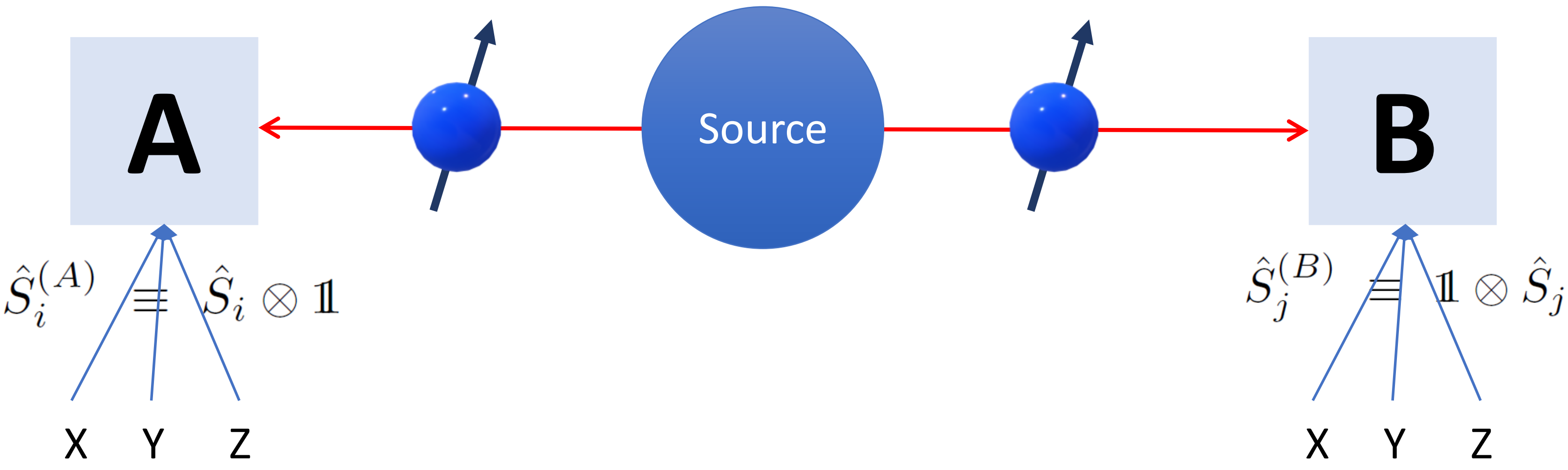
However, there exist states, such as

$$|s = 2, s_z = 1\rangle$$

for which the probability of measuring a projection zero along the directions x, y or z is 0.

a phenomenon that is analogous to the state-dependent contextuality

Bell-like scenario



The general form of the correlation inequality:

$$
\begin{aligned}
&c_{xx}\left\langle \hat{S}_x^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{xy}\left\langle \hat{S}_x^{(A)}\hat{S}_y^{(B)}\right\rangle + c_{xz}\left\langle \hat{S}_x^{(A)}\hat{S}_z^{(B)}\right\rangle \\
&+ c_{yx}\left\langle \hat{S}_y^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{yy}\left\langle \hat{S}_y^{(A)}\hat{S}_y^{(B)}\right\rangle + c_{yz}\left\langle \hat{S}_y^{(A)}\hat{S}_z^{(B)}\right\rangle \\
&+ c_{zx}\left\langle \hat{S}_z^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{zy}\left\langle \hat{S}_z^{(A)}\hat{S}_y^{(B)}\right\rangle + c_{zz}\left\langle \hat{S}_z^{(A)}\hat{S}_z^{(B)}\right\rangle \geq \beta
\end{aligned}
$$

Example – spin-1 $\vec{s} \in \{-1,0,1\}$

$$-\left\langle \hat{S}_x^{(A)} \hat{S}_x^{(B)} \right\rangle - \left\langle \hat{S}_x^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_y^{(A)} \hat{S}_x^{(B)} \right\rangle + \left\langle \hat{S}_y^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_z^{(A)} \hat{S}_z^{(B)} \right\rangle \geq \beta$$

$$\beta = -3$$

- ✗ $(\pm 1, \pm 1, \pm 1)$
- ✓ $(0, \pm 1, \pm 1)$ $(\pm 1, 0, \pm 1)$ $(\pm 1, \pm 1, 0)$
- ✗ $(\pm 1, 0, 0)$ $(0, \pm 1, 0)$ $(0, 0, \pm 1)$
- ✗ $(0,0,0)$

Example – spin-1 $\vec{s} \in \{-1,0,1\}$

$$-\left\langle \hat{S}_x^{(A)} \hat{S}_x^{(B)} \right\rangle - \left\langle \hat{S}_x^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_y^{(A)} \hat{S}_x^{(B)} \right\rangle + \left\langle \hat{S}_y^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_z^{(A)} \hat{S}_z^{(B)} \right\rangle \geq \beta$$

$$\beta = -2$$

✓ $(0,\pm 1,\pm 1)\ (\pm 1,0,\pm 1)\ (\pm 1,\pm 1,0)$

Example – spin-1 $\vec{s} \in \{-1,0,1\}$

$$-\left\langle \hat{S}_x^{(A)} \hat{S}_x^{(B)} \right\rangle - \left\langle \hat{S}_x^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_y^{(A)} \hat{S}_x^{(B)} \right\rangle + \left\langle \hat{S}_y^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_z^{(A)} \hat{S}_z^{(B)} \right\rangle \geq \beta$$

$\beta = -3$
(HV without conservation)

$\beta = -\frac{(1+\sqrt{17})}{2} \approx -2.5616$
(quantum)

$\beta = -2$
(HV with conservation)

Example – spin-1

$$
\begin{aligned}
&-\left\langle \hat{S}_x^{(A)} \hat{S}_x^{(B)} \right\rangle - \left\langle \hat{S}_x^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_x^{(A)} \hat{S}_z^{(B)} \right\rangle - \left\langle \hat{S}_y^{(A)} \hat{S}_x^{(B)} \right\rangle \\
&- \left\langle \hat{S}_y^{(A)} \hat{S}_y^{(B)} \right\rangle - \left\langle \hat{S}_y^{(A)} \hat{S}_z^{(B)} \right\rangle - \left\langle \hat{S}_z^{(A)} \hat{S}_x^{(B)} \right\rangle - \left\langle \hat{S}_z^{(A)} \hat{S}_y^{(B)} \right\rangle \\
&+ 3 \left\langle \hat{S}_z^{(A)} \hat{S}_z^{(B)} \right\rangle \geq \beta
\end{aligned}
$$

$$
-\sqrt{17} \approx -4.1231 \geq \begin{cases} -7 \text{ (without conservation)} \\ -4 \text{ (with conservation)} \end{cases}
$$

Example – spin-2

$$\frac{5}{2}\left\langle \hat{S}_x^{(A)}\hat{S}_x^{(B)}\right\rangle + 2\left\langle \hat{S}_x^{(A)}\hat{S}_y^{(B)}\right\rangle - \left\langle \hat{S}_x^{(A)}\hat{S}_z^{(B)}\right\rangle$$
$$+\frac{5}{2}\left\langle \hat{S}_y^{(A)}\hat{S}_x^{(B)}\right\rangle - 2\left\langle \hat{S}_y^{(A)}\hat{S}_y^{(B)}\right\rangle - \left\langle \hat{S}_y^{(A)}\hat{S}_z^{(B)}\right\rangle$$
$$-\frac{3}{2}\left\langle \hat{S}_z^{(A)}\hat{S}_x^{(B)}\right\rangle - 3\left\langle \hat{S}_z^{(A)}\hat{S}_z^{(B)}\right\rangle \geq \beta$$

$$-20.1897 \geq \begin{cases} -34 \text{ (without conservation)} \\ -20 \text{ (with conservation)} \end{cases}$$

Bell-like scenario :: for general s

The inequality:

$$c_{xx}\left\langle \hat{S}_x^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{xy}\left\langle \hat{S}_x^{(A)}\hat{S}_y^{(B)}\right\rangle + c_{xz}\left\langle \hat{S}_x^{(A)}\hat{S}_z^{(B)}\right\rangle + c_{yx}\left\langle \hat{S}_y^{(A)}\hat{S}_x^{(B)}\right\rangle$$
$$+ c_{yy}\left\langle \hat{S}_y^{(A)}\hat{S}_y^{(B)}\right\rangle + c_{yz}\left\langle \hat{S}_y^{(A)}\hat{S}_z^{(B)}\right\rangle + c_{zx}\left\langle \hat{S}_z^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{zy}\left\langle \hat{S}_z^{(A)}\hat{S}_y^{(B)}\right\rangle$$
$$+ c_{zz}\left\langle \hat{S}_z^{(A)}\hat{S}_z^{(B)}\right\rangle \geq \beta$$

can be rewritten as:

$$\langle \vec{S}^{(A)} \cdot C \cdot \vec{S}^{(B)} \rangle \geq \beta$$

where

$$\vec{S}^{(A)} = \left(\hat{S}_x^{(A)}, \hat{S}_y^{(A)}, \hat{S}_z^{(A)}\right)$$
$$\vec{S}^{(B)} = \left(\hat{S}_x^{(B)}, \hat{S}_y^{(B)}, \hat{S}_z^{(B)}\right)$$

$$C = \begin{pmatrix} c_{xx} & c_{xy} & c_{xz} \\ c_{yx} & c_{yy} & c_{yz} \\ c_{zx} & c_{zy} & c_{zz} \end{pmatrix}$$

Bell-like scenario :: the singlet state

The singlet state of two spin-*s* particles:

$$|\psi_0\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |m\rangle \otimes |-m\rangle$$

- is maximally entangled
- the corresponding total angular momentum is zero
- the state is invariant under rotations generated by

$$e^{i\vec{n}\cdot\mathbf{S}^{(\mathbf{A})}\theta} \otimes e^{i\vec{n}\cdot\mathbf{S}^{(\mathbf{B})}\theta}$$

Bell-like scenario :: quantum value

The singlet state of two spin-*s* particles:

$$|\psi_0\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |m\rangle \otimes |-m\rangle$$

Let us calculate the correlation:

$$\langle\psi_0|\hat{S}_z^{(A)}\hat{S}_z^{(B)}|\psi_0\rangle = -\frac{1}{2s+1}\sum_{m=-s}^{s} m^2 = -\frac{s(s+1)}{3}$$

The rotational symmetry implies:

$$\langle\psi_0|\hat{S}_x^{(A)}\hat{S}_x^{(B)}|\psi_0\rangle = \langle\psi_0|\hat{S}_y^{(A)}\hat{S}_y^{(B)}|\psi_0\rangle$$

$$= \langle\psi_0|\hat{S}_z^{(A)}\hat{S}_z^{(B)}|\psi_0\rangle = -\frac{s(s+1)}{3},$$

Therefore

identity matrix

$$\sum_{j=x,y,z} \langle\psi_0|\hat{S}_j^{(A)}\hat{S}_j^{(B)}|\psi_0\rangle = \langle \mathbf{S}^{(\mathbf{A})}\cdot\mathbf{C}_{\mathbf{id}}\cdot\mathbf{S}^{(\mathbf{B})}\rangle_{\psi_0} = -s(s+1)$$

Bell-like scenario :: quantum value

Next, consider an Euler rotation of Bob's spin generated by

$$U^{(B)} \;=\; e^{i\hat{S}_z^{(B)}\theta} e^{i\hat{S}_y^{(B)}\varphi} e^{i\hat{S}_z^{(B)}\xi}$$

If this rotation is applied to the singlet state one gets $|\phi\rangle = U^{(B)}|\psi_0\rangle$

As a result

$$\sum_{j=x,y,z} \langle\phi|\hat{S}_j^{(A)} U^{(B)} \hat{S}_j^{(B)} U^{(B)\dagger}|\phi\rangle = -s(s+1)\text{-}1)$$

However, the above can be rewritten as

$$\langle \mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})} \rangle_\phi = -s(s+1)$$

where ***C*** is an orthogonal matrix whose entries are given by

$$U^{(B)}\hat{S}_j^{(B)}U^{(B)\dagger} = c_{jx}\hat{S}_x^{(B)} + c_{jy}\hat{S}_y^{(B)} + c_{jz}\hat{S}_z^{(B)}$$

Bell-like scenario :: quantum value

$$\langle \mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})} \rangle_\phi = -s(s+1)$$

For a pair of arbitrary s-spin particles the operator $\mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})}$ has an eigenvalue $-s(s+1)$ with the corresponding eigenvector $|\phi\rangle$

This is the smallest eigenvalue of the operator $\mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})}$, because it has the same spectrum as $\mathbf{S}^{(\mathbf{A})} \cdot \mathbf{S}^{(\mathbf{B})}$, for which $-s(s+1)$ is the smallest eigenvalue.

Bell-like scenario :: quantum value

$$\langle(\mathbf{S}^{(\mathbf{A})}+\mathbf{S}^{(\mathbf{B})})^2\rangle \;\geq 0$$

$$\langle(\mathbf{S}^{(\mathbf{A})}+\mathbf{S}^{(\mathbf{B})})^2\rangle = \underbrace{\langle(\mathbf{S}^{(\mathbf{A})})^2\rangle}_{=s(s+1)}+\underbrace{\langle(\mathbf{S}^{(\mathbf{B})})^2\rangle}_{=s(s+1)}+2\langle\mathbf{S}^{(\mathbf{A})}\cdot\mathbf{S}^{(\mathbf{B})}\rangle \geq 0$$

$$2s(s+1)+2\langle S^{(A)}\cdot S^{(B)}\rangle \geq 0$$

$$\langle\mathbf{S}^{(\mathbf{A})}\cdot\mathbf{S}^{(\mathbf{B})}\rangle \geq -s(s+1)$$

We know that if C is a rotation matrix, then in quantum theory the expression $\langle\mathbf{S}^{(\mathbf{A})}\cdot\mathbf{C}\cdot\mathbf{S}^{(\mathbf{B})}\rangle$ has a lower bound $-s(s+1)$.

In particular, there exists a state $|\phi\rangle$ which corresponds to the eigenvalue $-s(s+1)$

Bell-like scenario :: violation

an orthogonal matrix

the lowest quantum value

$$\langle S^{(A)} \cdot C \cdot S^{(B)} \rangle = -s(s-1) \geq \beta$$

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

including conservation

without conservation

s	β and $\bar{\beta}$	-s(s+1)
1	$-1-\frac{1}{\sqrt{2}} \approx -1.707$	-2
	$-1-\sqrt{2} \approx -2.414$	
2	$-1+\frac{1}{\sqrt{2}}-4\sqrt{2} \approx -5.949$	-6
	$-4-4\sqrt{2} \approx -9.657$	
3	$-4(1+\sqrt{2}) \approx -9.657$	-12
	$-9-9\sqrt{2} \approx -21.730$	
4	$-14\sqrt{2} \approx -19.799$	-20
	$-16-16\sqrt{2} \approx -38.627$	

Conclusions

In any HV-theory pertaining to measurements on a spin one can find situations in which

either HV-assignments do not represent a physical reality of a spin vector, but rather provide a deterministic algorithm for prediction of the measurement outcomes,

or HV-assignments represent a physical reality, but the spin cannot be considered as a vector of fixed length.

https://arxiv.org/abs/1806.06637