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Self-testing of two-qutrit quantum systems

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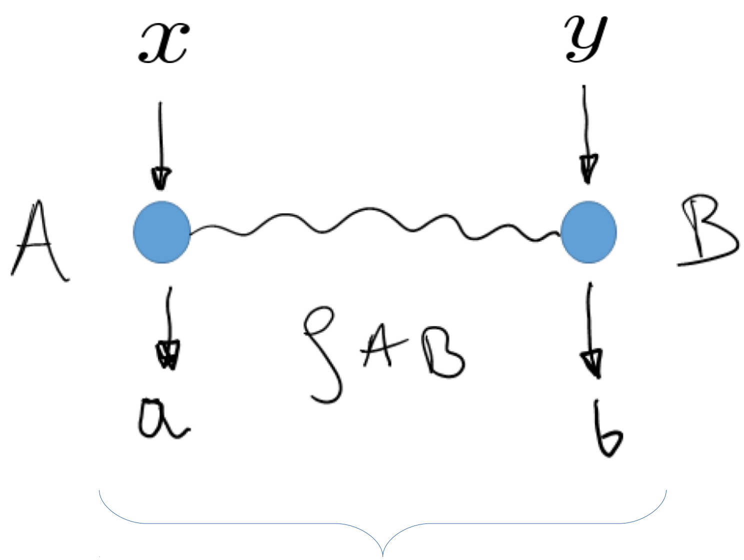
Preliminaries

- **Bell scenario:** two parties performing measurements on their local systems

(2,m,d) scenario



$$p(a,b|x,y)$$

measurement choices

$$x,y = 1,\ldots,m$$

outcomes

$$a,b = 0,\ldots,d-1$$

- Correlations are described by a collection of probability distributions

$$\{p(a,b|x,y)\}$$

$$p(a,b|x,y) = \mathrm{Tr}[\rho_{AB}(M_x^a \otimes N_y^b)]$$

$\{p(a,b|x,y)\}$

correlations

Preliminaries

Non- locality and Bell inequalities

- **Local/classical correlations**

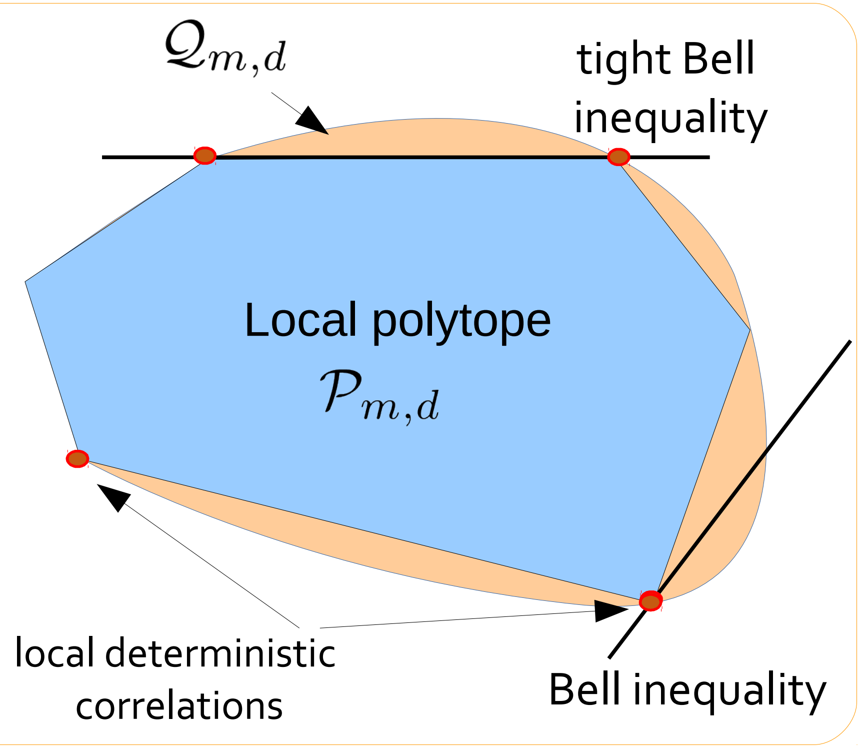
$$p(a,b|x,y) = \sum_\lambda p(\lambda) p_{\text{det}}(a|x,\lambda) p_{\text{det}}(b|y,\lambda)$$

deterministic

- Otherwise they are called **nonlocal (nonlocality)**

$$\mathcal{P}_{m,d} \subsetneq \mathcal{Q}_{m,d}$$

[J. S. Bell, Physics **1**, 195 (1964)]



- **Bell inequalities**: Hyperplanes constraining the local set

$$I := \sum_{a,b,x,y} T^{a,b}_{x,y} p(a,b|x,y) \leq \beta_C$$

$$\beta_C = \max_{\mathcal{P}_{m,d}} I$$

$$\beta_Q = \max_{\mathcal{Q}_{m,d}} I$$

- **Example**: the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$$

[Clauser, Horne, Shimony, Holt (1969)]

$$|\psi_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$

$$A_{0/1} = X/Z \qquad B_{0/1} = \frac{1}{\sqrt{2}}(X \pm Z)$$

Non-locality

- Non-locality is a resource for device-independent applications

- Quantum key distribution
  [Ekert, PRL (1991); A. Acín *et al.*, PRL (2007)]

- Randomness certification/amplification
  [Pironio *et al.*, Nature (2010); Colbeck, Renner, Nat. Phys. (2012)]

- Device-independent entanglement certification
  [J.-D. Bancal *et al.*, PRL (2011)]

- Self-testing
  [Mayers, Yao, QIC (2004)]

Self-testing

[Mayers, Yao, 2004]

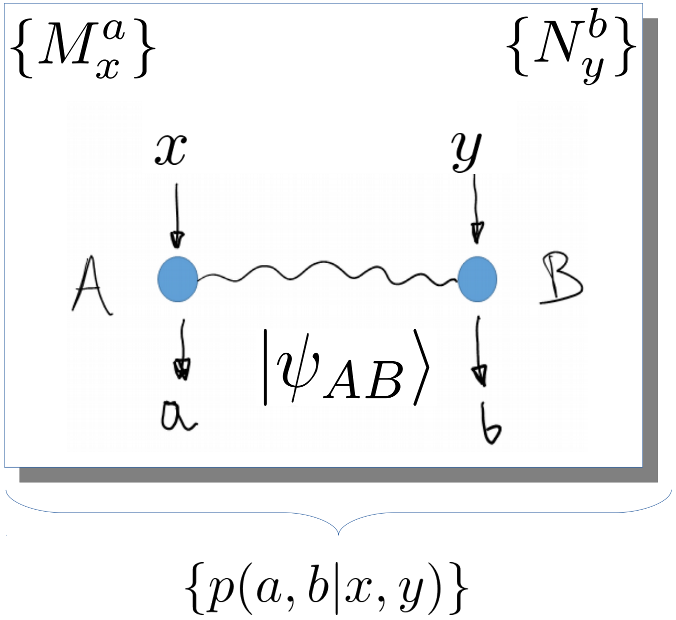
The idea of device-independent certification

- Given $\{p(a,b|x,y)\}$
- or violation of some Bell inequality

$$\sum_{a,b,x,y} T_{x,y}^{a,b}\, p(a,b|x,y) = \beta$$

- **deduce properties of** $|\psi_{AB}\rangle$ **and** $\{M_x^a\},\ \{N_y^b\}$

Quantum device



Self-testing

- **decide** whether there exist
  - local isometry $\Phi = \Phi_A \otimes \Phi_B$
  - a state $|\widetilde{\psi}_{AB}\rangle$ and measurements $\{\widetilde{M}_x^a\},\ \{\widetilde{N}_y^b\}$

$$\Phi([M_x^a \otimes N_y^b]|\psi_{AB}\rangle) = |\mathrm{aux}\rangle \otimes (\widetilde{M}_x^a \otimes \widetilde{N}_y^b|\widetilde{\psi}_{AB}\rangle)$$

$|\widetilde{\psi}_{AB}\rangle$ and $\{\widetilde{M}_x^a\},\ \{\widetilde{N}_y^b\}$ are self-tested

Self-testing

- **Seems like a hopeless task!**

maybe not …

often one can deduce **everything**!

- All two-qubit pure entangled states

  Tsirelson 93, Meyers, Yao, 2004;
  M. McKague *et al.*, 2012; J. Kaniewski, 2017

- Maximally entangled states of two qudits

  Yang, Navascués, 2014

- All bipartite pure entangled states

  A. Coladangelo *et al.*, 2017

Self-testing from projections

+

the CHSH Bell inequality

- **Questions:**
  - Self-testing with genuinely *d*-outcome Bell inequalities (e.g. maxent state)
  - self-testing of higher-dimensional measurements (MUBs)

Results

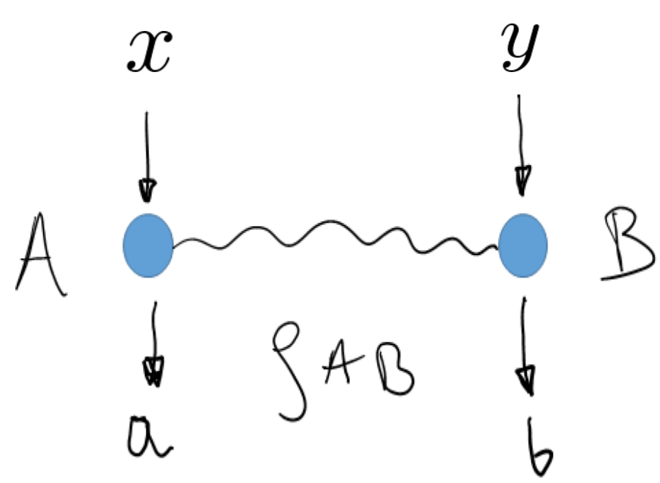
[arXiv:1807.03332]

▶ **Result I**: Bell inequalities maximally violated by the maxent state and MUBs

▶ The original CHSH inequality [(2,2,2) scenario]

$$I_2 := \sum_{x,y=0}^{1} (-1)^{xy} \langle A_x B_y \rangle \leq 2$$

nonlocal game $\Leftrightarrow$ $\begin{cases} x + y + a \cdot b = 0 \pmod 2 \\ a, b, x, y \in \{0, 1\} \end{cases}$



▶ A generalisation to *d* outcomes – CHSH-*d* inequality [(2,*d*,*d*) scenario]

$$I_d := \sum_{n=1}^{d-1} \sum_{x,y=0}^{d-1} \omega^{nxy} \langle A_x^n B_y^n \rangle \leq \beta_C \quad \Leftrightarrow \quad \begin{cases} x + y + a \cdot b = 0 \pmod d \\ a, b, x, y \in \{0, 1, \ldots, d-1\} \end{cases}$$

A_x, B_y – unitary observables with eigenvalues $1, \omega, \ldots, \omega^{d-1}$ $[\omega = \exp\left(\frac{2\pi \mathrm{i}}{d}\right)]$

[Buhrman, Massar, (2005); Ji et al., (2008); Bavarian, Shor (2013), Liang et al. (2009)]

Results

[arXiv:1807.03332]

- Modifying the CHSH-*d* inequality [prime *d*]

$$\tilde{I}_d := \sum_{n=1}^{d-1} \underbrace{\lambda_{n,d}} \sum_{x,y=0}^{d-1} \omega^{nxy} \langle A_x^n B_y^n \rangle \leq \beta_C$$

phases chosen so that $\frac{\lambda_{n,d}}{\sqrt{d}} A_i^n \otimes \bar{B}_i^{(n)} |\psi_d\rangle = |\psi_d\rangle$

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i,i\rangle$$

$$\bar{B}_i^{(n)} = \sum_j \omega^{nij} B_j^n$$

- Easier to characterise: direct computation of the max. quantum value

$$\beta_Q = \max_{\mathcal{Q}} \tilde{I}_d = d\sqrt{d}\,(d-1)$$

- sum of squares decomposition

$$\beta_Q \mathbb{1} - \mathcal{B} \sim \sum_{n,i} L_i^{(n)\dagger} L_i^{(n)}$$

$$L_i^{(n)} = \mathbb{1} \otimes \mathbb{1} - A_i^n \otimes \bar{B}_i^{(n)}$$

- quantum realisation achieving β_Q

$$|\psi_d\rangle = \sum_{i=0}^{d-1} |i,i\rangle/\sqrt{d}$$ maxent state

$$B_k = \omega^{k(k+1)} X Z^k$$

$$A_k = \ldots$$

mutually unbiased bases

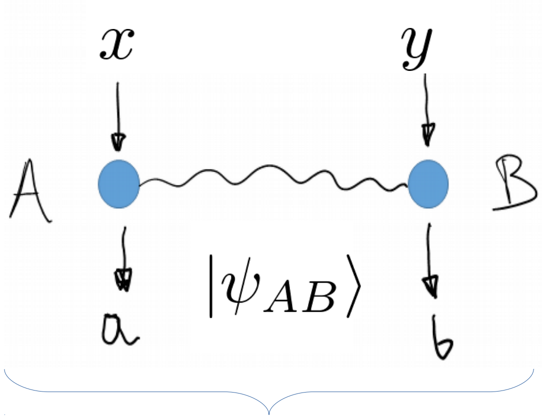
[Ji et al. (2008)]

Results

[arXiv:1807.03332]

- **Result II**: For *d=3* our inequality self-tests maxent state and MUBs!

$\{A_0, A_1, A_2\}$ $\{B_0, B_1, B_2\}$



$$\tilde{I}_3 = \max_{Q}$$

- Let $|\psi\rangle$ and $A_x,\ B_y$ violate maximally our inequality

sum of squares ↓

$$B_0^\dagger = -\omega\{B_1, B_2\}$$ (+ permutations)

some algebra ↓

$$B_0 = U_B(X \otimes \mathbb{1})U_B^\dagger \qquad B_1 = U_B(X^2 Z \otimes \mathbb{1})U_B^\dagger \qquad B_2 = U_B(Z^2 \otimes \mathbb{1})U_B^\dagger$$

(+ their transpositions)

- and analogously for Alice

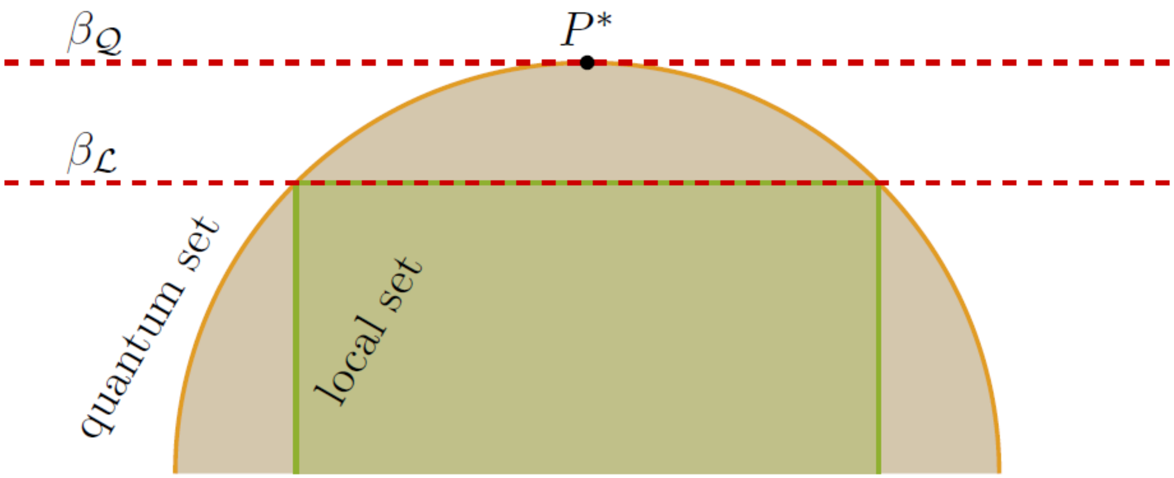
$$U_A \otimes U_B|\psi_{AB}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \otimes |\mathrm{ancilla}\rangle$$

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Results

[arXiv:1807.03332]

- Corollary 1: Our Bell expression has a unique maximiser



- Corollary 2: Maximal violation of our inequality certifies $\log 3$ of local randomness

- For *d>3* the problem gets more complicated

$$C_j^{(n)} = \left[C_j^{(1)}\right]^n \qquad (n = 2, 3, \ldots, d-1)$$

$$C_j^{(n)} \sim \sum_k \omega^{njk} B_k^n$$

$$\omega^q B_0 B_1 = B_1 B_0 \qquad (q = 0, \ldots, d-1)$$

not equivalent under unitary operations and transpose

Conclusion/Open questions

J. Kaniewski, I. Šupić, J. Tura, F. Baccari, A. Salavrakos, R. A., arXiv:1807.03332

- A new family of Bell inequalities maximally violated by the maxent state and MUB measurements for any prime *d*

A. Salavrakos, R. A., J. Tura, P. Wittek, A. Acin, S. Pironio, arXiv:1607.04578

- For *d=3* our inequality is a **self-test**
  - $\log_2 3$ bits of randomness
  - our proof does not rely on self-testing of qubit subspaces
- Not a self-test in the standard sense for higher *d*

Further research

(in progress)

- Extend the self-testing statement to any prime *d*
- Investigate **robustness** of our self-testing statement
- Generalizations of our approach to **more complex scenarios**:
  - partially entangled states
  - multipartite scenario (e.g. graph states)