Device-independent witnesses of entanglement depth from two-body correlators

Institut de Ciències Fotòniques

Albert Aloy

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Goal:

- Certify the amount of entanglement present on a quantum many-body system
- Computational and experimentally tractable, also for large systems

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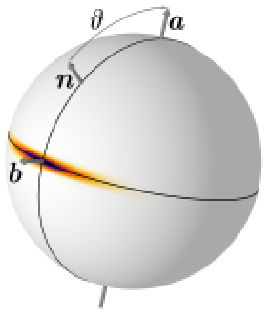
2-body Permutation Invariant Bell Inequalities



[J. Tura *et al. Science* **344** 1256 (2014)]
[R. Schmied *et al*. *Science* **352** 6284 (2016)]]

Outline

❶ Many particle entanglement
- Entanglement depth and *k*-producibility

❷ Detecting entanglement and nonlocality
- Device independent approach
- 2-body permutation invariant Bell inequalities

❸ DI Witnesses of Entanglement Depth
- Optimization problem: find *k*-prod bounds
- Class of states approximation: asymptotic behaviour
- Comparison to other criteria and experimental data

Many particle entanglement

Bi- and multipartite entanglement

- A bipartite state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff admits the convex decomposition $\rho = \sum_i p_i \, \rho_i^A \otimes \rho_i^B, \qquad \sum_i p_i = 1, \quad p_i \geq 0$
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- A multipartite state ρ acting on $\mathcal{H}_A \otimes \ldots \otimes \mathcal{H}_K$ is *K*-separable iff

$$\rho = \sum_{\mathcal{P} \in \mathrm{Partitions}^K} \lambda_{\mathcal{P}} \sum_i \mu_{i,\mathcal{P}} \bigotimes_{\mathcal{A} \in \mathcal{P}} \rho_i^{\mathcal{A}}$$

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Partially separable states:

More parties → more types of entanglement → more complex

Entanglement depth and *k*-producibility

- A pure state is *k*-producible if it can be written as $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes \ldots$ where $|\psi_i\rangle$ are states of at most *k* parties

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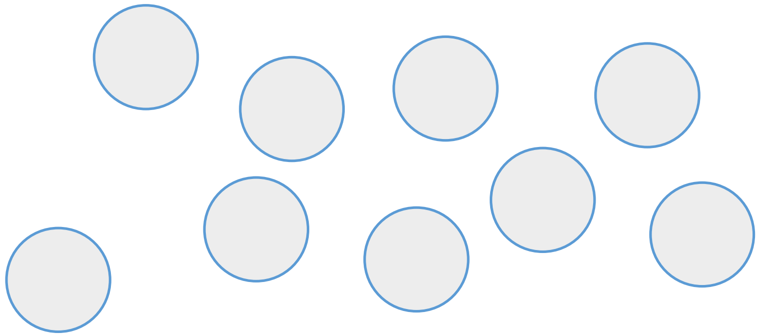
e.g. $n = 9$

fully separable
1-producible



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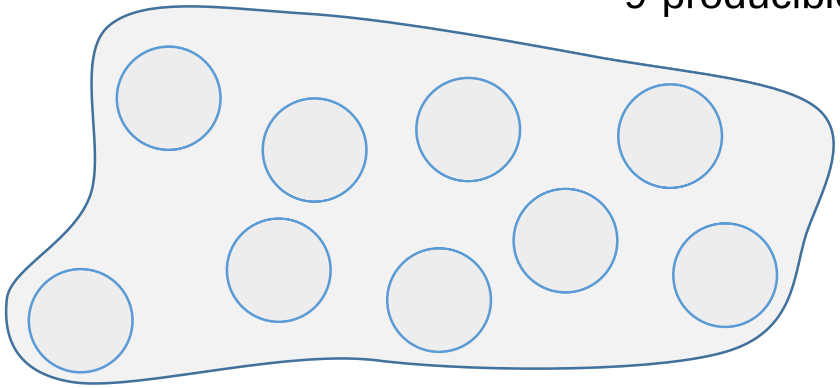
GME
9-producible



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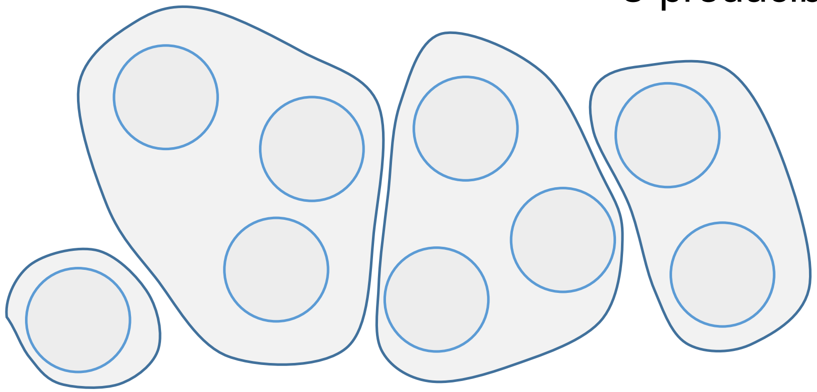
3-producible



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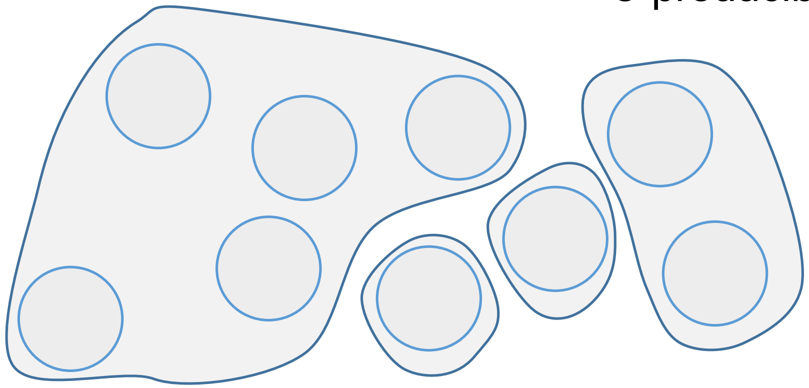

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1 2 3

Detecting entanglement and nonlocality

Characterizing entanglement

Common challenges/issues:

- Require faithful characterization of measurements, states and relevant degrees of freedom

- Exponential growth of the Hilbert space with n

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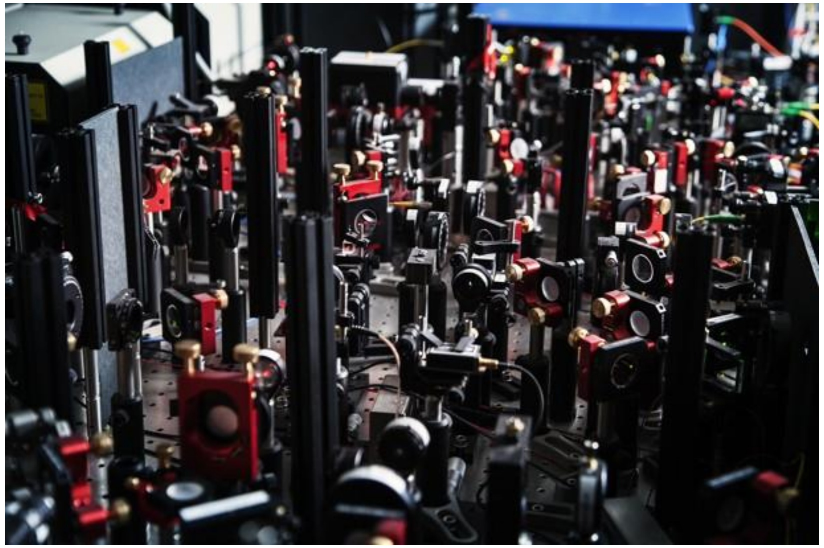
Counter: 2-body permutation invariant Bell inequalities

Device independent approach

- Treat experiment setup as a blackbox:
    - #inputs, #outputs
    - inhere physical properties from statistical correlations



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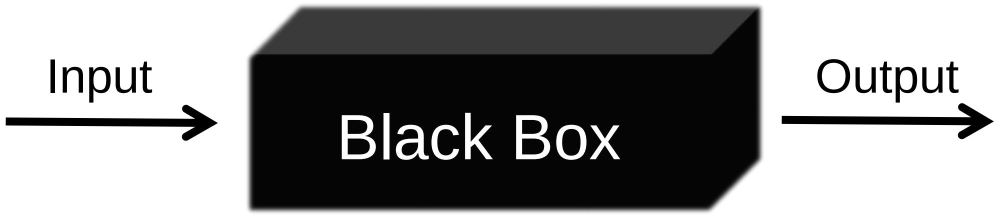

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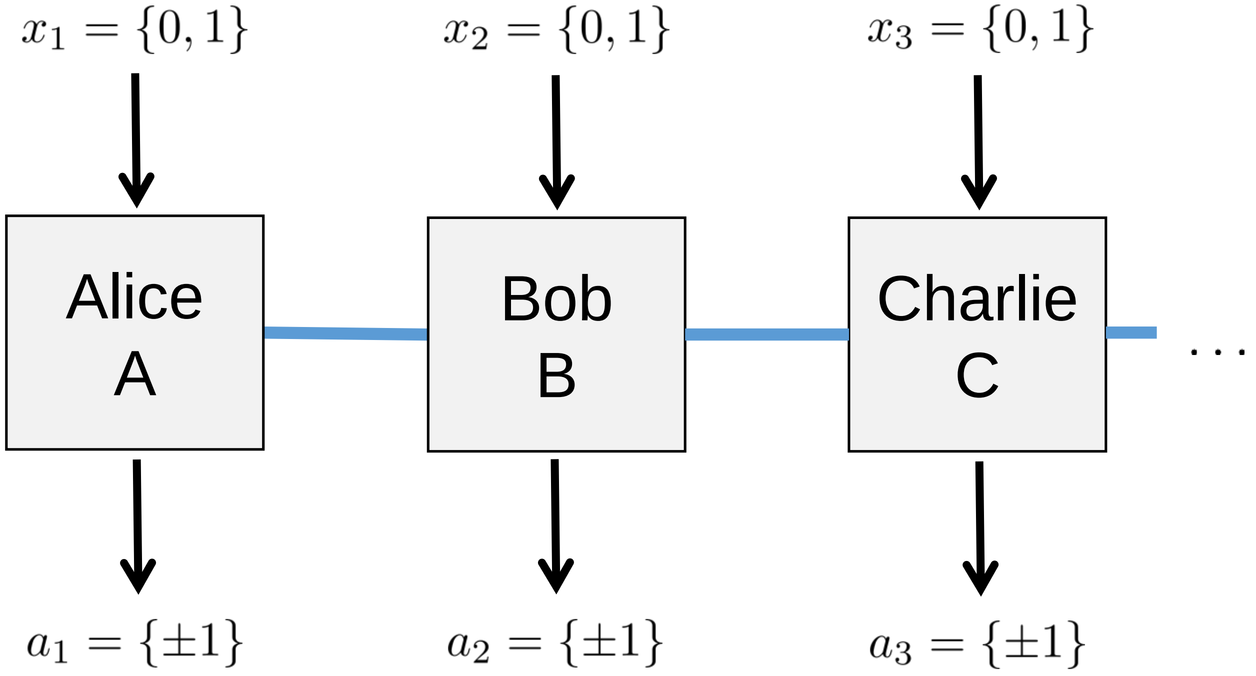
Bell inequalities from 2-body correlators

Bell-type experiment

Simplest multipartite scenario (n,2,2):



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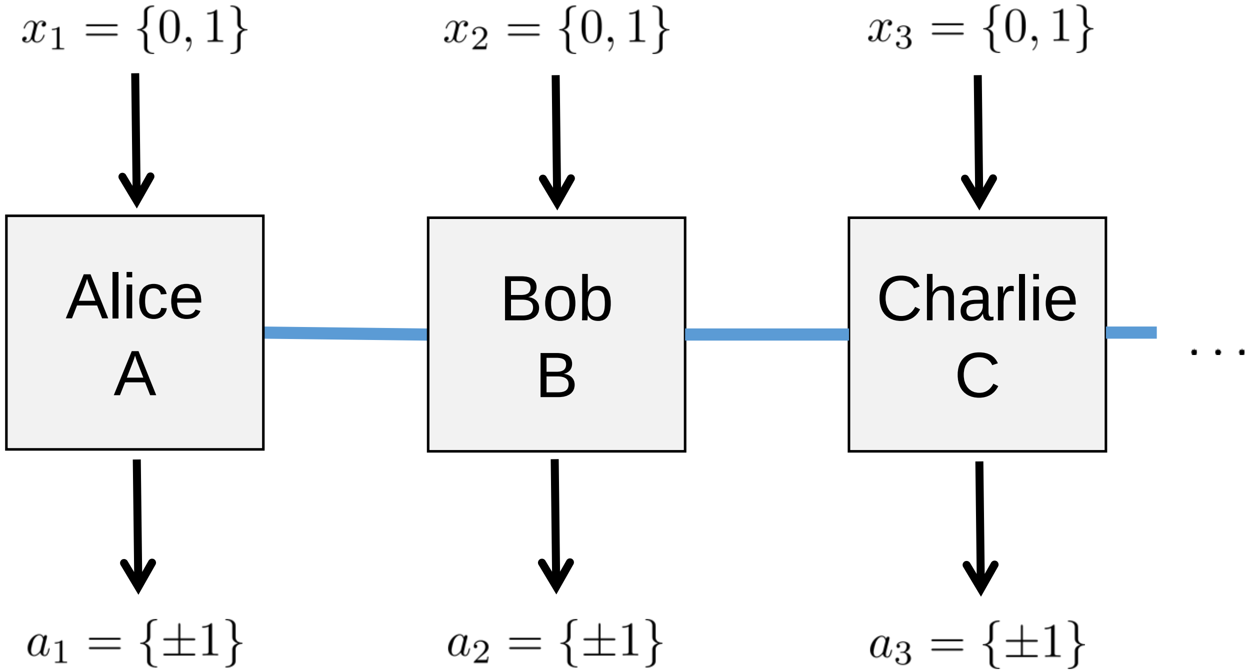
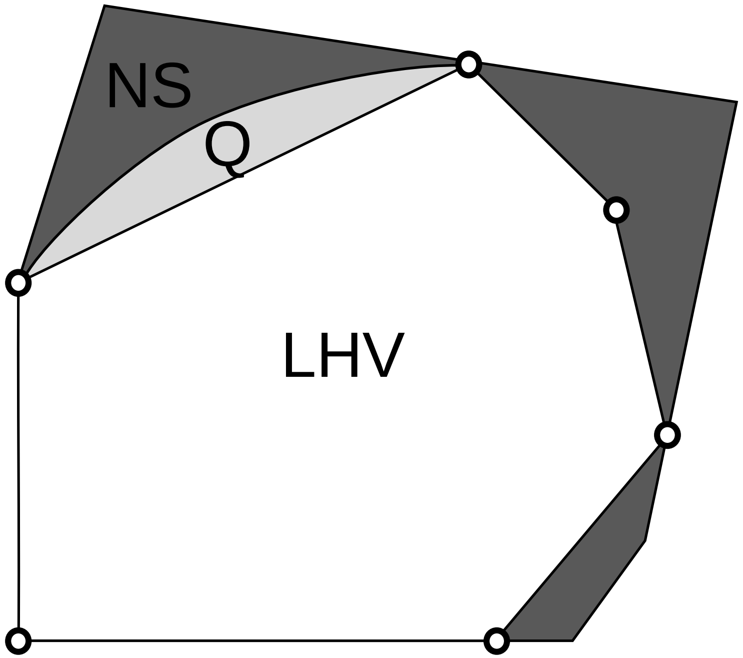


$$p(a_1, a_2, \ldots, a_n | x_1, x_2, \ldots, x_n)$$

$$\{p(\vec{a}|\vec{x}), \forall \vec{a}, \vec{x}\}$$



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$\mathbb{P}_L$ $\mathcal{Q}$ $\mathbb{P}_{NS}$

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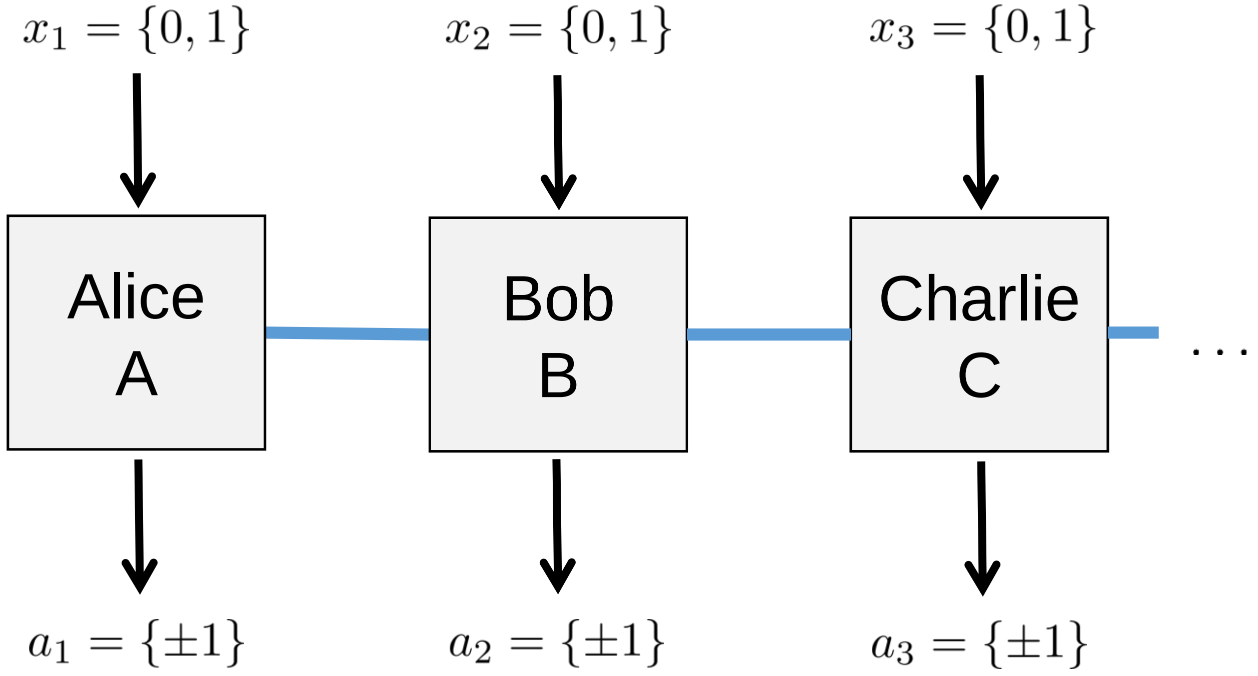

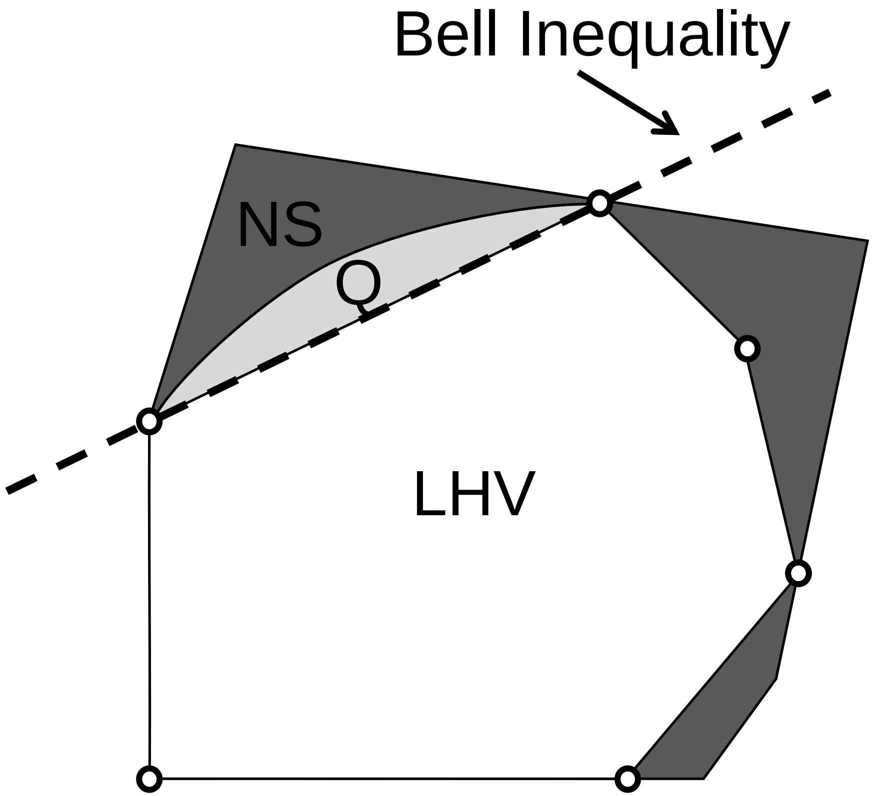

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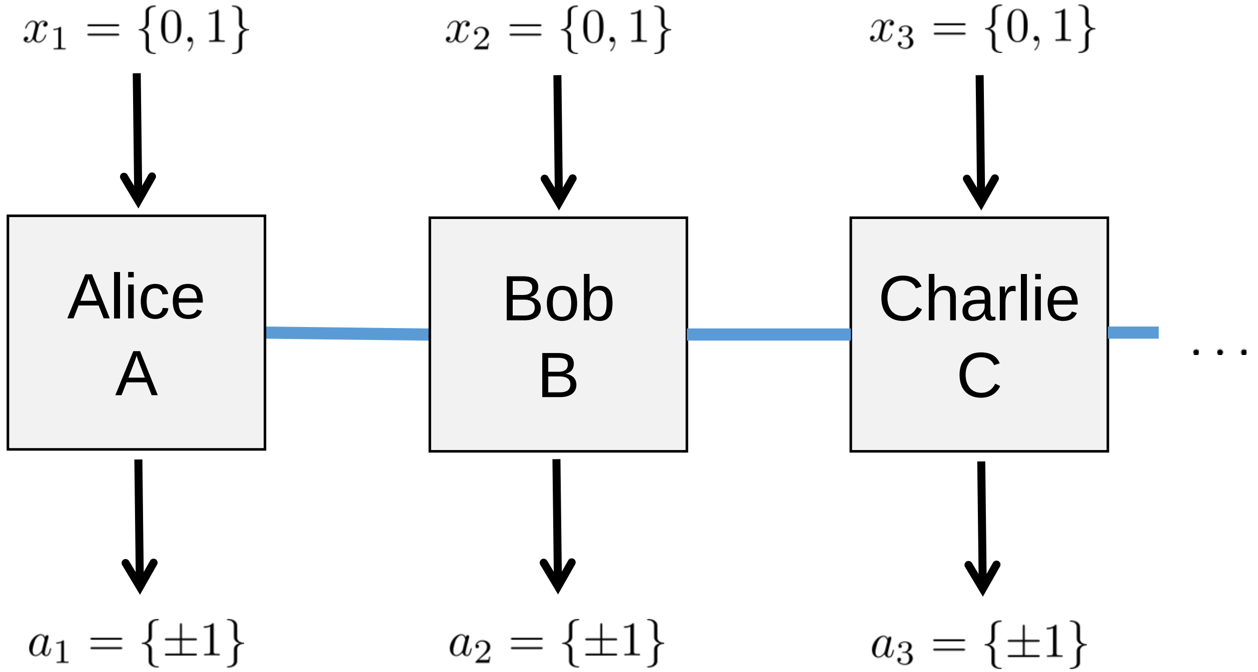
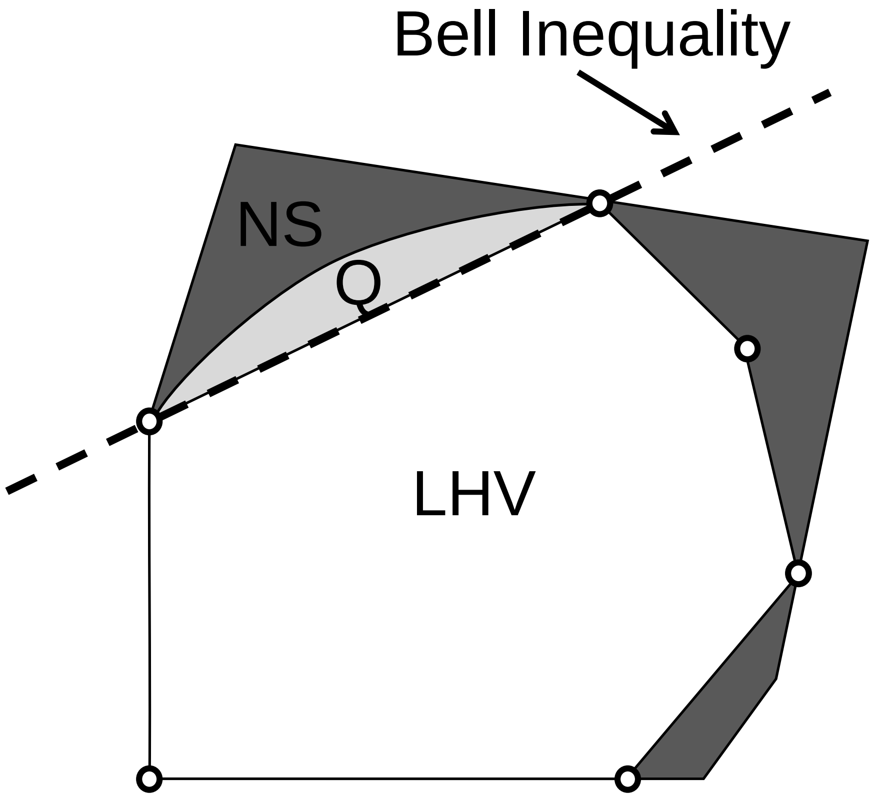


$$\langle \mathcal{M}^A_{x_1} \mathcal{M}^B_{x_2} \rangle = \sum_{a_1 a_2} a_1 a_2 \, p(a_1 a_2 | x_1 x_2)$$

$$\left\{ \langle \mathcal{M}^{i_1}_{x_{i_1}} \cdots \mathcal{M}^{i_r}_{x_{i_r}} \rangle \right\}$$



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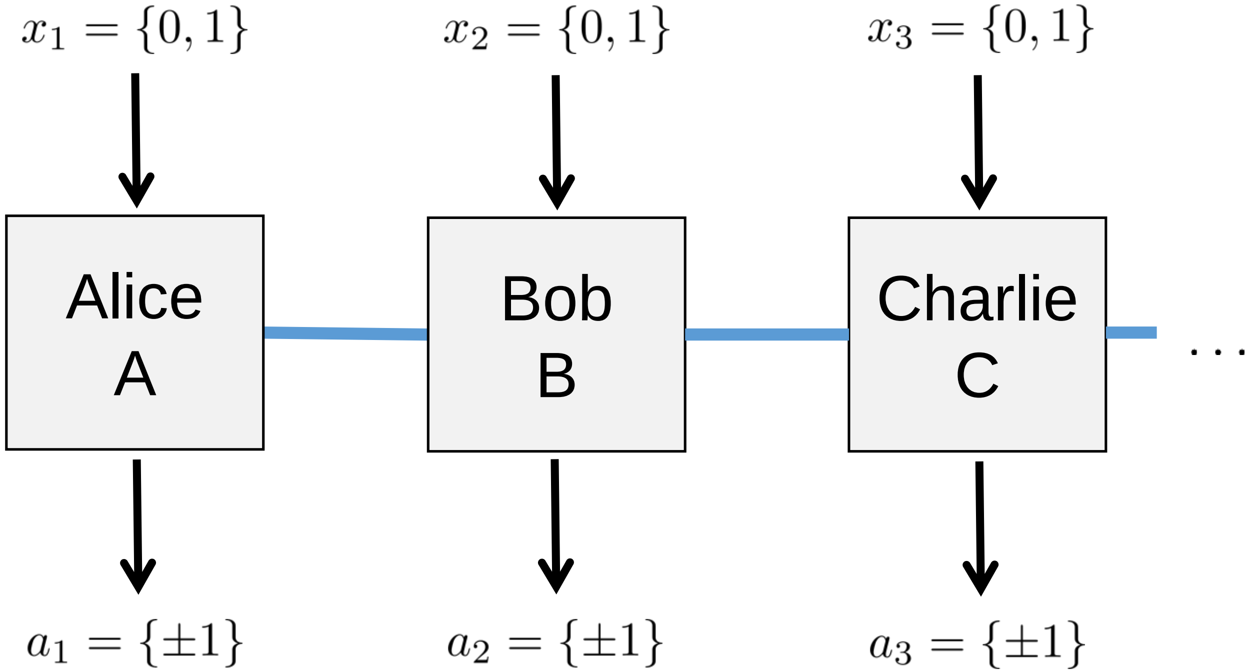

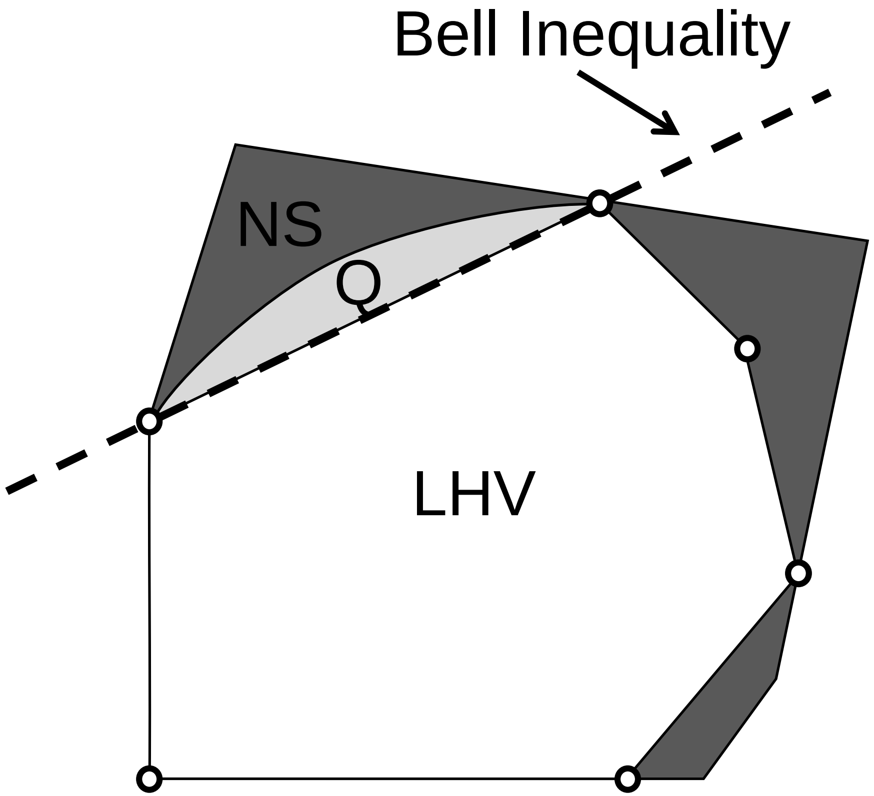

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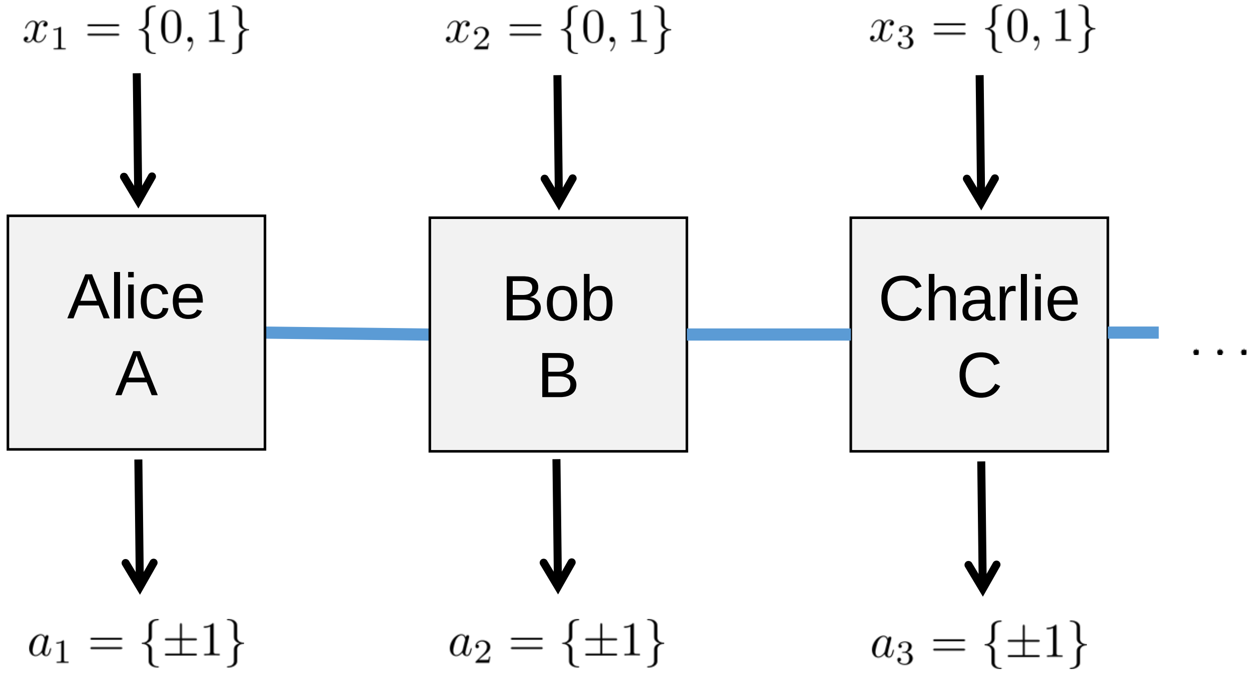

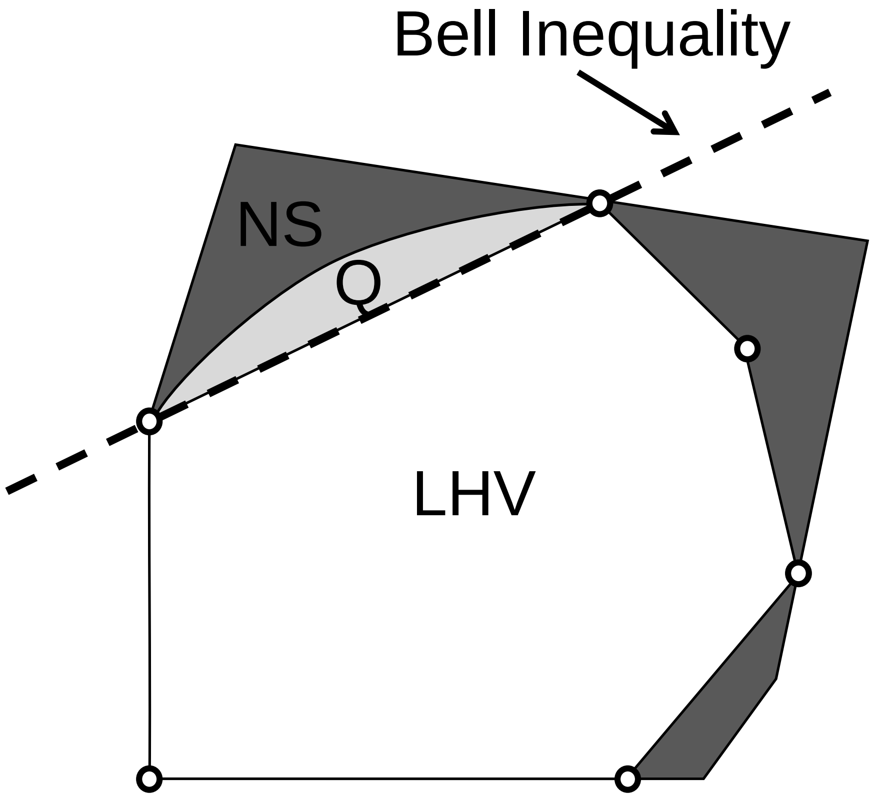

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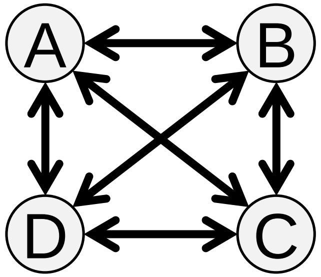
2-body Permutation Invariant Bell Inequality

Permutationally invariant



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[J. Tura *et al.* *Science* **344** 1256 (2014)]

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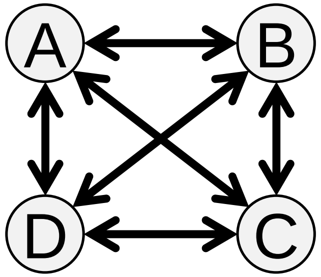

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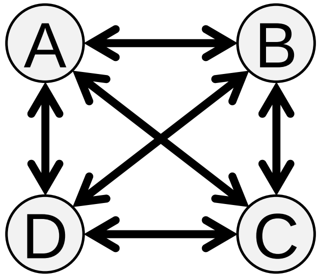

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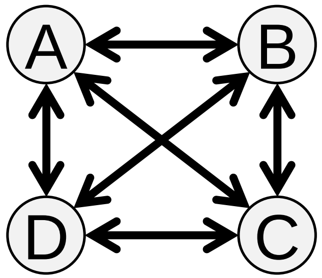

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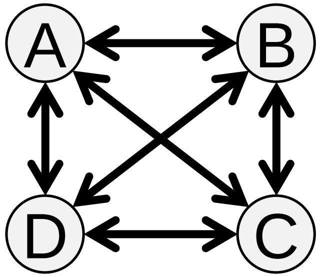

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- Consider a partition $\mathcal{P}$ of $[n] := \{1, 2, \ldots, n\}$

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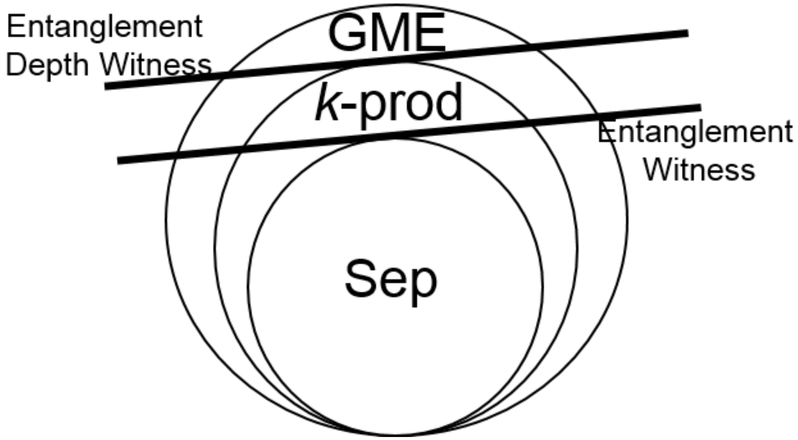
- Optimize Bell operator over states and measurements

$$\boxed{\beta_{\text{k-prod}} = \min_{|\Psi\rangle, \boldsymbol{\varphi}, \boldsymbol{\theta}} \langle\Psi|\hat{\mathcal{B}}|\Psi\rangle}$$

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_{|\mathcal{P}|}\rangle$$



Finding the bounds

Finding the bounds: Key points

- In the (*n*,2,2) scenario, maximal quantum violation can be achieved with:

⇨ Pure qubit states

⇨ Real observables

$$\mathcal{M}_0 = \cos(\varphi)\sigma_z + \sin(\varphi)\sigma_x$$
$$\mathcal{M}_1 = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x$$

[B. Toner & F. Verstraete, *arXiv quant-ph/0611001* (2006)]

Finding the bounds: Key points

$$\hat{\mathcal{B}}(\boldsymbol{\varphi},\boldsymbol{\theta}) := \alpha\hat{\mathcal{S}}_0 + \beta\hat{\mathcal{S}}_1 + \gamma\hat{\mathcal{S}}_{00} + \delta\hat{\mathcal{S}}_{01} + \epsilon\hat{\mathcal{S}}_{11}$$

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$$B_k^{\mathcal{A}} := \sum_{i\in\mathcal{A}} \mathcal{M}_k^{(i)}$$

$$B_{kl}^{\mathcal{A}} := \sum_{i\in\mathcal{A}} \sum_{j\in\mathcal{A}\setminus\{i\}} \mathcal{M}_k^{(i)}\mathcal{M}_l^{(j)}$$

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Same region
1-body terms

Same region
2-body terms

Crossed regions
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Same region 1-body terms

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Crossed regions 2-body terms

- Expectation value involves at most 2-body terms

$$\begin{aligned}\langle\Psi|\,\hat{\mathcal{B}}\,|\Psi\rangle \;&=\; \sum_{\mathcal{A}\in\mathcal{P}}\sum_{k}\alpha_k\langle\psi_{\mathcal{A}}|\,B_k^{\mathcal{A}}\,|\psi_{\mathcal{A}}\rangle + \sum_{\mathcal{A}\in\mathcal{P}}\sum_{k\leq l}\alpha_{kl}\langle\psi_{\mathcal{A}}|\,B_{kl}^{\mathcal{A}}\,|\psi_{\mathcal{A}}\rangle \\ &+ \sum_{\mathcal{A}\neq\mathcal{A}'\in\mathcal{P}}\sum_{k\leq l}\alpha_{kl}\langle\psi_{\mathcal{A}}|\,B_k^{\mathcal{A}}\,|\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}'}|\,B_k^{\mathcal{A}'}\,|\psi_{\mathcal{A}'}\rangle\end{aligned}$$

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$$
\begin{aligned}
\mathrm{Tr}[\mathcal{B}(\boldsymbol{\theta})\rho] \;&=\; \sum_{\mathcal{A}\in\mathcal{P}} \left(\sum_{k} \alpha_k \mathrm{Tr}[\mathcal{B}^{\mathcal{A}}_{k}\rho_{\mathcal{A}}] + \sum_{k\leq l} \alpha_{kl} \mathrm{Tr}[\mathcal{B}^{\mathcal{A}}_{kl}\rho_{\mathcal{A}}] \right) \\
&+ \sum_{\mathcal{A}\neq\mathcal{A}'\in\mathcal{P}} \sum_{k\leq l} \alpha_{kl} \mathrm{Tr}[\mathcal{B}^{\mathcal{A}}_{k} \otimes \mathcal{B}^{\mathcal{A}'}_{l} \rho_{\mathcal{A}\cup\mathcal{A}'}]
\end{aligned}
$$

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$$\beta^L_{\text{k-prod}} = \min \mathrm{Tr}[\mathcal{B}(\boldsymbol{\theta})\rho] \quad \text{s.t.} \quad \begin{array}{ll} \rho_{\mathcal{A}} \succeq 0,\ \rho_{\mathcal{A}\cup\mathcal{A}'} \succeq 0, & \mathrm{Tr}_{\mathcal{A}'}[\rho_{\mathcal{A}\cup\mathcal{A}'}] = \rho_{\mathcal{A}}, \\ \mathrm{Tr}[\rho_{\mathcal{A}}] = \mathrm{Tr}[\rho_{\mathcal{A}\cup\mathcal{A}'}] = 1, & \rho^{T_{\mathcal{A}}}_{\mathcal{A}\cup\mathcal{A}'} \succeq 0 \end{array}$$

Note: $\rho_{\mathcal{A}\cup\mathcal{A}'} = |\psi_{\mathcal{A}}\rangle\langle\psi_{\mathcal{A}}| \otimes |\psi_{\mathcal{A}'}\rangle\langle\psi_{\mathcal{A}'}|$ satisfies all previous constraints

Hence, $\min_{|\Psi\rangle} \langle\Psi|\,\mathcal{B}\,|\Psi\rangle \geq \min_{\sigma_{pq}} \beta^{\mathrm{L}}_{\text{k-prod}}$

Finding the bounds: Numerical insights

Optimal $\beta_{\text{k-prod}}$ is achieved when:

All "Alices" set the same measurement input;
All "Bobs" set the same measurement input;
…

$\Longrightarrow$ The whole system is not PI, but the regions become PI

Schur-Weyl duality $\Longrightarrow$ Block-diagonal decomposition

[T. Moroder *et al. New J. Phys* **14** (2012)]

[J. Tura *et al. Annals of Physics* **352** 370 (2015)]

Project each region from a $2^{|\mathcal{A}|}\text{-dim subspace}$ to an $(|\mathcal{A}|+1)\text{-dim subspace}$

Case of study

$$\alpha\mathcal{S}_0 + \beta\mathcal{S}_1 + \gamma\mathcal{S}_{00} + \delta\mathcal{S}_{01} + \epsilon\mathcal{S}_{11} - \beta_c \geq 0$$

$$-2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} + 2n \geq 0$$

[J. Tura *et al. Science* **344** 1256 (2014)]

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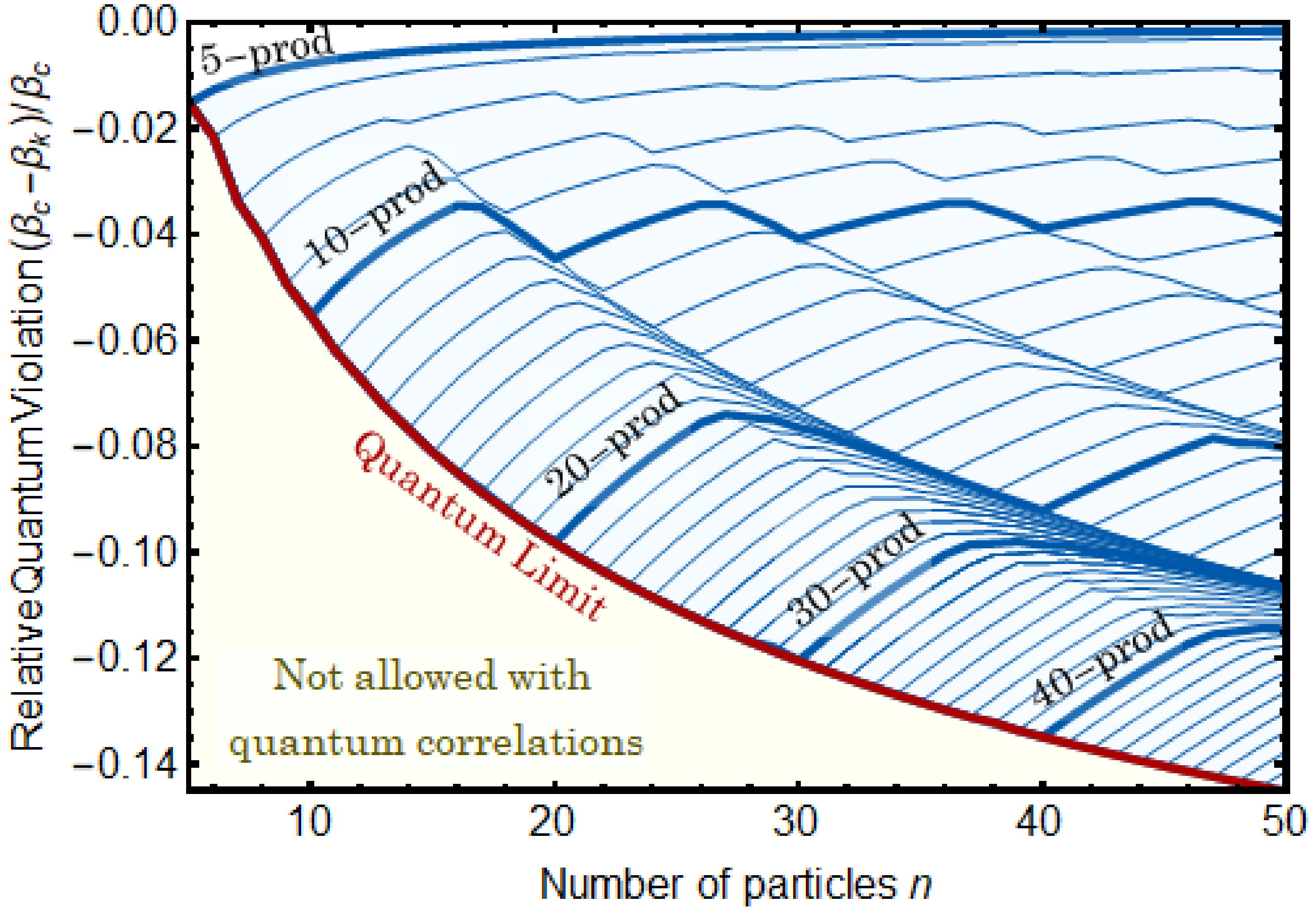
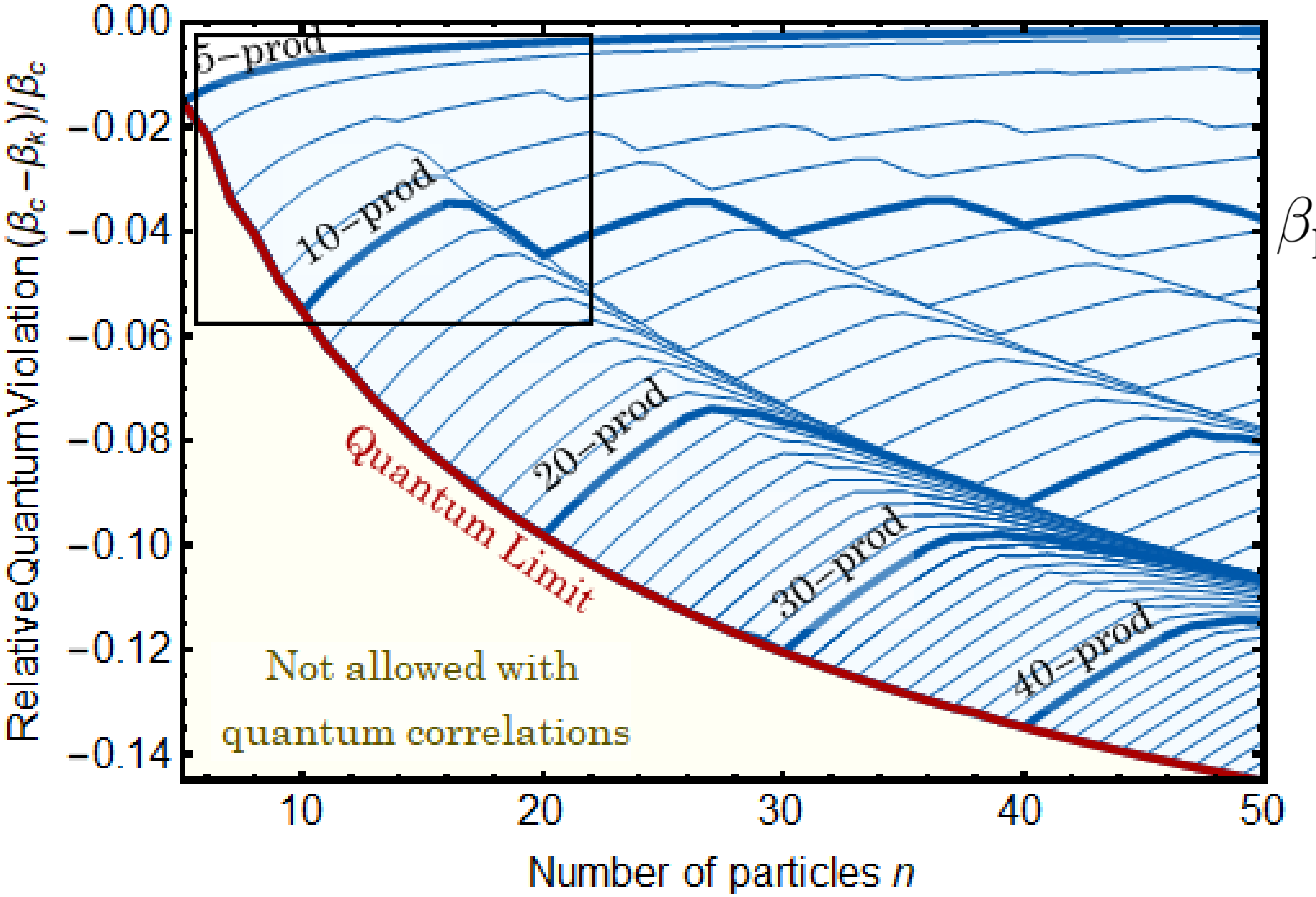
$$\boxed{-2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} - \beta_{\text{k-prod}} \geq 0}$$

2-body DI Witnesses of Entanglement Depth



2-body DI Witnesses of Entanglement Depth



$$\beta_{\mathrm{k}}^{U} - \beta_{\mathrm{k}}^{L} \approx 10^{-7}$$

Class of states approximation

Analytical approximation

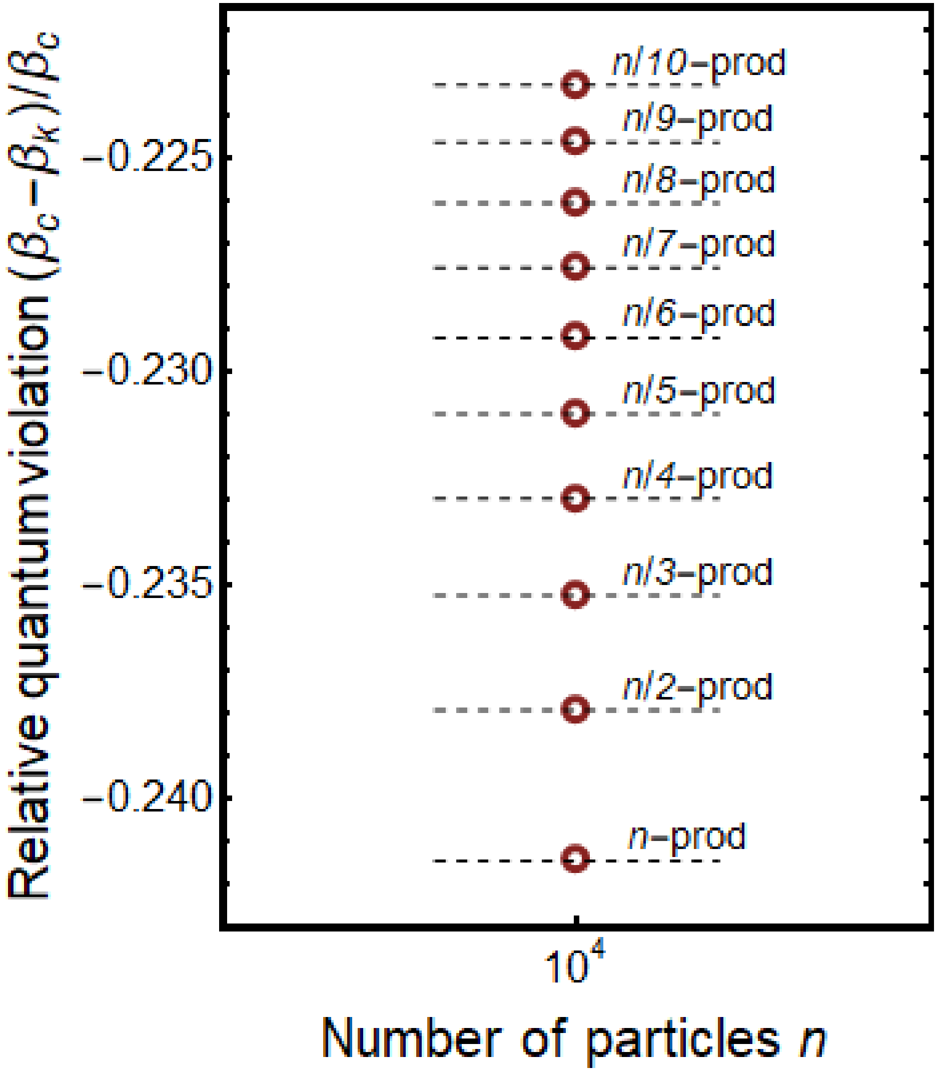
- Class of states: Gaussian superposition of Dicke states for each region

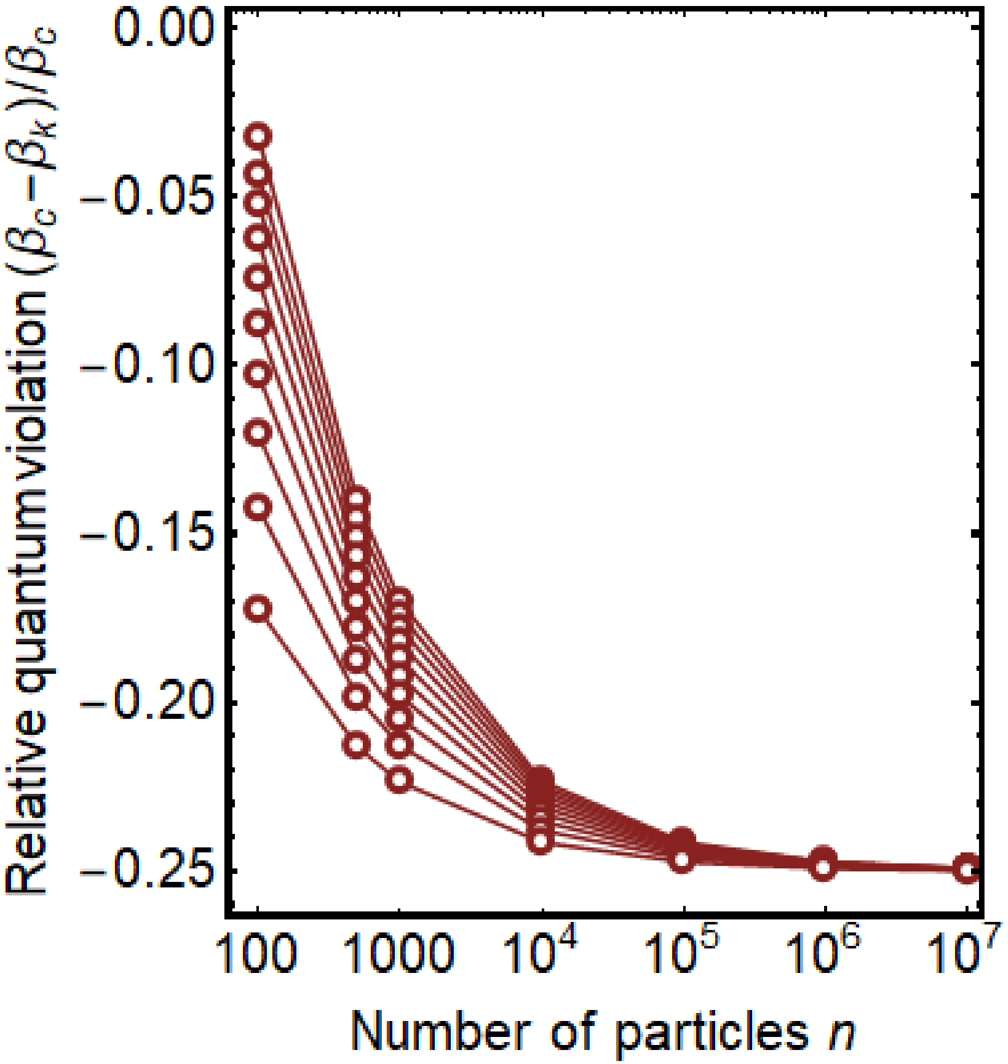
$$|\Psi\rangle = \bigotimes_{\mathcal{A}\in\mathcal{P}} \Big(\sum_{0\leq k_{\mathcal{A}}\leq|\mathcal{A}|} \psi^{\mathcal{A}}_{k_{\mathcal{A}}} \, |D^{k_{\mathcal{A}}}_{|\mathcal{A}|}\rangle \Big) \quad \text{where, } \psi^{\mathcal{A}}_{k_{\mathcal{A}}} := \frac{e^{-(k_{\mathcal{A}}-\mu_{\mathcal{A}})^2/4\sigma_{\mathcal{A}}}}{\sqrt[4]{2\pi\sigma_{\mathcal{A}}}}$$

- $\langle\Psi|\, \hat{B} \,|\Psi\rangle =$ polynomial depending on $\mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \vec{\theta}$

  For large k converges to the β_{k-prod}

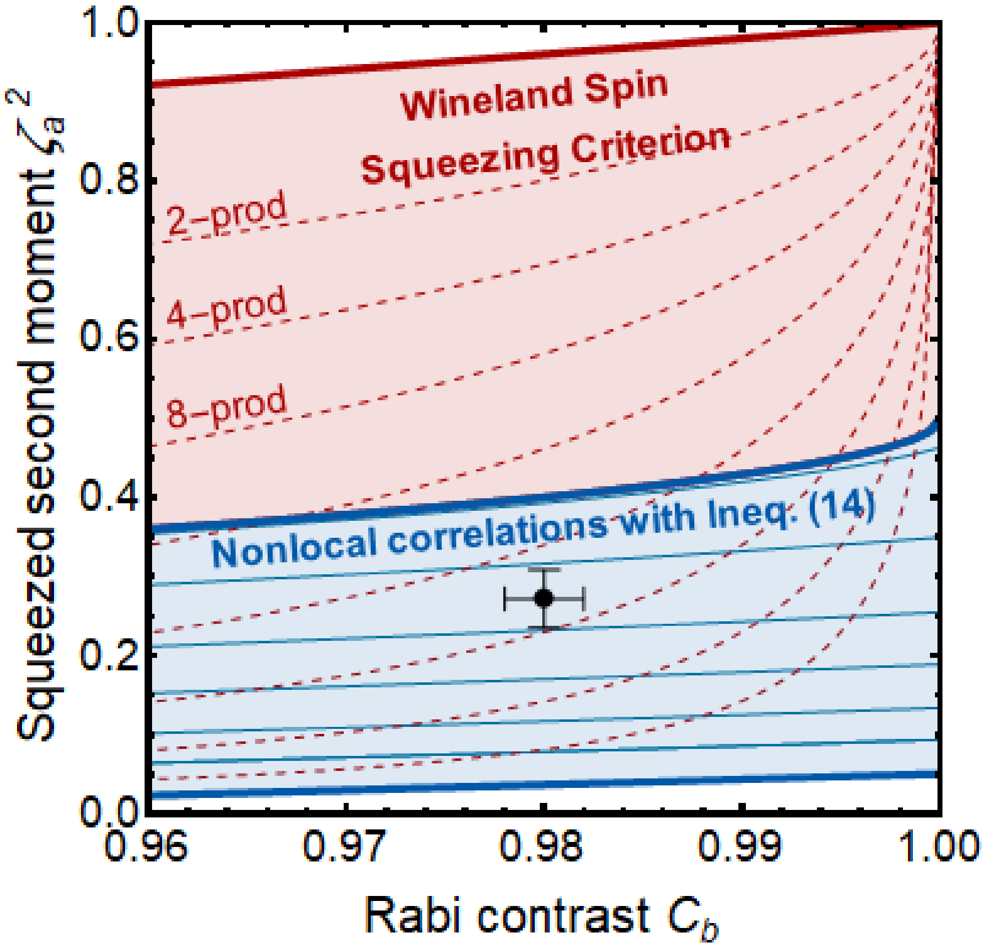
Analytical approximation: asymptotic behaviour

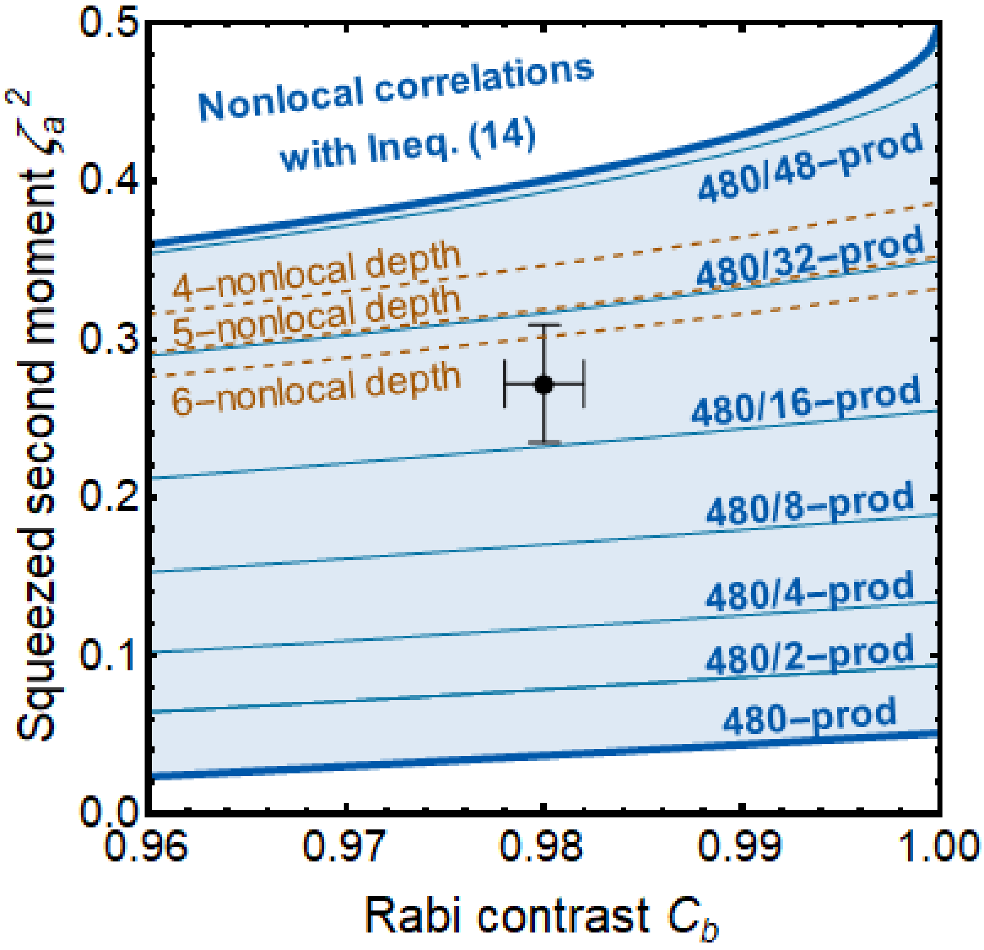




Comparison with other criteria

Comparison to other criteria and experimental data





[R. Schmied *et al*. *Science* **352** 6284 (2016)]]

[D.J. Wineland *et al. Phys. Rev. A* **50** 67 (1994)]

[F. Baccari *et al. arXiv:* 1802.09516 (2018)]

Conclusions

- We present a method to derive DIWEDs from many-body Bell inequalities
  - In the (n,2,2) Bell scenario

- When applied to 2-body PI Bell inequalities:
  - Obtain DIWEDs that involve at most 2-body correlation functions
  - Numerically characterize a hierarchy of bounds that certify amount of entanglement

- Experimentally, the techniques proposed are within reach of current technology
  - For instance they can be applied on systems where the total spin components can be accessed

Thanks for your attention

[*arXiv:*1807.06027]