Northeastern University
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CCSB

Universal critical behavior in the entanglement entropy of the quantum Ising model

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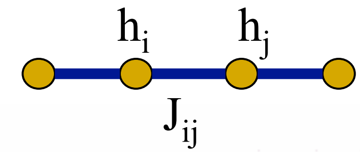
WIGNER

A change of perspectives

- So far:
  - ‘How to find out the properties and eventually a realization of some interesting entanglement patterns?’
- Tonight:
  - ‘What kind of interesting entanglement patterns emerge in known realizations of physical systems?’

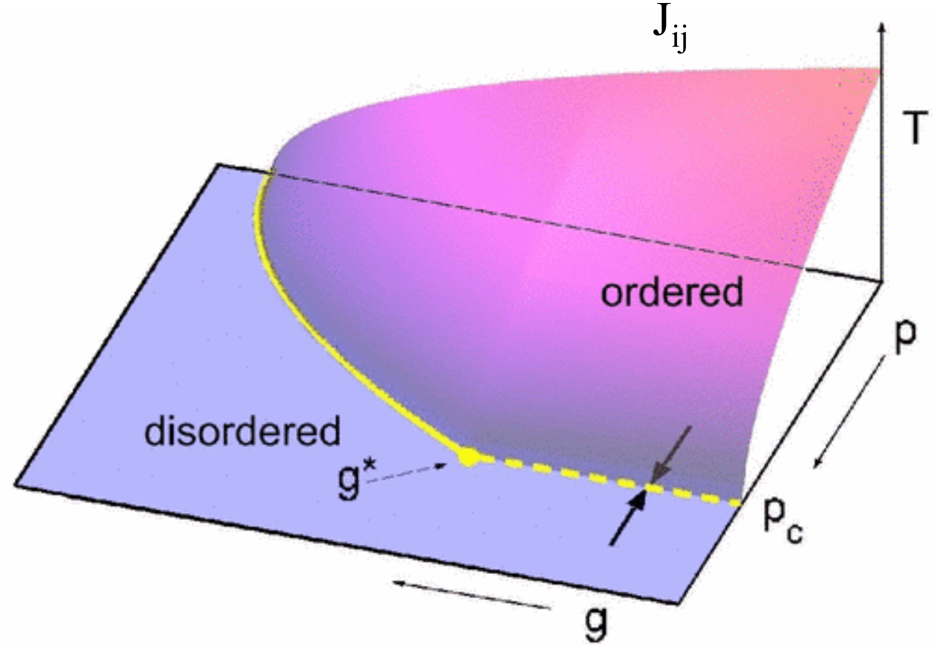
Quantum Ising model

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$



- Ferromagnetic couplings only
- Continuous phase transition: d≥1
- Control parameter: g=<h> or p
- Critical behavior is related to the classical d+1 dimensional (anisotropic) model
- d<4 Harris-criteria: weak disorder is a relevant perturbation

thermal, quantum and disorder fluctuations



T. Vojta and J. Schmalian Phys. Rev. Lett. 95, 237206

- Effects of disorder:
  - ‘infinitely strong’ at criticality
  - ‘strong’ in the surrounding Griffiths-phase

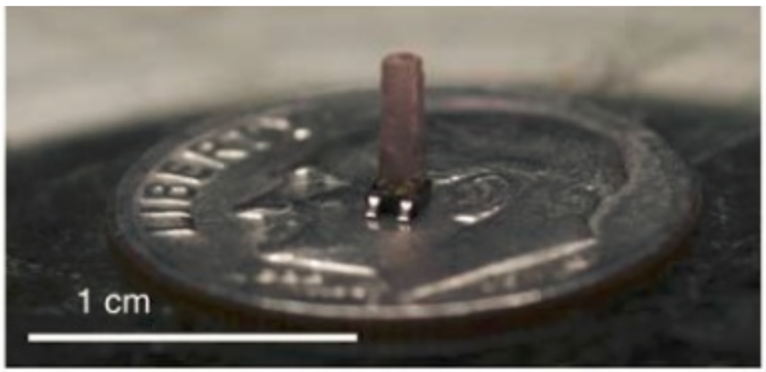
Frustration (negative interactions) irrelevant at criticality

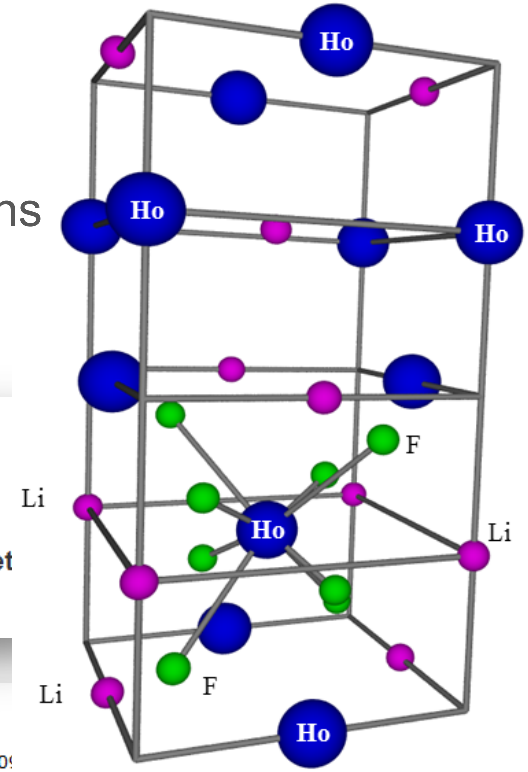
Experimental Realizations

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$

- $K(H_xD_{1-x})_2PO_4$
- $Rd_{1-x}(NH_4)_xH_2PO_4$
- $CoNb_2O_6$
- **$LiHo_xY_{1-x}F_4$**

Ultracold gases, i.e. Rb with heterogeneous interactions





$a = b = 5.176$ Å

$c = 10.75$ Å

Science 30 April 1999:
Vol. 284 no. 5415 pp. 779-781
DOI: 10.1126/science.284.5415.779

REPORT

Quantum Annealing of a Disordered Magnet

J. Brooke[1], D. Bitko[1], T. F., Rosenbaum[1,*] and G. Aeppli[2]

Vol 448|2 August 2007|doi:10.1038/nature06050 nature

LETTERS

A ferromagnet in a continuously tunable random field

D. M. Silevitch[1], D. Bitko[2], J. Brooke[3], S. Ghosh[4], G. Aeppli[5] & T. F. Rosenbaum[1]

Letters to Nature

Nature **413**, 610-613 (11 October 2001) | doi:10.1038/350
Accepted 16 August 2001

Tunable quantum tunnelling of magnet domain walls

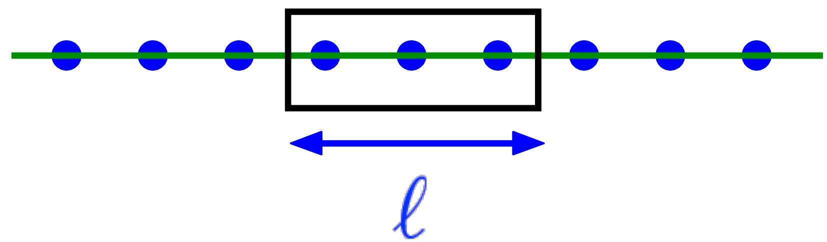
J. Brooke[1,2], T. F. Rosenbaum[1] & G. Aeppli[2]

$$\mathcal{H}_{\text{Dipolar}} = \frac{1}{2}\frac{\mu_0}{4\pi}\mu_B^2 g_J^2 \sum_{ij}\left[\frac{\mathbf{J}_i\cdot\mathbf{J}_j}{r_{ij}^3} - \frac{3(\mathbf{J}_i\cdot\mathbf{r}_{ij})(\mathbf{J}_j\cdot\mathbf{r}_{ij})}{r_{ij}^5}\right]$$

combined effects of dilutions and disorder

1d: logarithm

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$



Jordan-Wigner transformation → well studied free-fermion system

- Clean: CFT

$$S_L^{(p)}(\ell) = \frac{c}{3}\log_2\left[\frac{L}{\pi}\sin\left(\frac{\ell\pi}{L}\right)\right] + c_1$$

$$c = 1/2$$

C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B (1994);
V. E. Korepin, PRL (2004)
P. Calabrese and J. Cardy, J. Stat. Mech. (2004)

- Disordered

$$S_\infty(\ell) = \frac{c}{3}\log_2 \ell + c_1$$

$$c_{\mathrm{eff}} = \ln 2/2$$

Free fermions: numerically
Strong disorder RG: analitically treatable

G. Refael and J. E. Moore, PRL (2004)
F. Iglói, Yu-Ch. Lin, J. Stat. Mech. (2008)

Exact connection between XY (XX) and the Ising model: F. Iglói, R. Juhász, arXiv:0709.3927

Entanglement maximal at the critical point

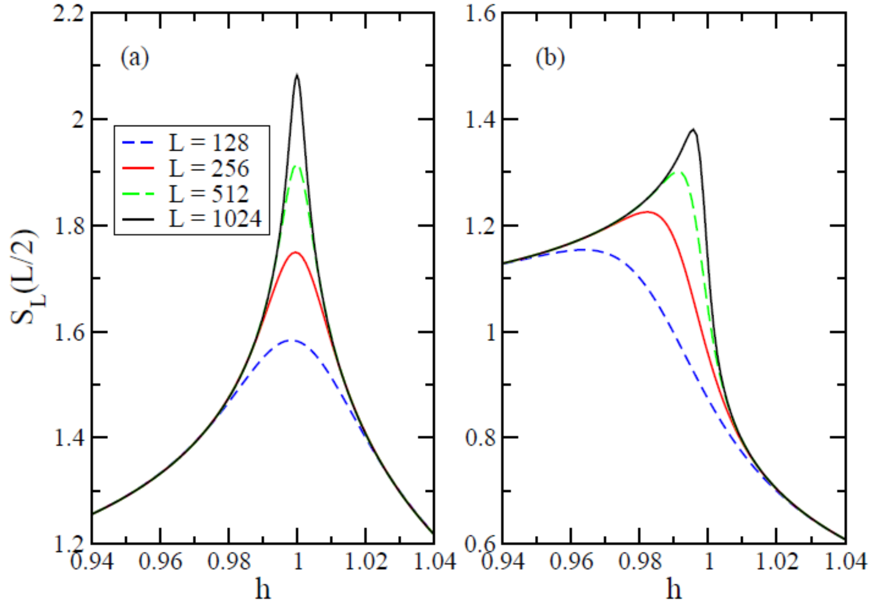


Figure 3. The entanglement entropy of a half of a chain with $J = 1$ as a function of the strength of the transverse field h, for periodic (a) and open (b) boundary conditions. On increasing the system size L, the maximum gets more pronounced, and the position of the maximum tends towards the critical point $h_c = 1$.

F. Iglói and Y.-C. Lin, J. Stat. Mech. P06004 (2008)

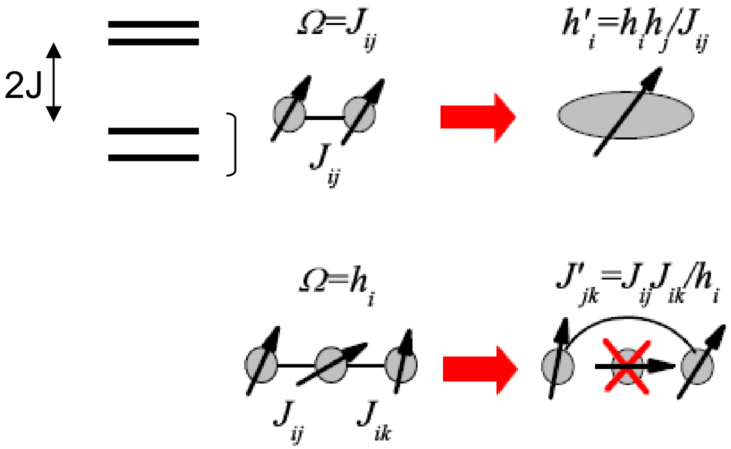
Log singularities pinpoint critical points with a universal prefactor

No need to uncover the order parameter

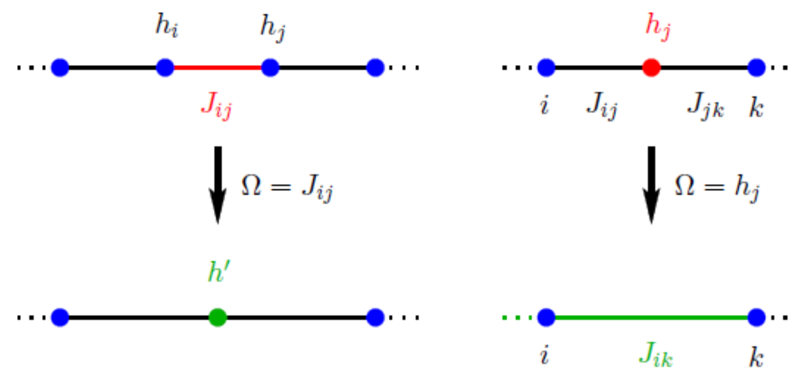
Strong disorder RG

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$

- RG rules:



- chain → chain



D. S. Fisher (1994): analytical solution in 1D

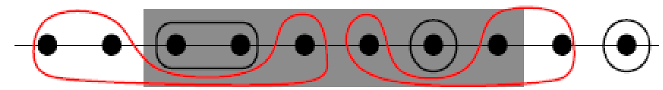
- nonlinear, integro-differential system of equations
- turns out to be asymptotically exact at criticality
- strength and shape of disorder is irrelevant

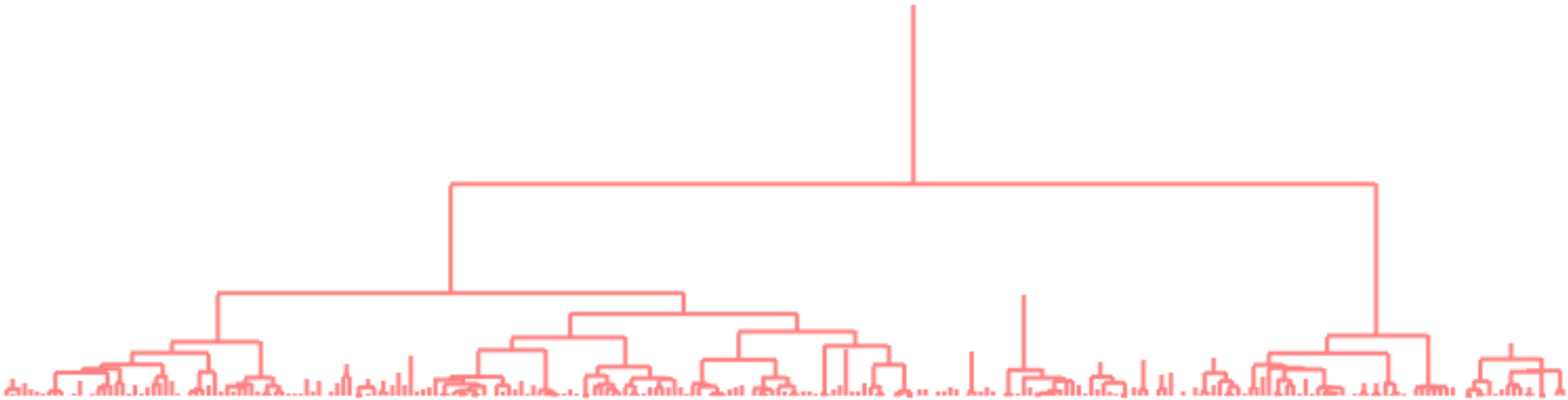
$$\frac{\mathrm{d}R}{\mathrm{d}\Omega} = R(J,\Omega)\left[P(\Omega,\Omega) - R(\Omega,\Omega)\right] - P(\Omega,\Omega)\int_J^\Omega \mathrm{d}J' R(J',\Omega)R(\frac{J}{J'}\Omega,\Omega)\frac{\Omega}{J'},$$

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = P(h,\Omega)\left[R(\Omega,\Omega) - P(\Omega,\Omega)\right] - R(\Omega,\Omega)\int_h^\Omega \mathrm{d}h' P(h',\Omega)P(\frac{h}{h'}\Omega,\Omega)\frac{\Omega}{h'},$$

Entanglement: due to shared clusters

GHZ: $\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle\right)$





$$S_\infty(\ell) = \frac{c}{3}\log_2 \ell + c_1$$

c = ½ ln2

G. Refael and J. E. Moore, PRL (2004)

Starting point: 'area law'

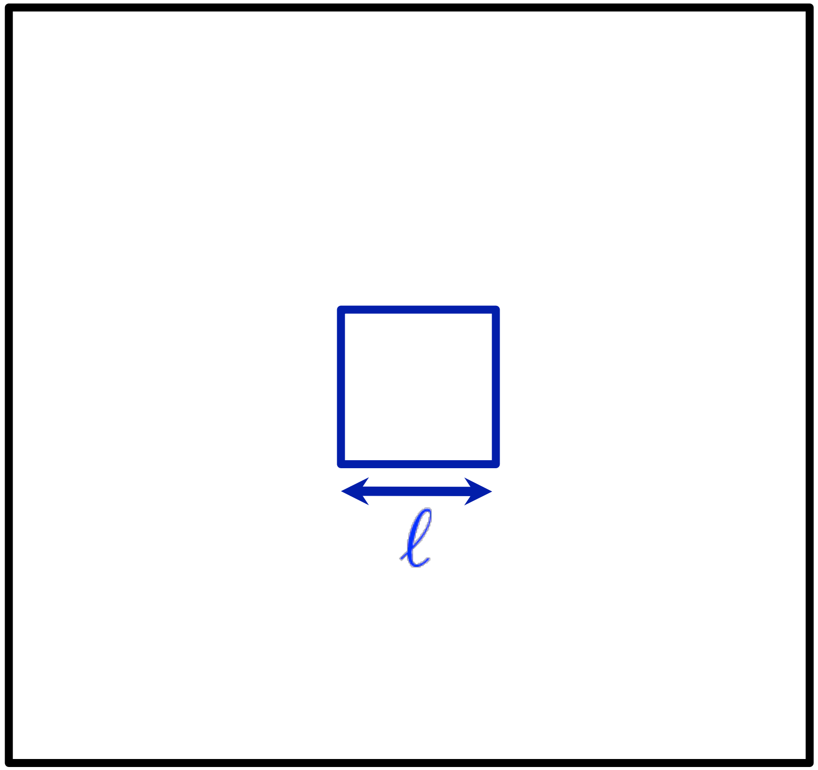
$$\mathcal{S}(\ell) \sim \ell^{d-1}$$

von-Neumann entropy

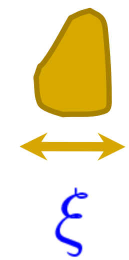
$$\mathcal{S} = -\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}})$$

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}} |\Psi\rangle\langle\Psi|$$

Simplest case:
S = # of shared clusters



Clusters of a finite correlation length



How does it scale with subsystem size?

$$\mathcal{S}_{\mathrm{2D}}(\ell) \sim \ell$$

M. Srednicki, PRL 1993

Starting point: 'area law' $\mathcal{S}(\ell) \sim \ell^{d-1}$

von-Neumann entropy

$$\mathcal{S} = -\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}})$$

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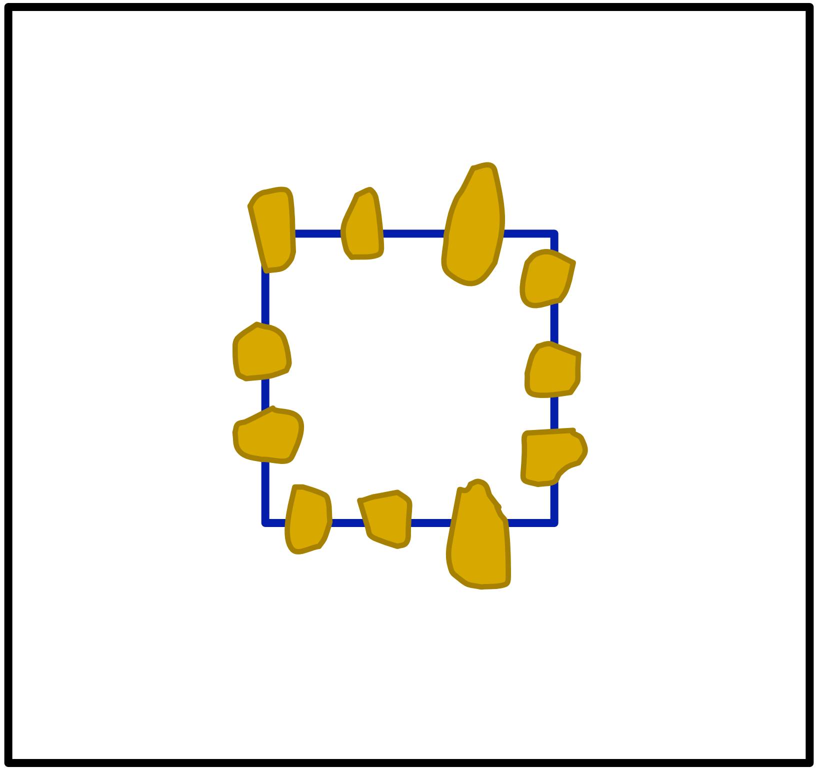
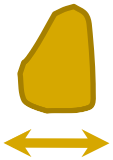


Clusters of a finite correlation length



ξ

At criticality:

$$\xi \sim |\delta|^{-\nu}$$

How does it scale with subsystem size? $\mathcal{S}_{\mathrm{2D}}(\ell) \sim \ell$

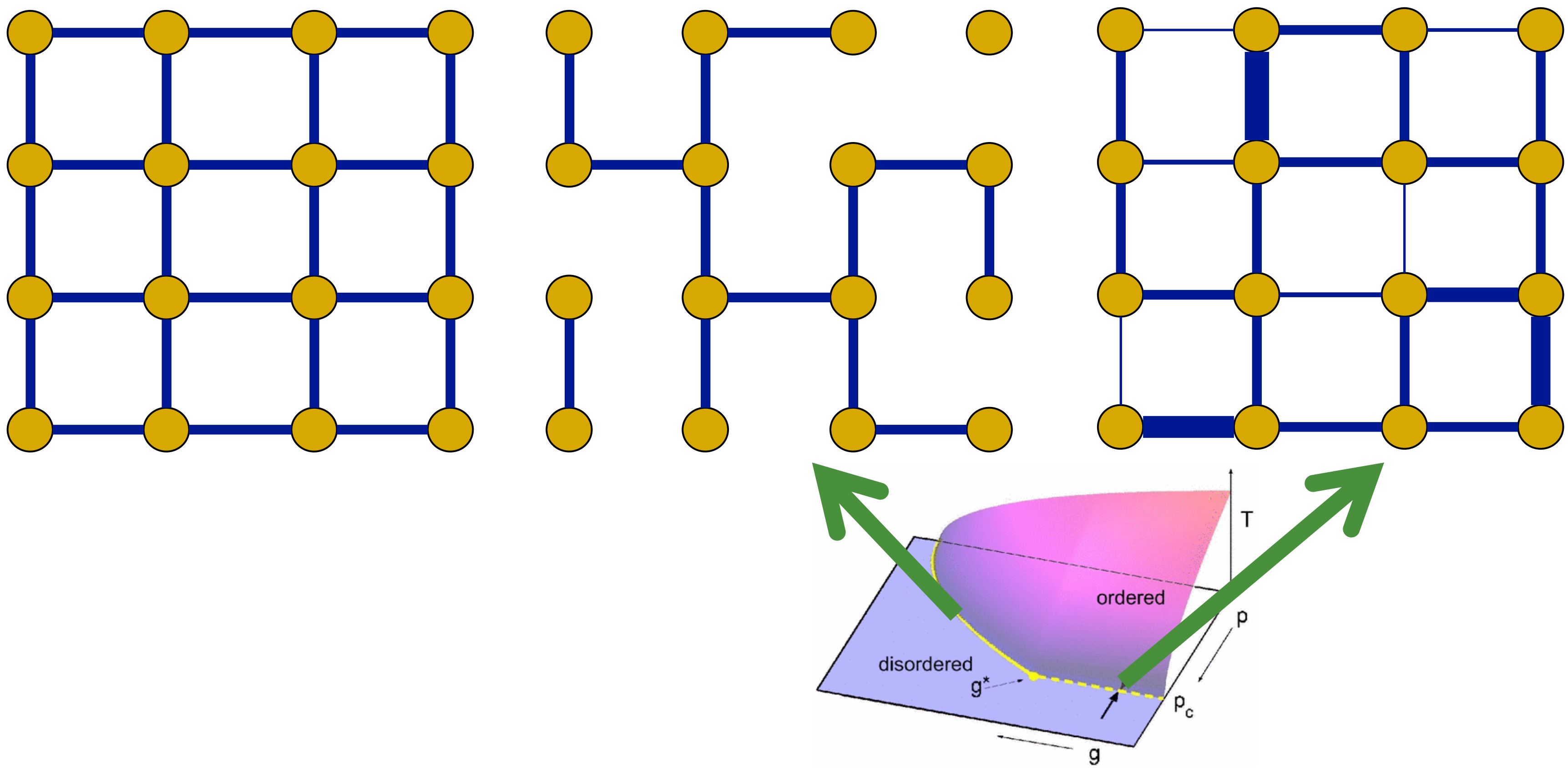
M. Srednicki, PRL 1993

Studied cases

$$\mathcal{H} = -\sum_{(i,j)} J_{ij}\sigma_i^x\sigma_j^x - \sum_i h_i\sigma_i^z$$

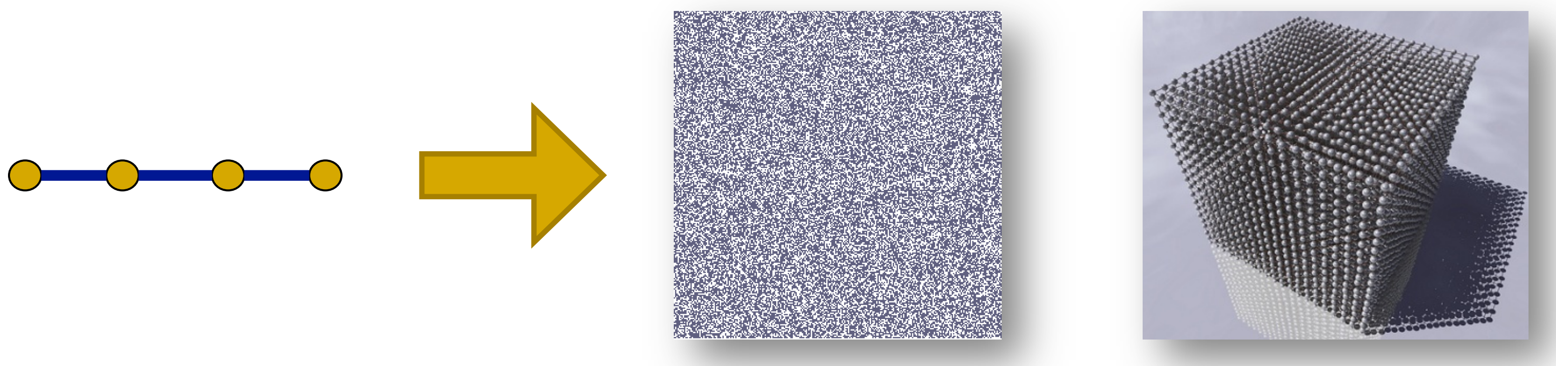
- clean
- diluted
- disordered

Interestingly, the clean model is the hardest!

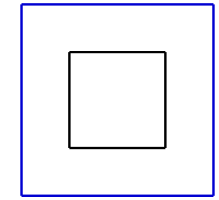


Challenges in higher dimensions

- Systems with continuous and discrete symmetry behave differently
- No free-fermion techniques in general
- RG methods applicable only numerically, with a high cost
- The entanglement entropy is not maximal at the critical point
- Leading orders in the entanglement entropy are not universal
- Shape of the subsystem might play a crucial role



2D results



$$\mathcal{S}_{2\mathrm{D}}(\ell) = a\ell + b\ln\ell$$

Area law with a universal corner log correction

- clean

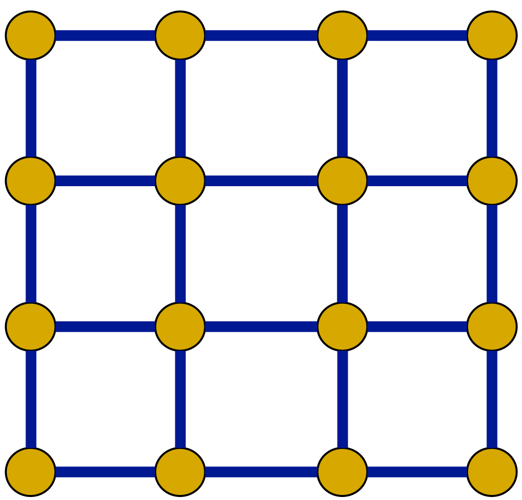
b=-0,0264(3)

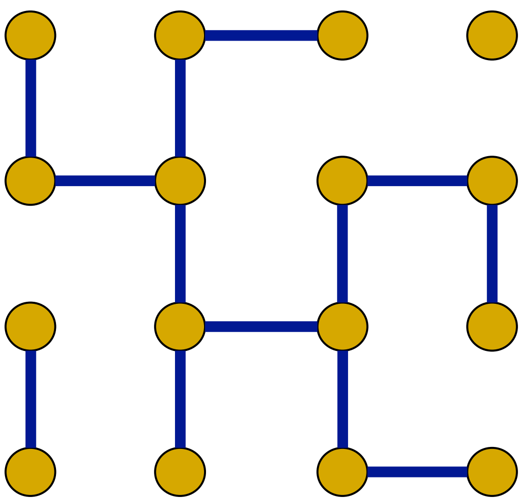
- diluted

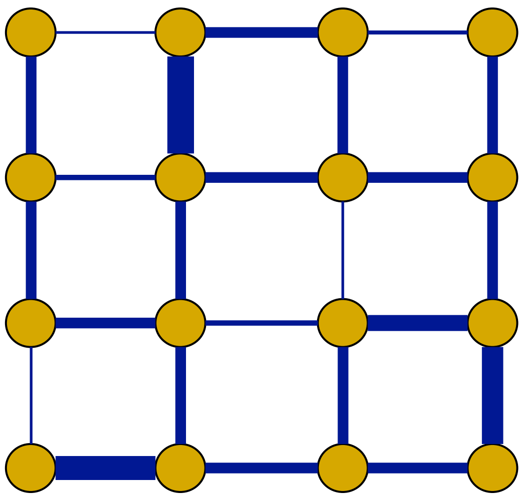
b=-0,0765735…

- disordered

b=-0,029(1)



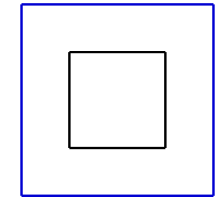




IAK, F. Iglói and J. Cardy PRB (2012)

IAK, F. Iglói, EPL (2012)

2D results



$$\mathcal{S}_{2\text{D}}(\ell) = a\ell + b\ln\ell$$

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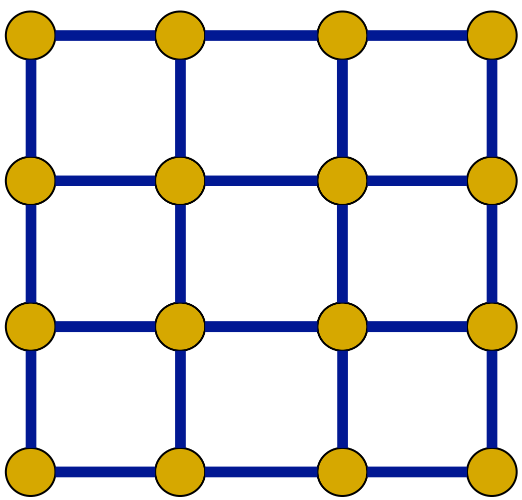

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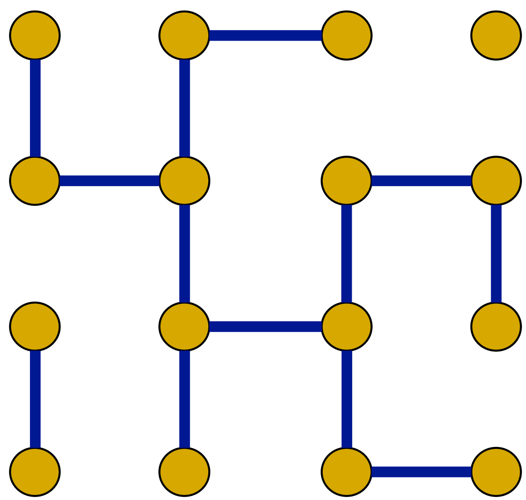

- disordered

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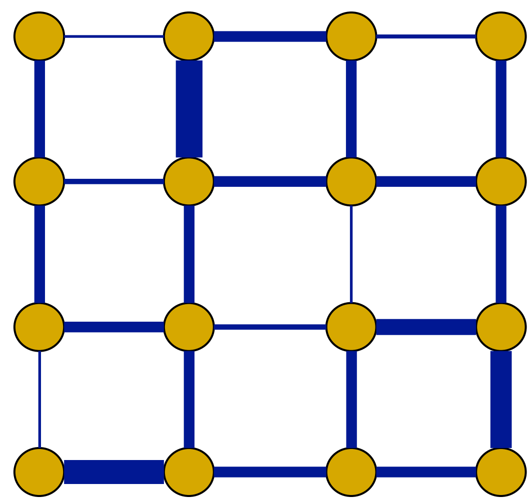
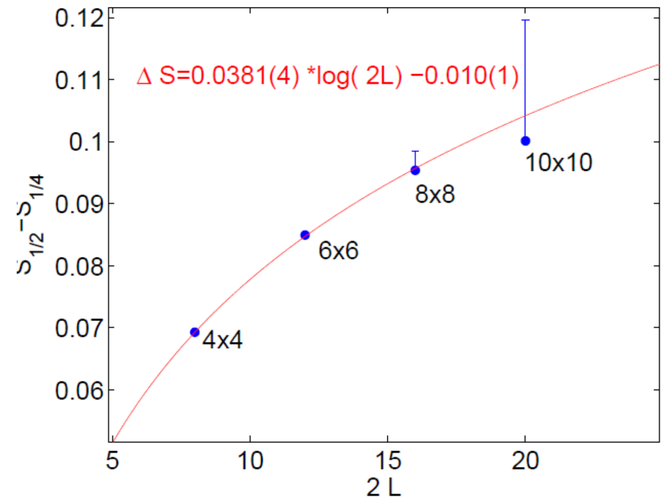


IAK, F. Iglói and J. Cardy PRB (2012)

IAK, F. Iglói, EPL (2012)

Clean, homogeneous model

$$S_{\rm 2D}(\ell) = a\ell + b\ln\ell$$



$$S_{1/4}(L) - S_{1/2}(L) = s_0 \log 2L + const.$$

TTNS: b = -0.0264(3)

L. Tagliacozzo, G. Evenbly and G. Vidal, Phys. Rev. B (2009)

Rényi entropies (mainly n=2):

$$S_n = \frac{1}{1-n}\ln \mathrm{Tr}\,\rho_A^n \quad 0 \le n \le \infty \qquad S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left[\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\right]$$

Expansions h/J és J/h: b_2 = -0.0305(28)

R. R. P. Singh, R. G. Melko and J. Oitmaa, Phys. Rev. B (2012)

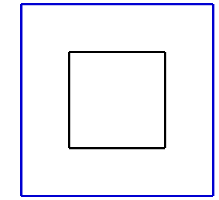
QMC (XXX, XX, XY, Ising): Ising: b_2 = -0.021(7)

S. Humeniuk and T. Roscilde, Phys. Rev. B (2012)

Free scalar field: $b_2 \approx$ -0.025 b ≈ -0.05

H. Casini and M. Huerta, Nucl. Phys. B (2007)

2D results



$$\mathcal{S}_{2D}(\ell) = a\ell + b \ln \ell$$

Area law with a universal corner log correction

- clean

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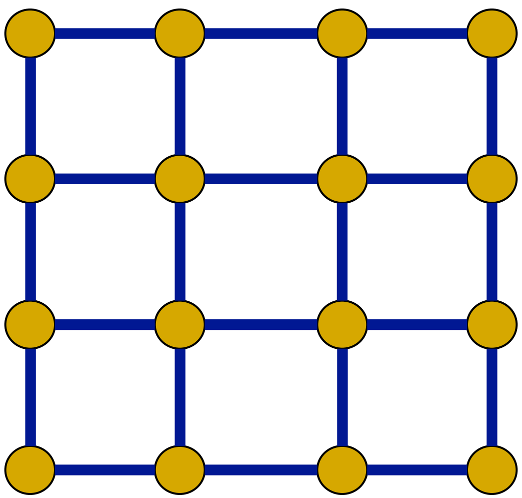
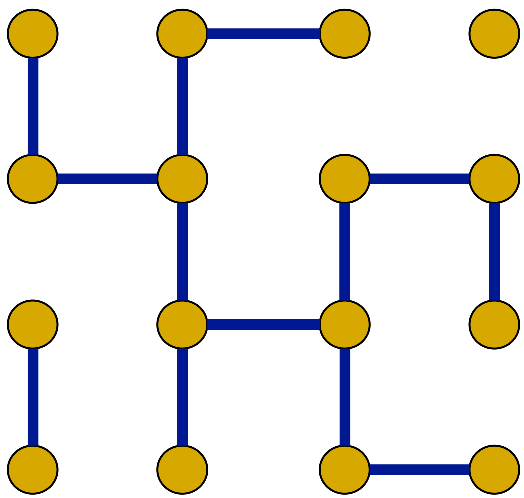


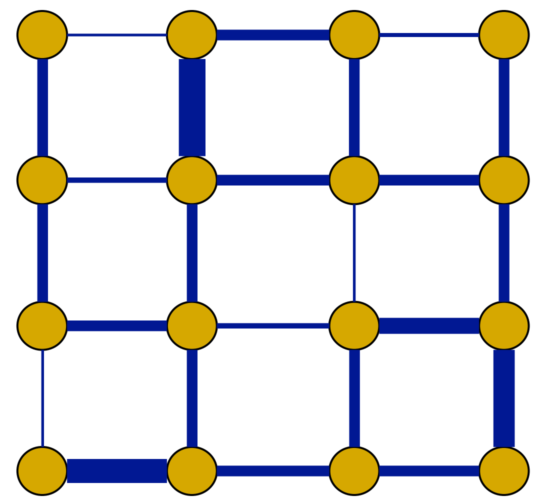
- diluted

b=-0,0765735…



- disordered

b=-0,029(1)



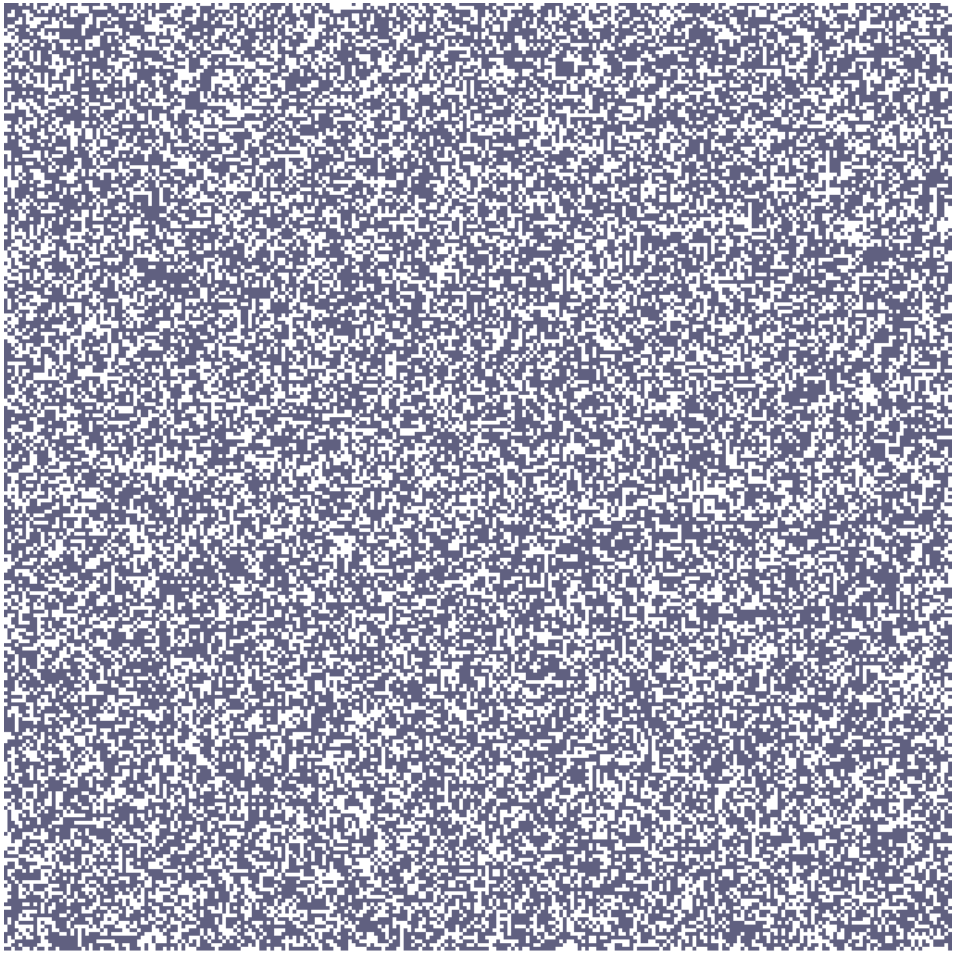
IAK, F. Iglói and J. Cardy PRB (2012)

IAK, F. Iglói, EPL (2012)

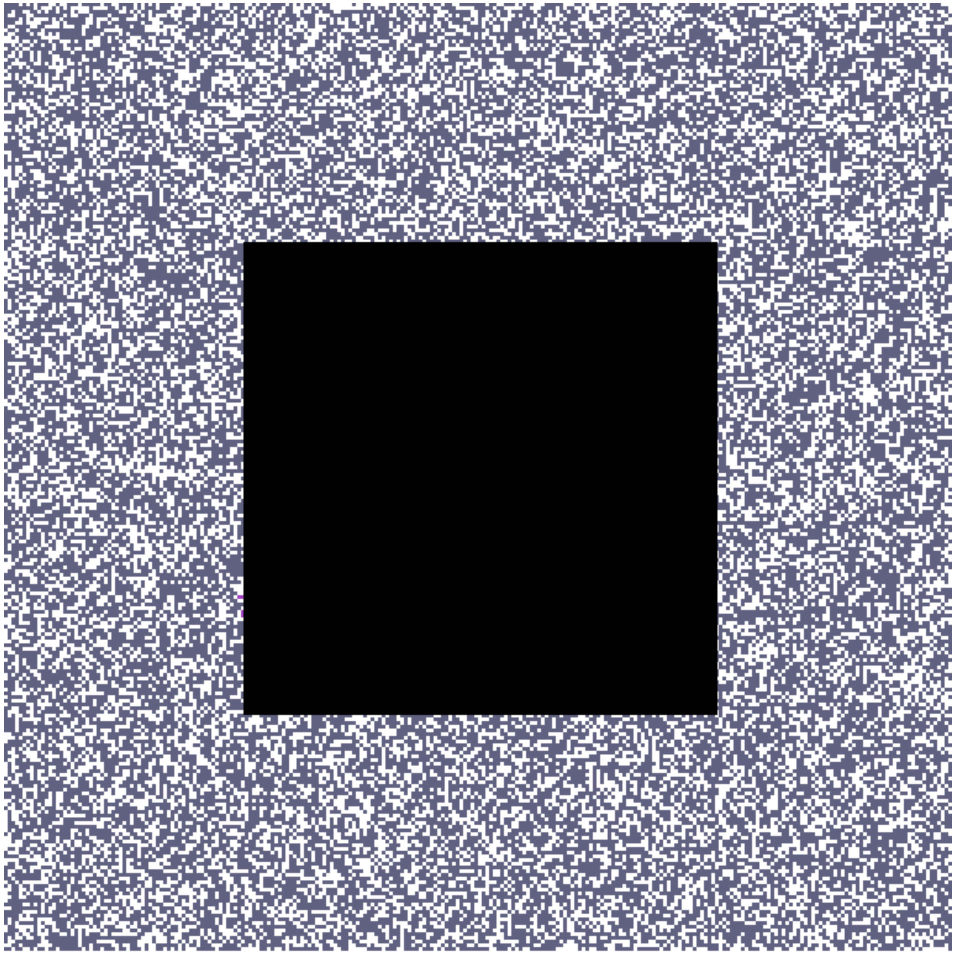
Percolation

2D: $p_c \approx 0.592746$

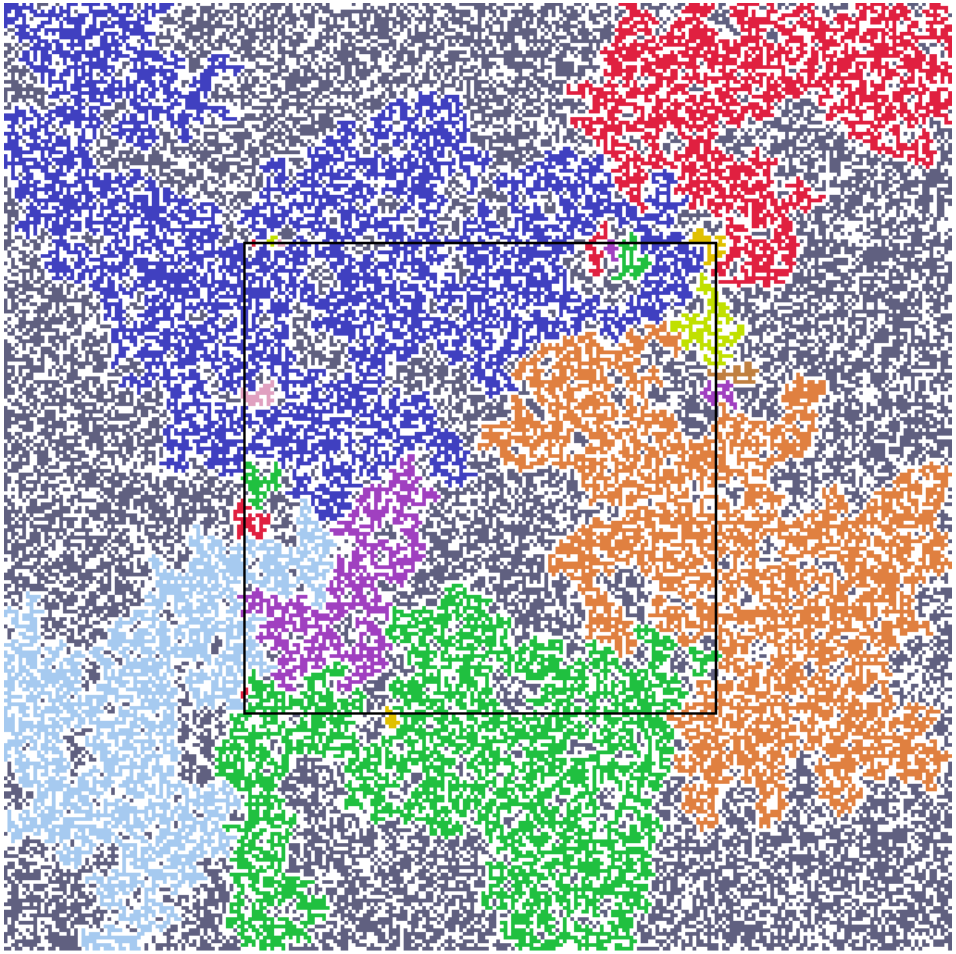
3D: $p_c \approx 0.3116$



Square subsystem: corners



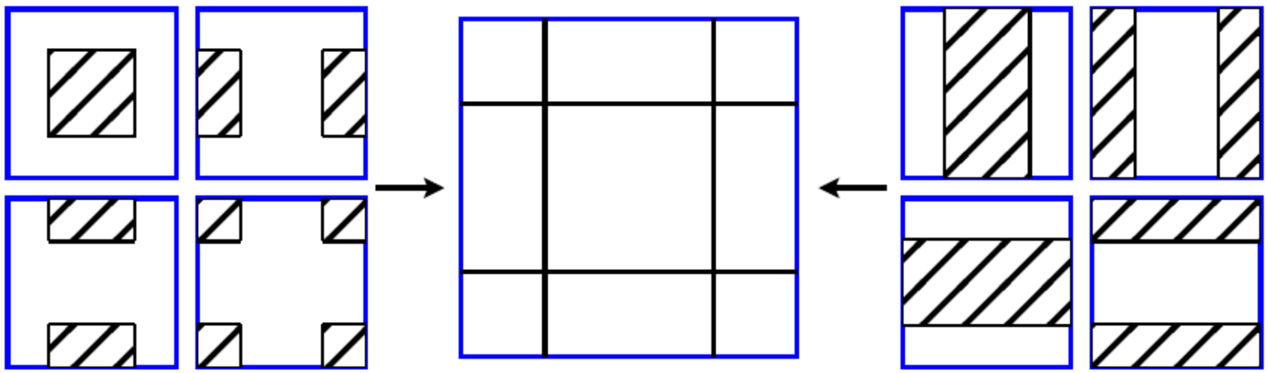
Number of crossed clusters? $\mathcal{S}_{2\mathrm{D}}(\ell) = a\ell + b\ln\ell$



First study: R. Yu, H. Saleur and S. Haas, Phys. Rev. B 77, 140402(R) (2008).

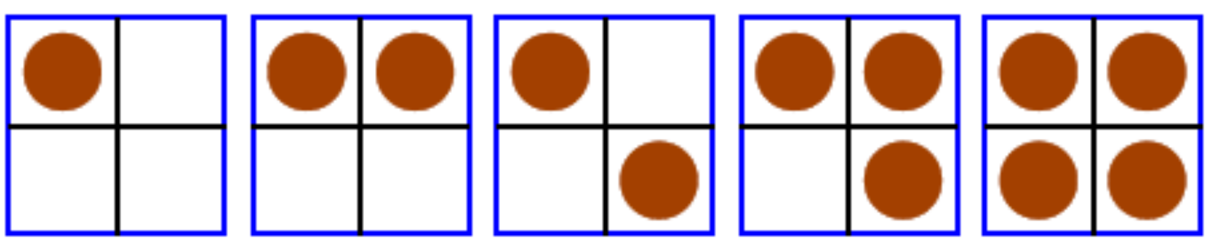
Logarithm emerges due to corners

How to show this?



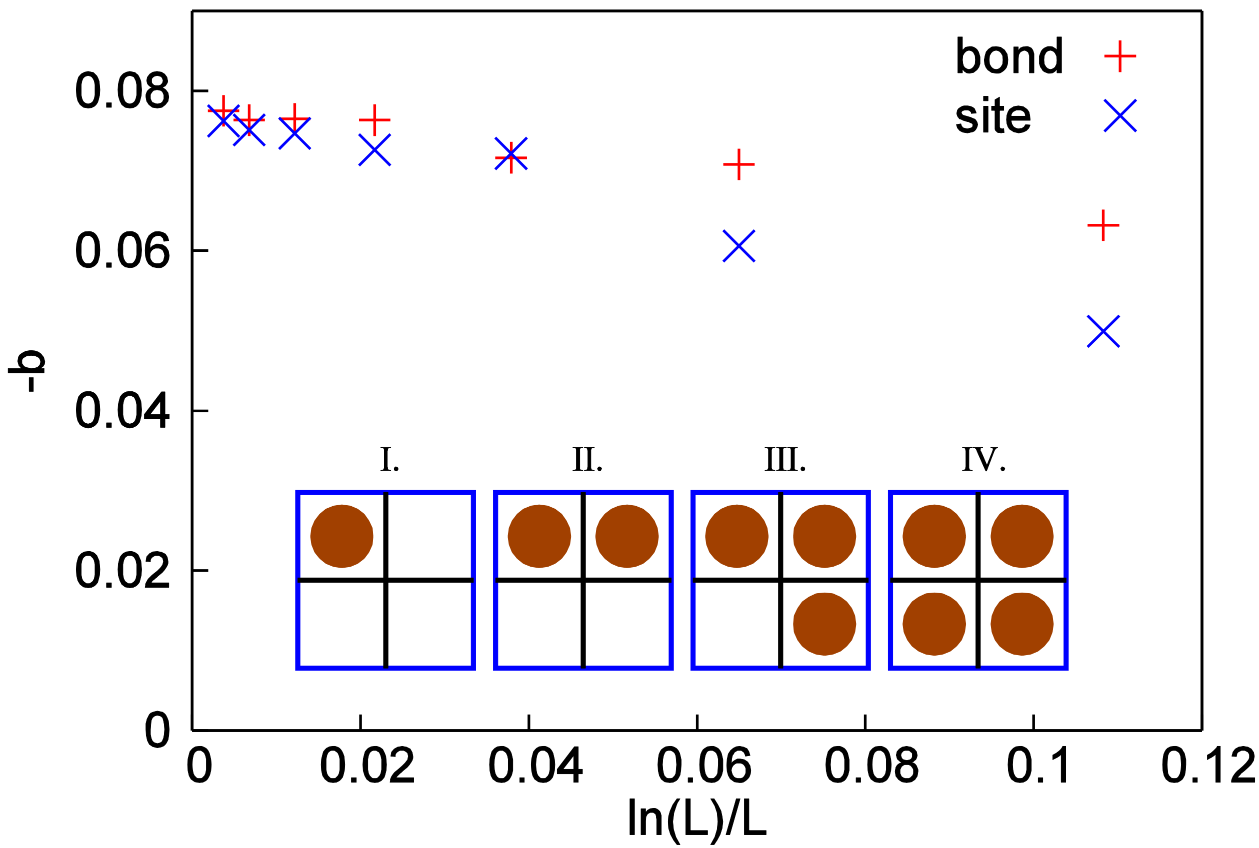
$$\ell = L/2$$

Difference comes from corners only



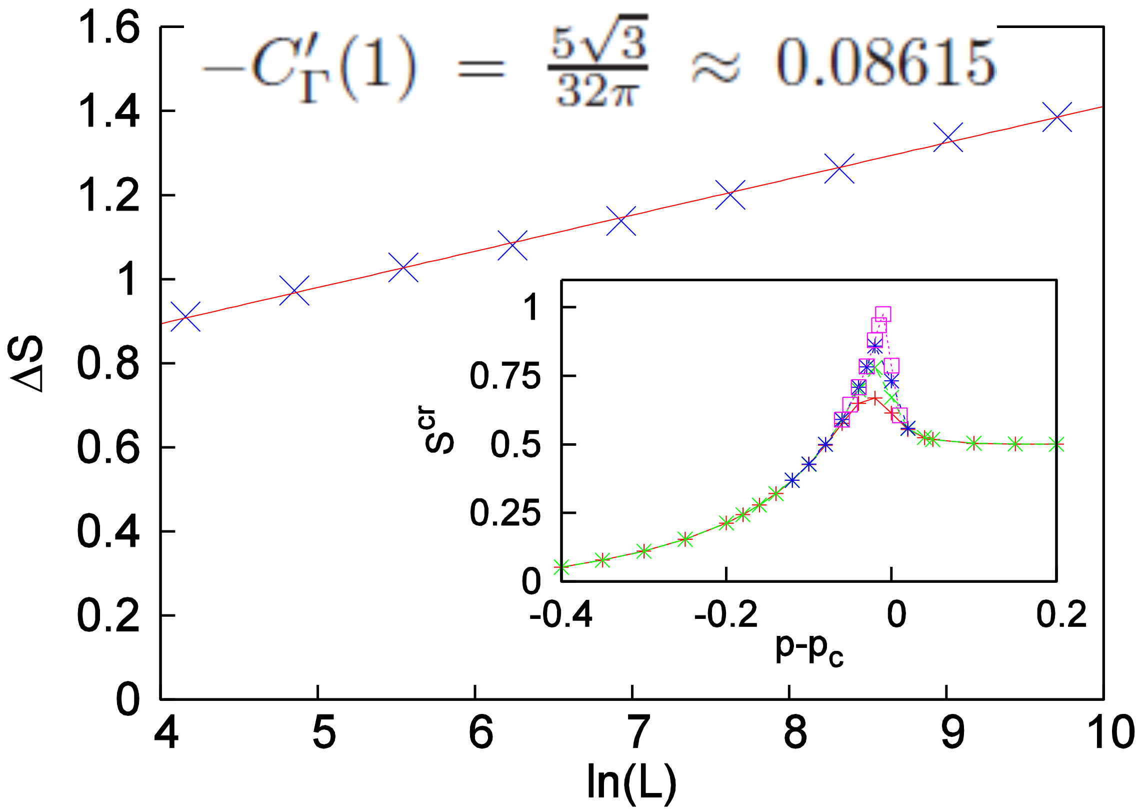
	1.	2.	3.*	4.*	5.
$D=0$	0	1/2	1/2	3/4	1
$D=1$	0	1/2	1	1	1

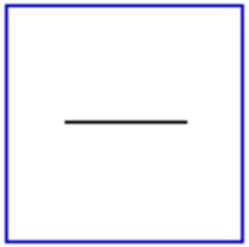
Log correction is universal



b=-0.077(1)

Line segment as a subsystem

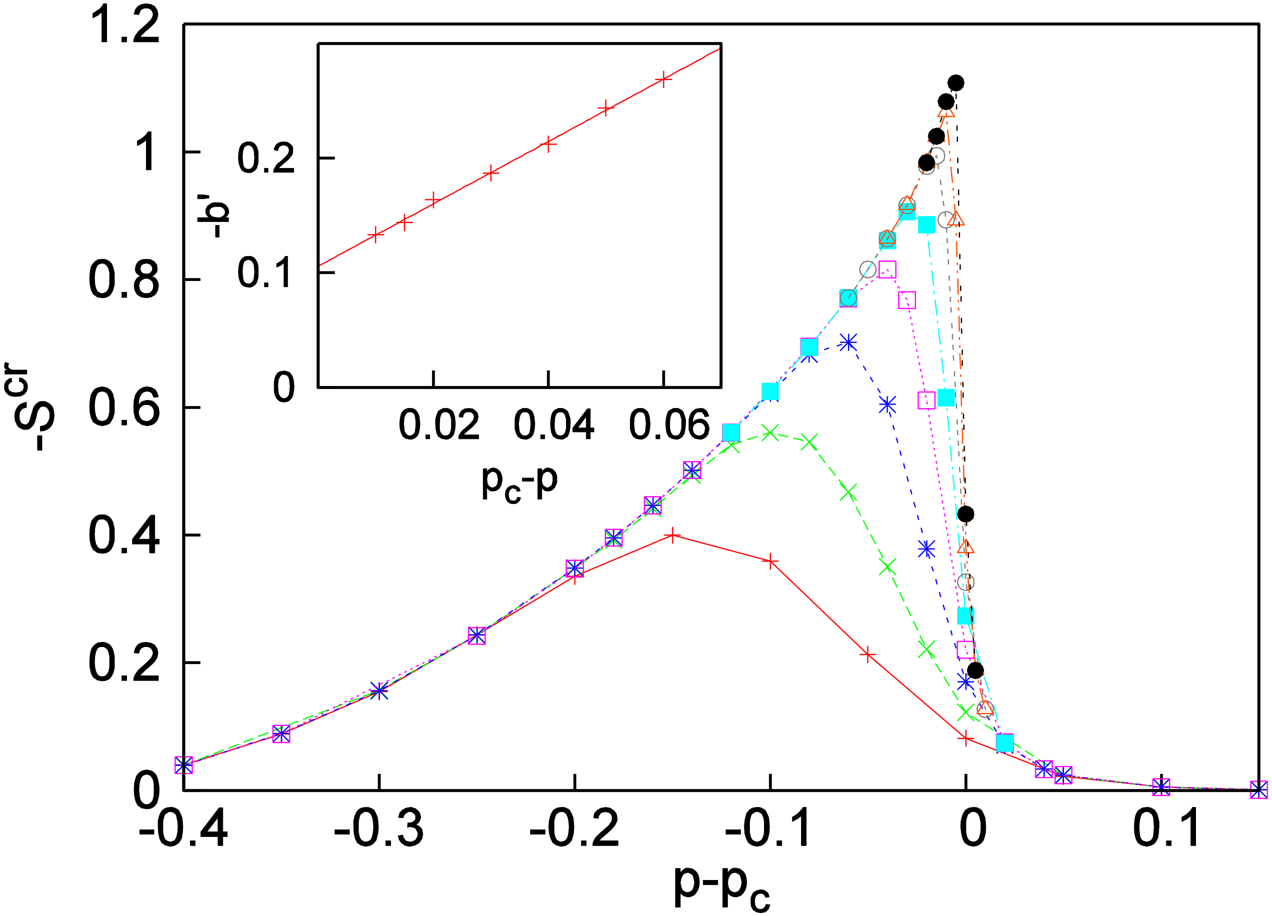




Different result on the surface: $b = \frac{\sqrt{3}}{4\pi} \approx 0.1378$

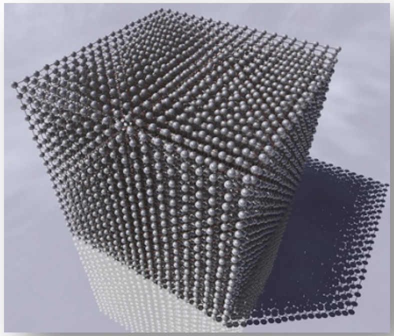
R. Yu, H. Saleur and S. Haas, Phys. Rev. B 77, 140402(R) (2008).

Emerges due to diverging correlation length



b'=0.106(10) b'=bν=0.102..

Free energy of a finite system



$$f = a\ell^3 + b\ell^2 + c\ell + d\ln\ell$$

volume surface edge corner

Scaling argument: V. Privman, Phys. Rev. B 38, 9261 (1988)

- Q-state Potts model

$$H_p = -J_p \sum_{(i,j)} \delta(s_i, s_j) \quad Z(Q) = \sum_s \prod_{\langle ij \rangle} \exp\left(K\delta_{s_i,s_j}\right)$$

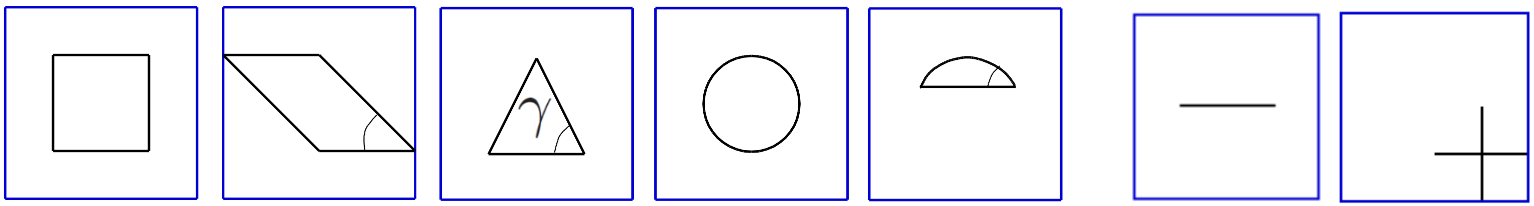
$$Z(Q) = \langle Q^{N_{\text{tot}}} \rangle = \sum_{\text{clusters}} Q^{N_{\text{tot}}} p^{N_{\text{open}}} (1-p)^{N_{\text{closed}}}$$

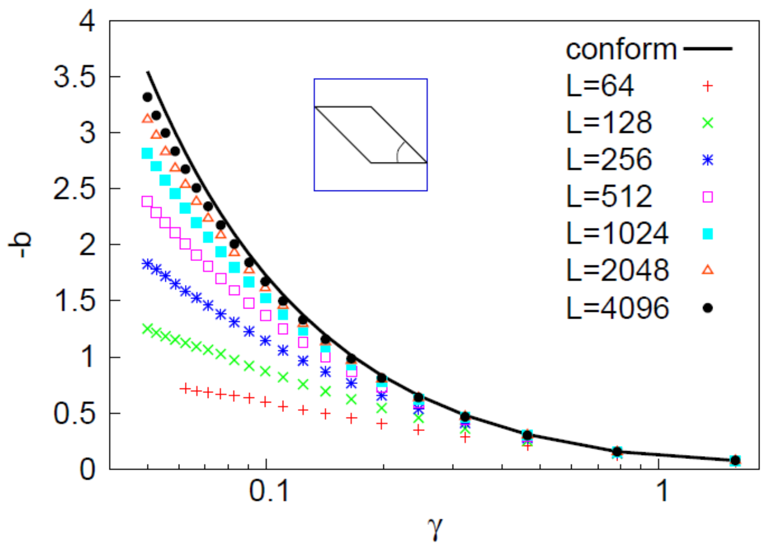
- Q→1: percolation
- Q=2: Ising model
- Continuous transition for: Q≤4 (2D), or Q<2,3 (3D)

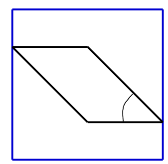
$$\langle N_{\text{tot}} \rangle = Q \frac{\partial \ln Z(Q)}{\partial Q}$$

Further geometries: nontrivial angle-dependence

Cardy-Peschel formula: $$b = \frac{\beta(Q)}{24} \sum_k \left(\frac{\gamma_k}{\pi} - \frac{\pi}{\gamma_k} \right)$$ $a\ell + b \ln \ell$







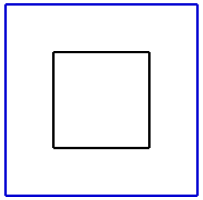
$$-\frac{c'(1)}{12}\left[4-\pi\left(\frac{1}{\gamma}+\frac{1}{\pi-\gamma}+\frac{1}{\pi+\gamma}+\frac{1}{2\pi-\gamma}\right)\right]$$

$$-\frac{c'(1)}{24}\left[6-\pi\left(\frac{2}{\gamma}+\frac{1}{\pi-2\gamma}+\frac{1}{\pi+2\gamma}+\frac{2}{2\pi-\gamma}\right)\right]$$

Conformal field theory, Fortuin-Kasteleyn clusters

2D results

$$\mathcal{S}_{2\mathrm{D}}(\ell) = a\ell + b\ln\ell$$



Area law with a universal corner log correction

- clean

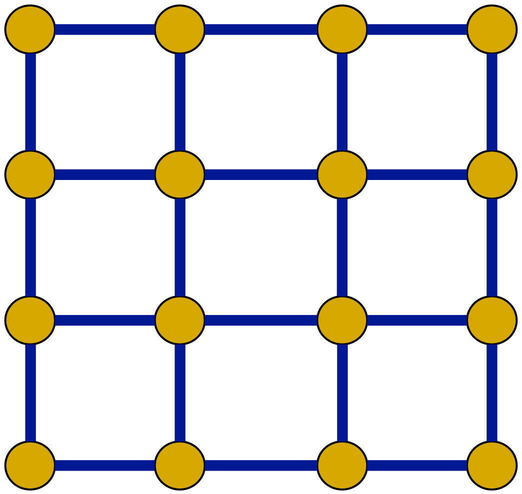
b=-0,0264(3)

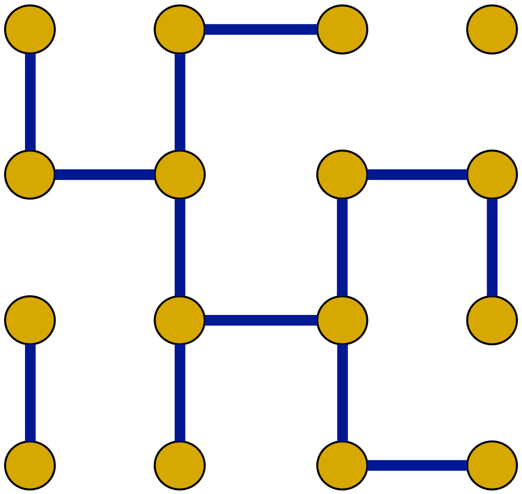
- diluted

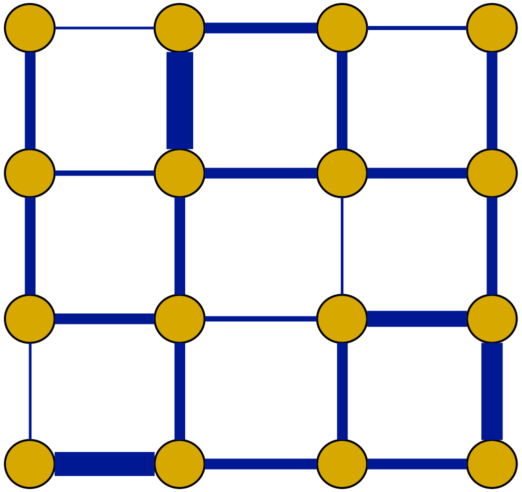
b=-0,0765735…

- disordered

b=-0,0291(3)







IAK, F. Iglói and J. Cardy PRB (2012)

IAK, F. Iglói, EPL (2012)

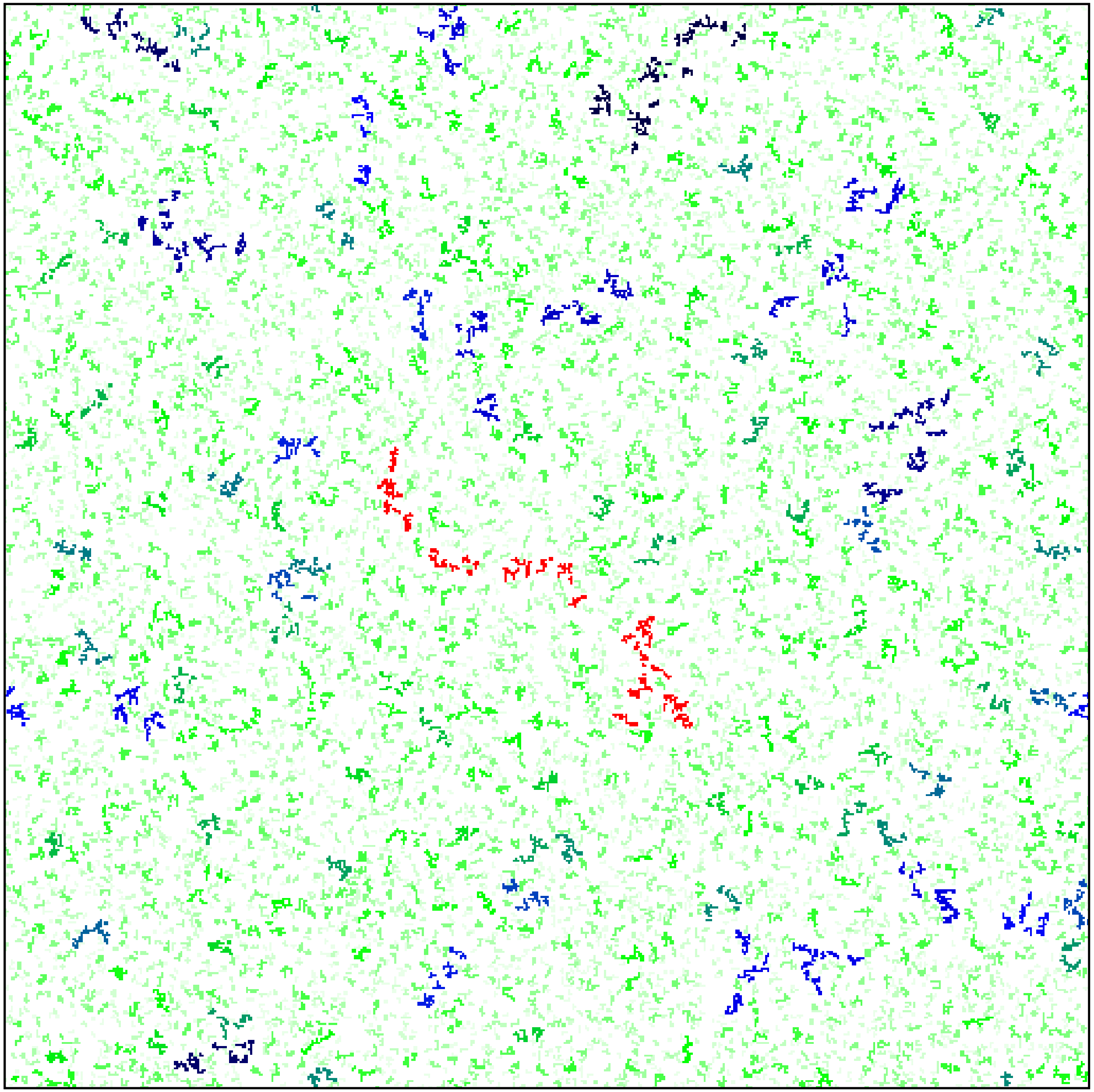
Runtime:

Naively:

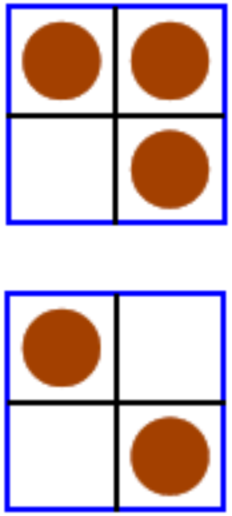
$t \sim N^3$

Mapping to graph theory:

$t \sim N \log N$



Configurations:



$b = -0.0291(3)$

http://teo.elte.hu/minosites/ertekezes2012/kovacs_istvan.pdf

Independent of the shape and strength of disorder

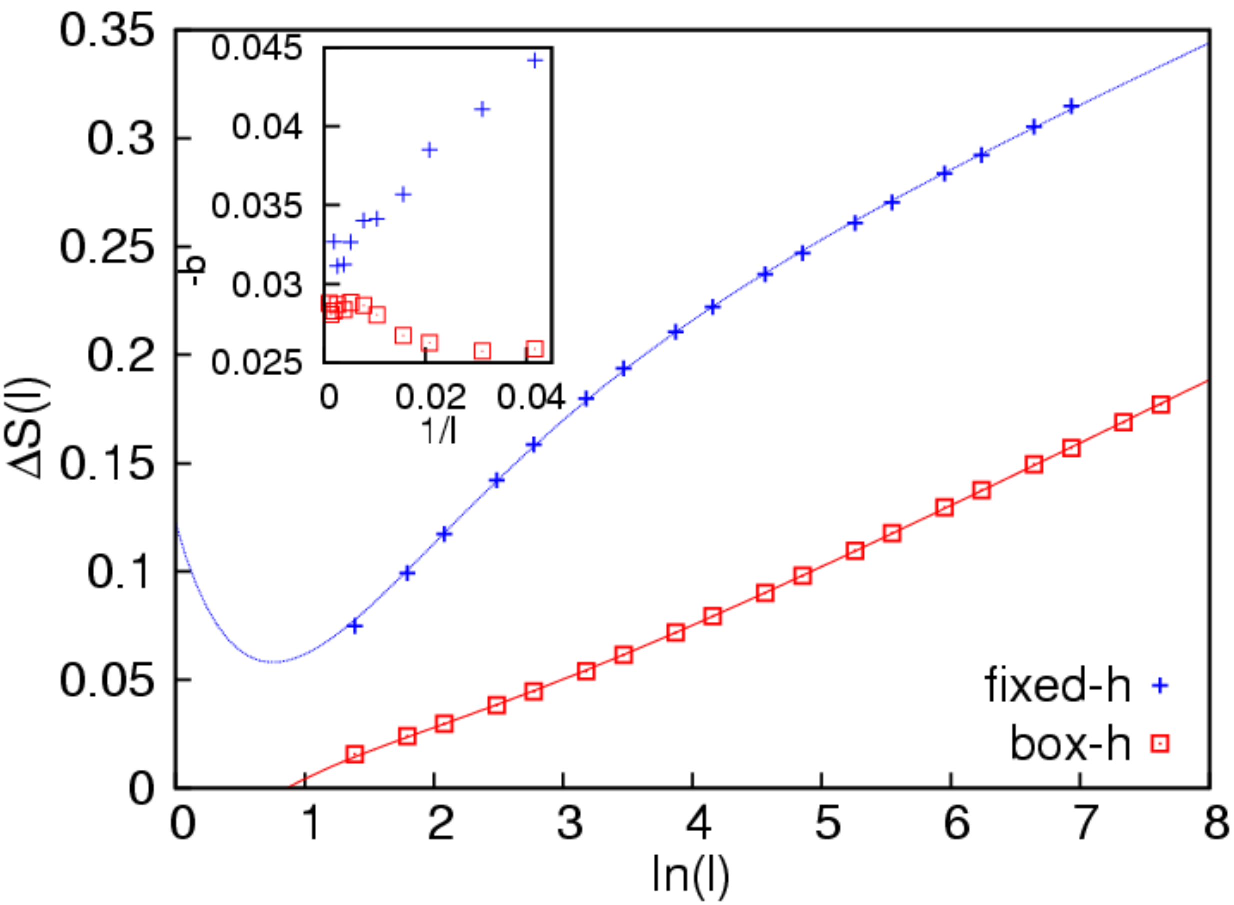
Runtime:

Naively:

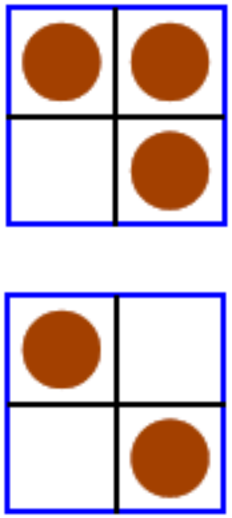
$t \sim N^3$

Mapping to graph theory:

$t \sim N \log N$



Configurations:



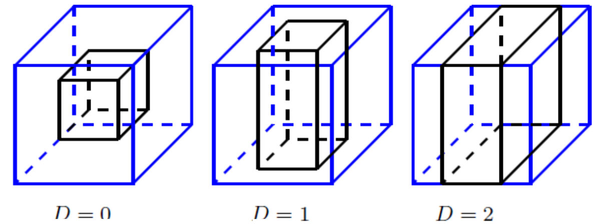
$b = -0.0291(3)$

http://teo.elte.hu/minosites/ertekezes2012/kovacs_istvan.pdf

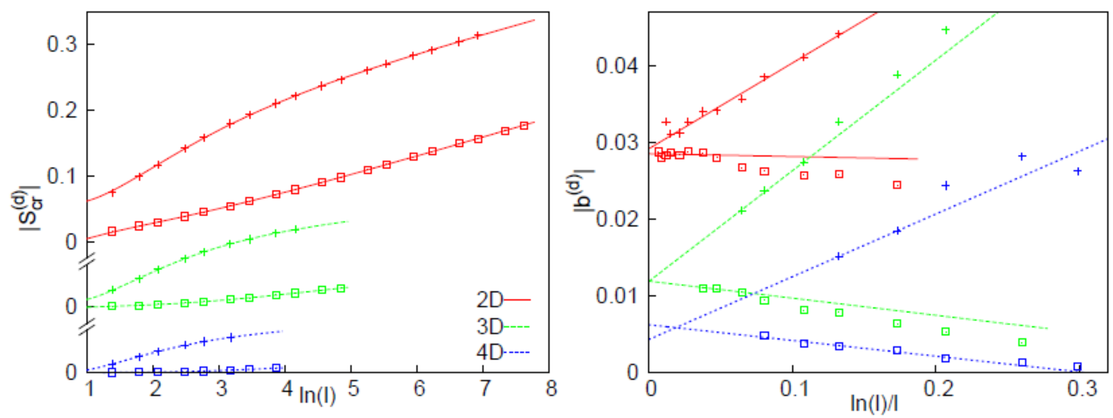
Higher dimensions

We need to compare all subsystems:

$$\mathcal{S}_0^{(d)}(0) = \sum_{D=0}^{d-1} \left(-\frac{1}{2}\right)^D \binom{d}{D} \mathcal{S}_D^{(d)}$$

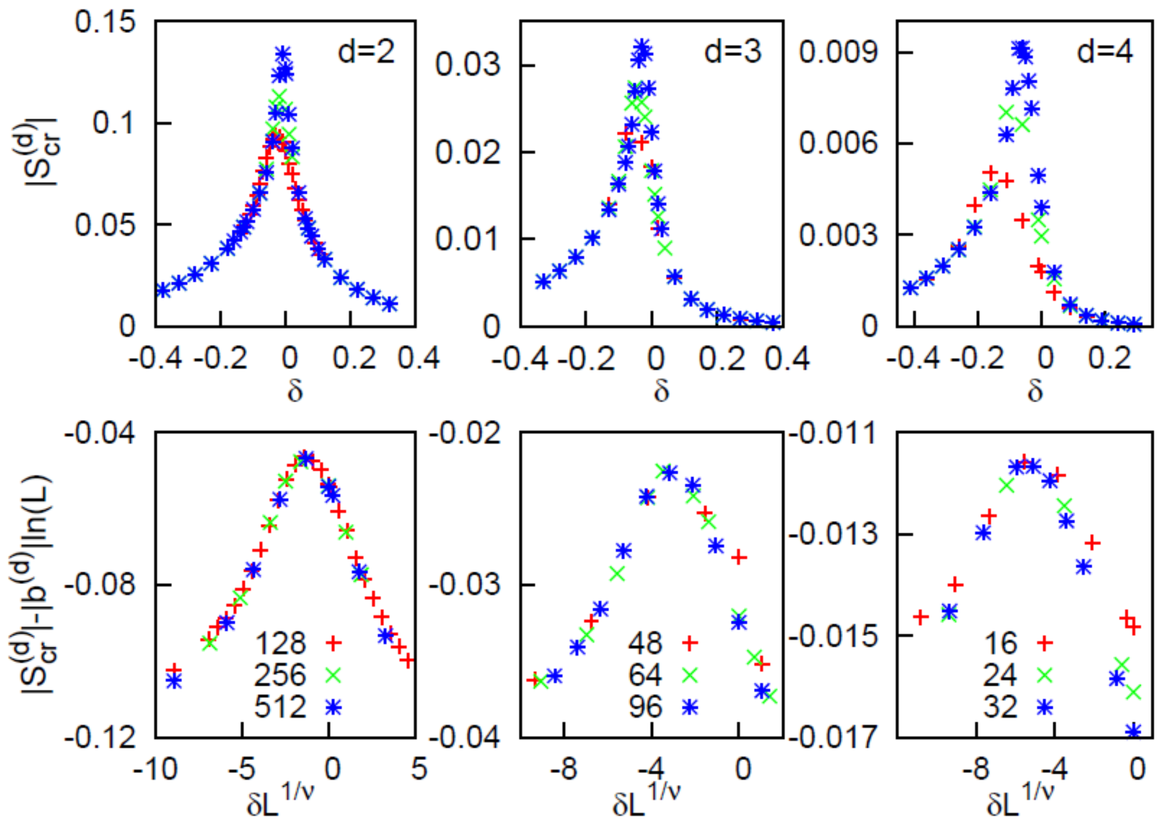


- **3D:** L=128 $\quad \mathcal{S}_{3D}(\ell) = a\ell^2 + b\ell + c\ln\ell$
- **4D:** L=48 $\quad \mathcal{S}_{4D}(\ell) = a\ell^3 + b\ell^2 + c\ell + d\ln\ell$



The corner contribution is a universal logarithm

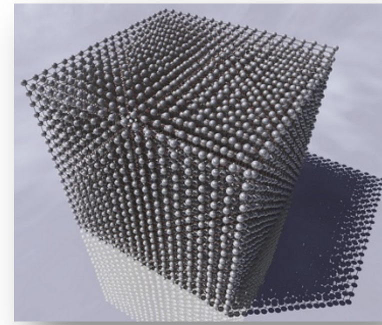
Logarithm emerges due to diverging correlation length



We can locate the critical point with the corner contribution

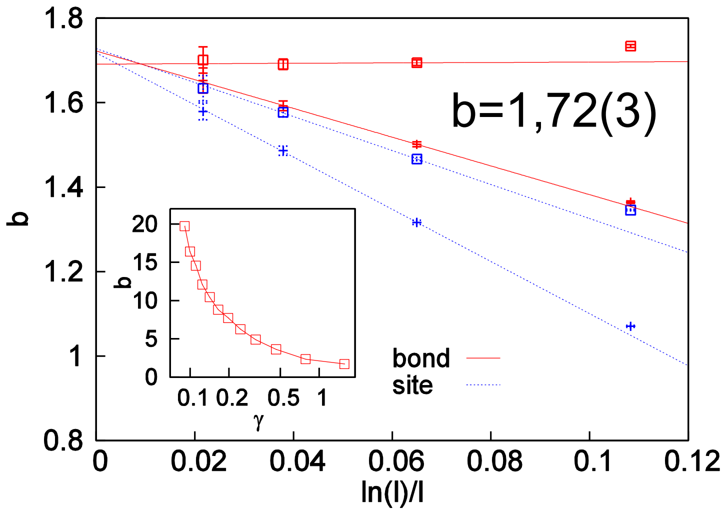
The full entanglement entropy is NOT maximal at the critical point

3D: universal corner logarithm

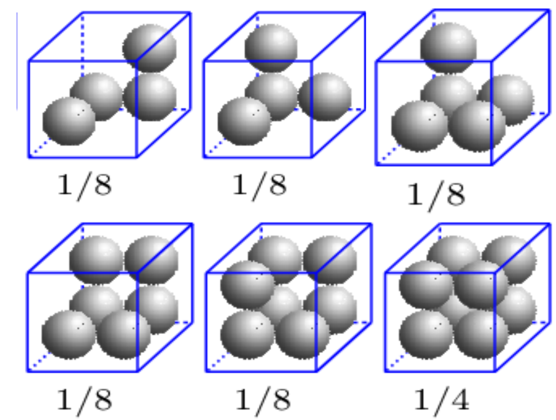


- clean
  b~0.012(?)
  T. Devakul, R. Singh
  arXiv:1407.0084
- diluted
  b=1,72(3)
  I. A. K. and F. Iglói
  arXiv:1402.6535
- disordered
  b=0,012(2)
  I. A. K. and F. Iglói
  arXiv:1108.3942

Diluted model:



Relevant configurations:



$$\langle N_\Gamma \rangle = a_2 \ell^2 + a_1 \ell + b \ln \ell$$

prefactor sign alternates with dimensionalty

Thank you for your attention!

- Critical entanglement depends on the shape of the subsystem
- Corner contributions are universal and can be used to determine the critical point and the universality class
- The disordered quantum Ising model offers an interesting playground for multipartite entanglement studies, including a scalable quantum network architecture
- Altogether, the disordered quantum Ising model is one of the best understood interacting systems

- For results on entanglement between a clean and a disordered system, **see the poster of Róbert Juhász!**