Bounding the set of classical correlations of a many-body system

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Outline

- Non-negative polynomials
- Sum-of-squares representations
- Characterizing many-body correlations

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Non-negative polynomials

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Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?

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Given a multivariate non-negative polynomial, does it admit a sum-of-squres representation?

Non-negative polynomials

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Given $p(\vec{x}) \in \mathbb{R}[\vec{x}]$ satisfying $p(\vec{x}) \geq 0 \; \forall \vec{x} \in \mathbb{R}^n$, does $p(\vec{x})$ admit a sum-of-squares (s.o.s.) representation $p(\vec{x}) = \sum_i q_i(\vec{x})^2$?

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- Converse is trivial. When does equivalence hold?
- How powerful is this representation?
- Can one extend it to subsets of $\mathbb{R}^n$?
  - Semialgebraic sets $\begin{cases} f_i(\vec{x}) & = & 0 \\ g_j(\vec{x}) & \geq & 0 \end{cases}$

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Non-constructive proof

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Example: Cauchy-Schwarz Inequality

$$||\vec{x}||^2 \cdot ||\vec{y}||^2 - \langle \vec{x}, \vec{y} \rangle^2 = \sum_{i<j} (x_i y_j - x_j y_i)^2$$

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Semidefinite Programming

- Primal-dual formulation

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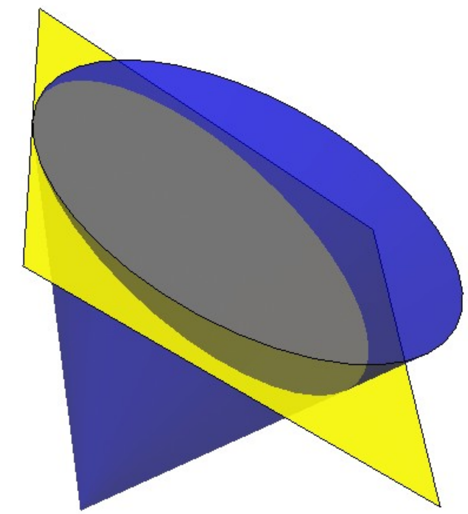

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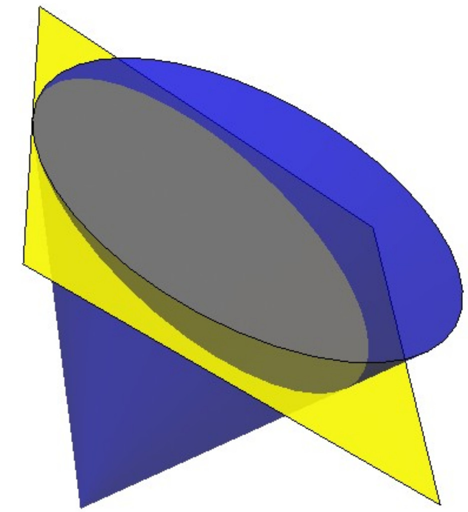

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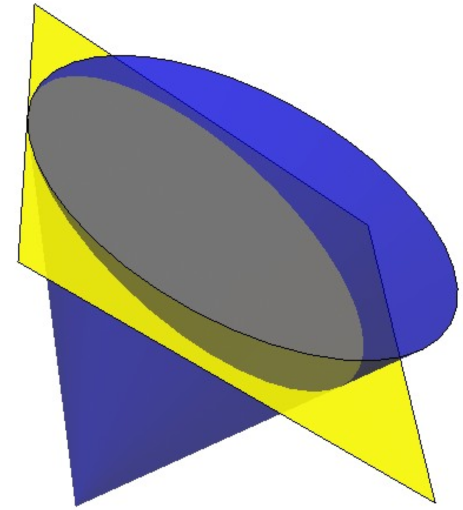

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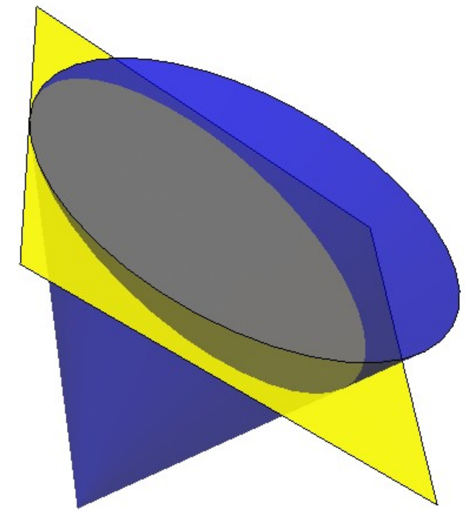

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Let's say they are efficiently solvable

Deciding if $p(\vec{x})$ admits a sos representation is simply an SdP in disguise

An easy one, since one can assume $\deg(q_i) \leq d$

$$(\deg(p) = 2d)$$

$$p(\vec{x}) = \sum_i q_i^2(\vec{x})$$

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$$Q = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} = L^T L \quad L = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

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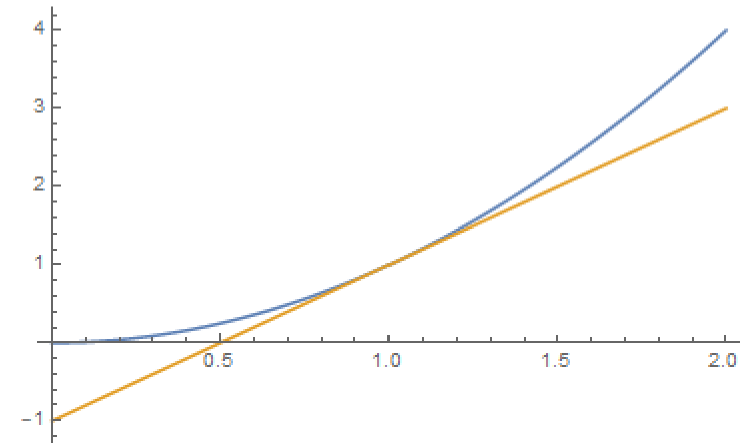

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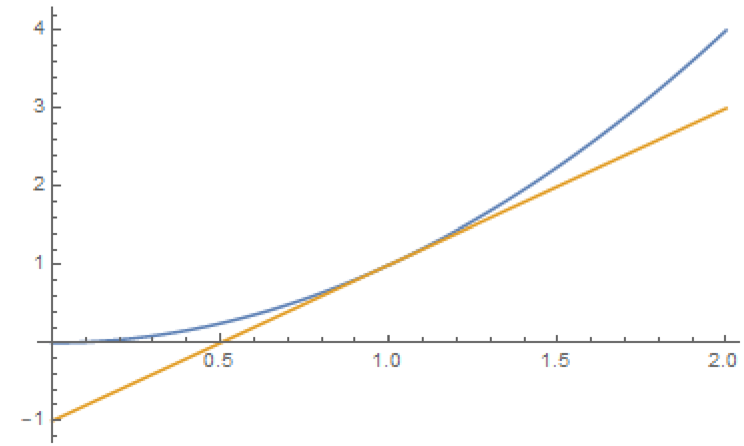

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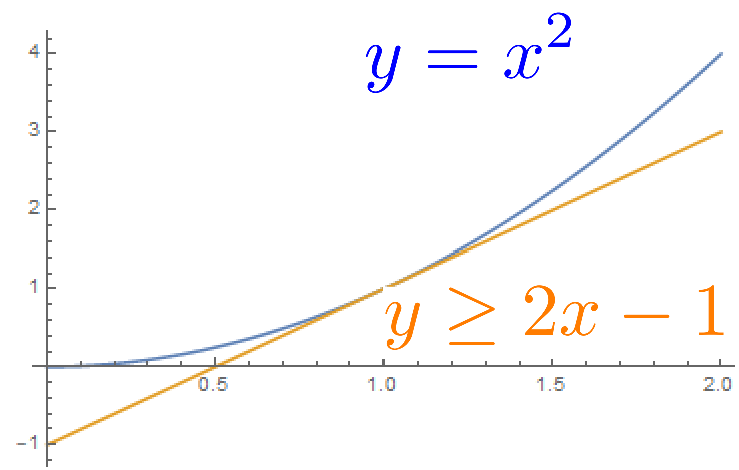

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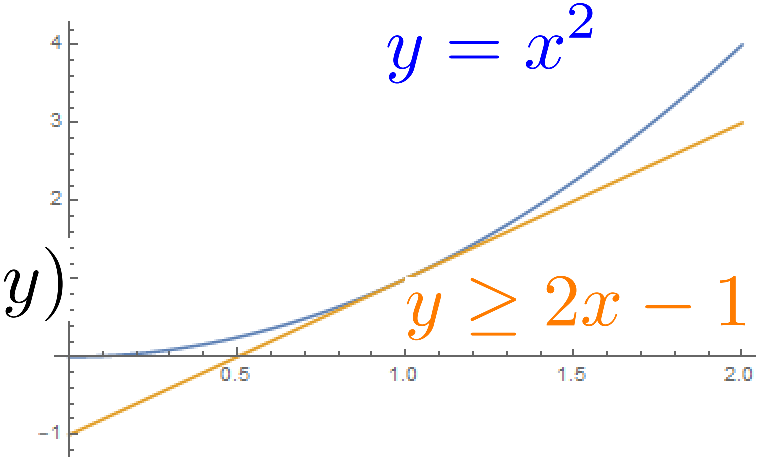

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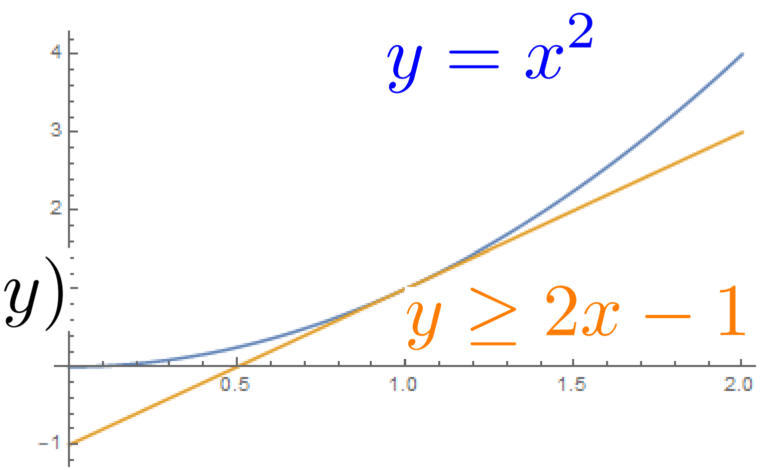

Sums-of-squares modulo ideals are powerful!

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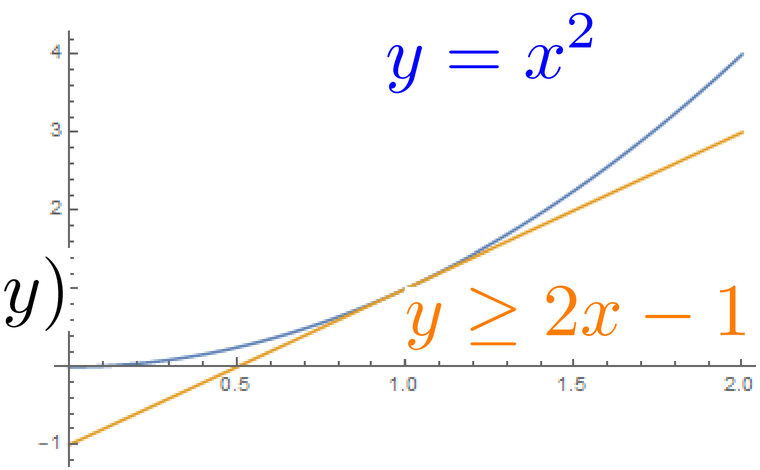

Sums-of-squares modulo ideals are powerful!

$$p(\vec{x}) = p(\vec{x}) + \sum_{i=1}^{k} f_i(\vec{x}) g_i(\vec{x}) \forall\ \vec{x} \in \mathcal{V}$$

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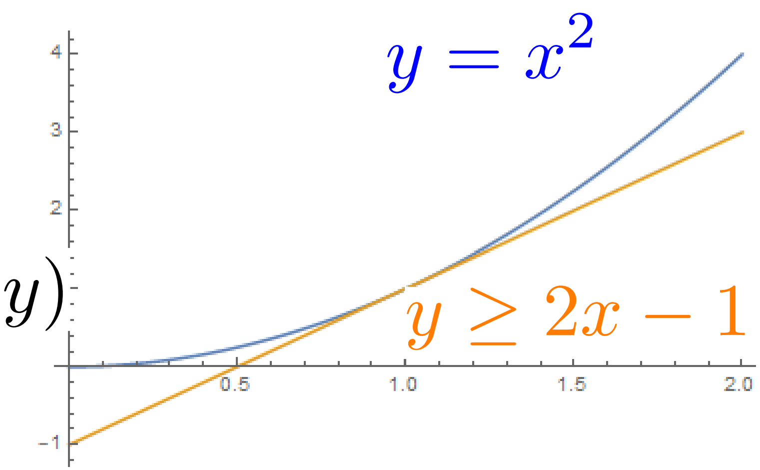

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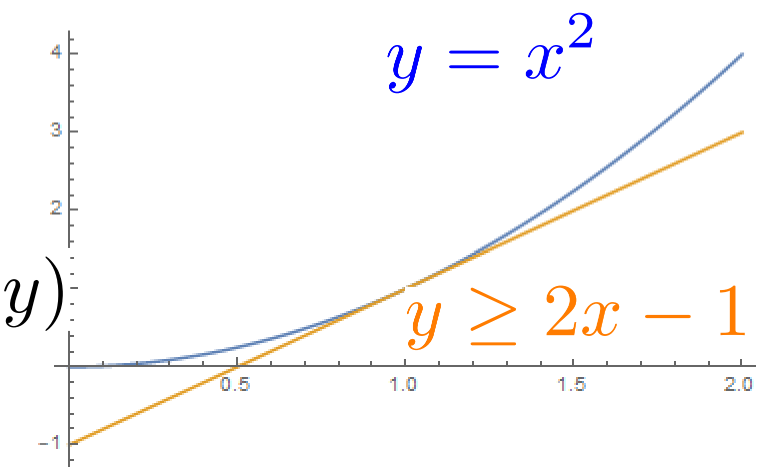

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Ideal generated by $\{f_i\}$

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- *But now the degree of the sos may be unbounded!!*

$$f(x,y) = x^4 - x^2 + y^2 = 0$$

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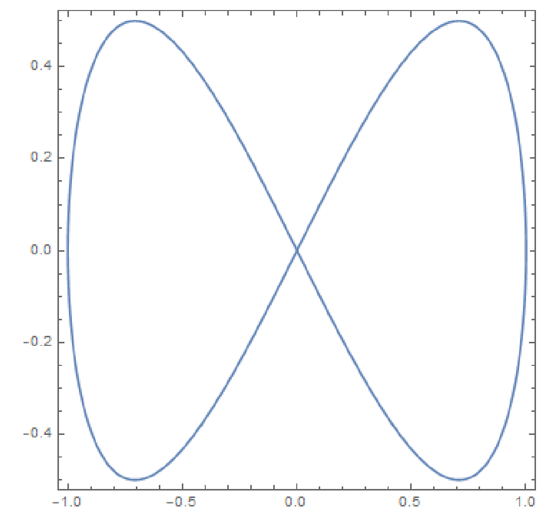

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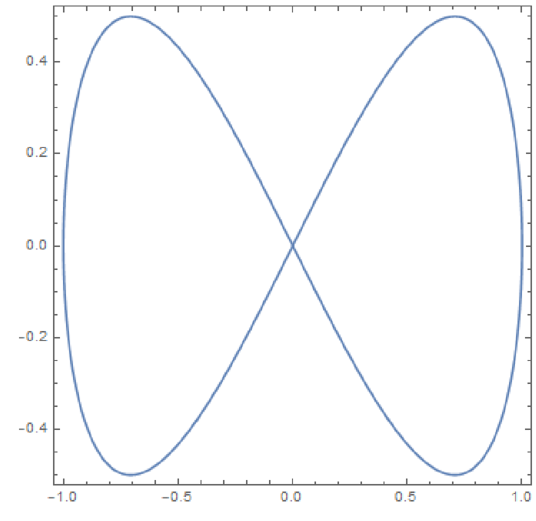

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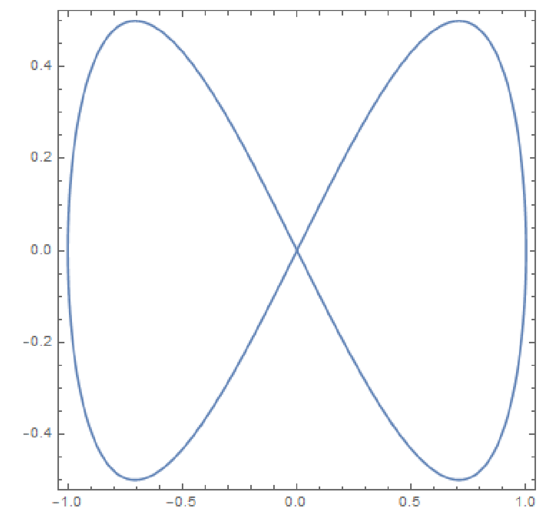

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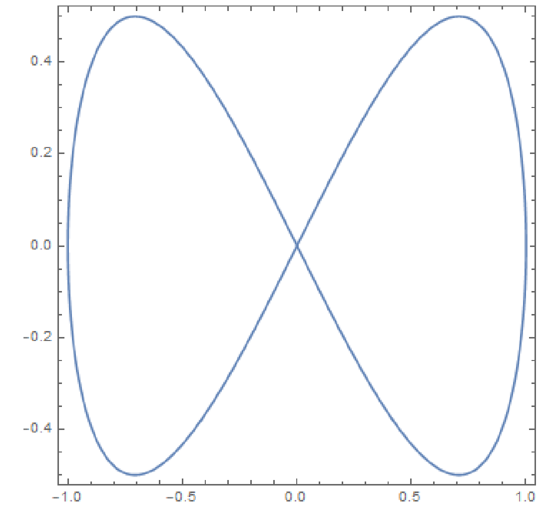

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Moment matrix, 2nd order

[Gouveia, Thomas, Convex Hulls of semialgebraic sets, 2012]
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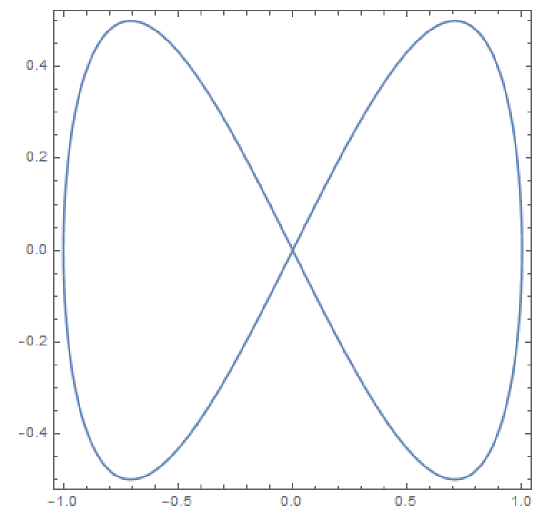

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w_i^j linearizes $x^i y^j$

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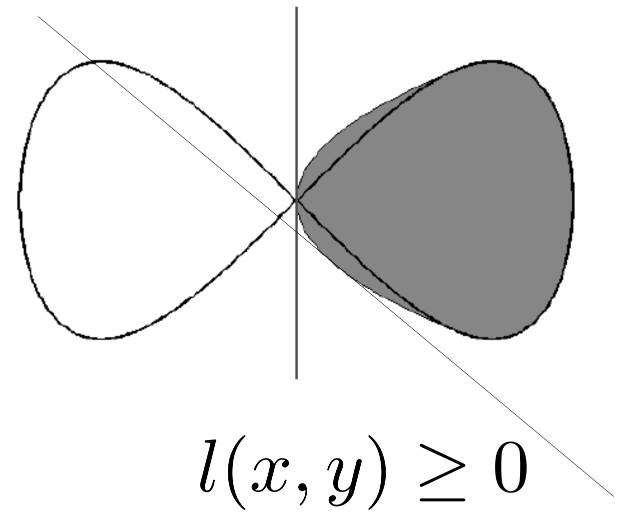

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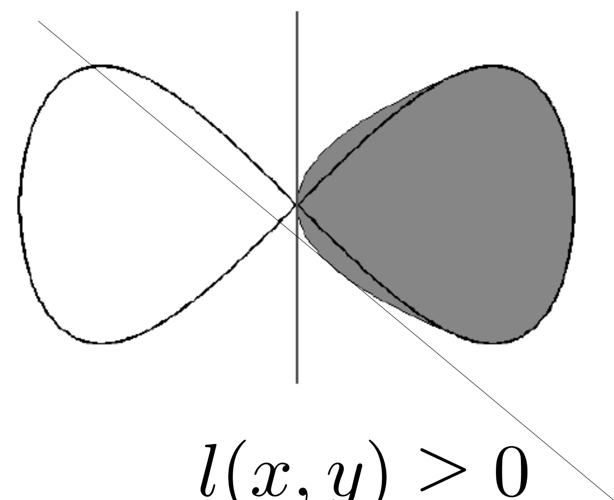

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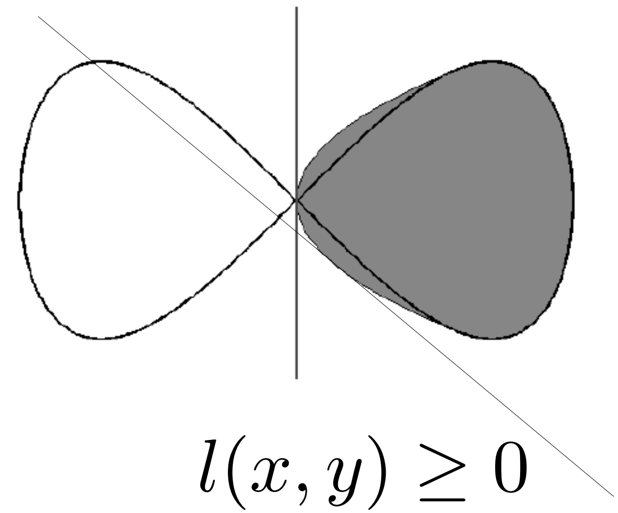

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Moment matrix, 2^{nd} order

$$\begin{pmatrix} x & w_2^0 & w_1^1 \\ w_2^0 & w_3^0 & w_2^1 \\ w_1^1 & w_2^1 & w_1^2 \end{pmatrix}$$

Shifted moment matrix by x, 1^{st} order

[Gouveia, Thomas, Convex Hulls of semialgebraic sets, 2012]
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sums-of squares

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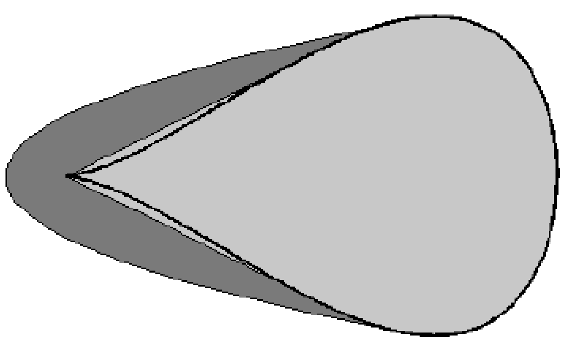

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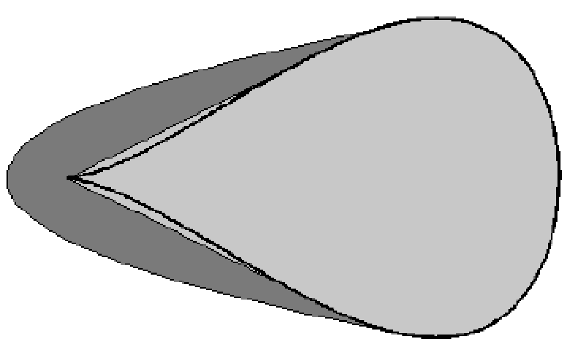

Increasing the degree of the sos σ_i gives more representability power

MAX PLANCK INSTITUTE OF QUANTUM OPTICS

Outline

- Non-negative polynomials
- Sum-of-squares representations
- Characterizing many-body correlations

A bit of Quantum Info

- Bell-type experiment

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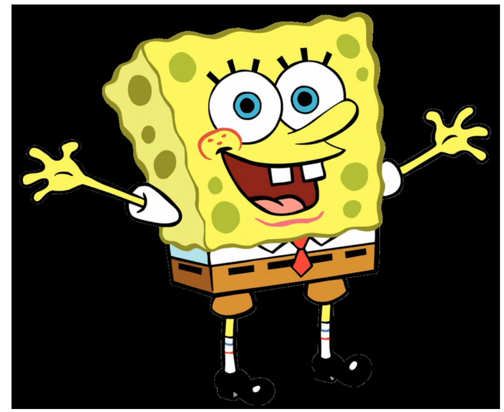

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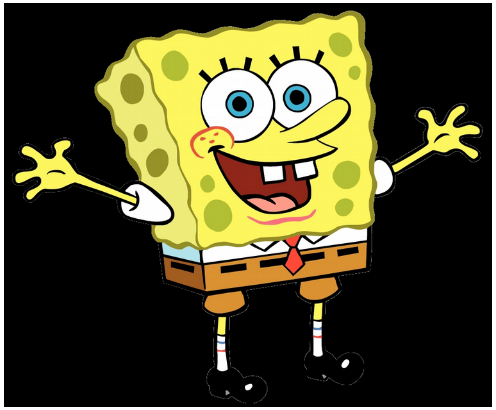

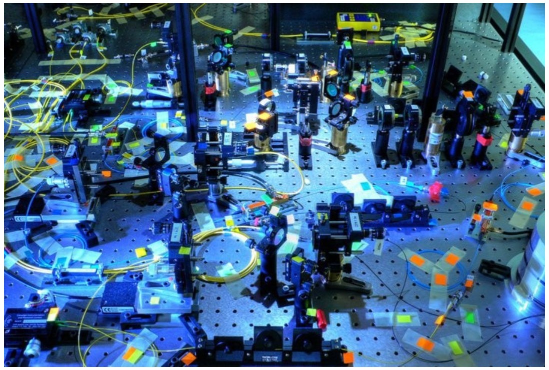

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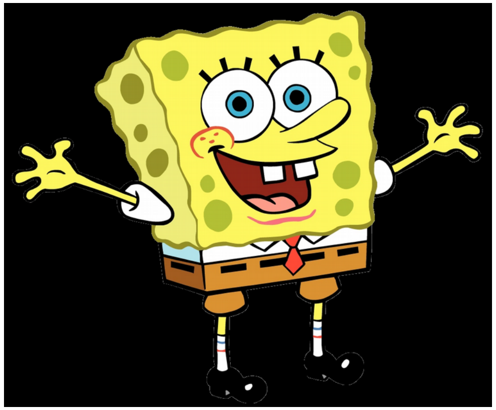

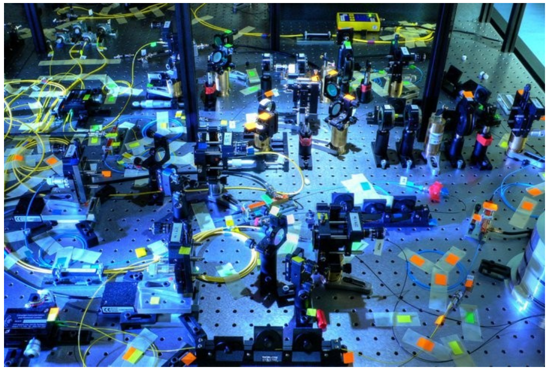

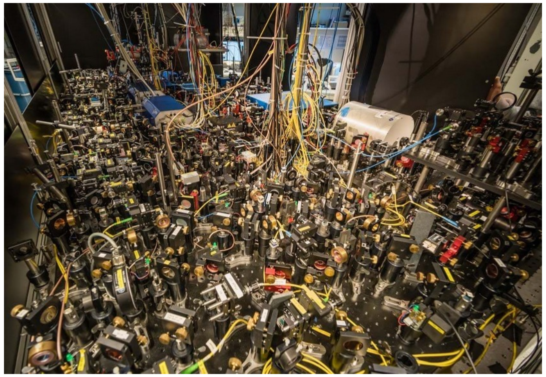

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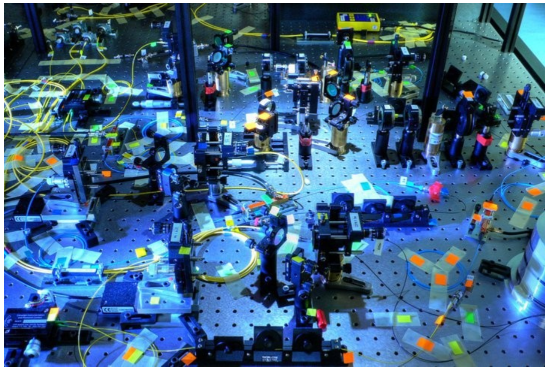

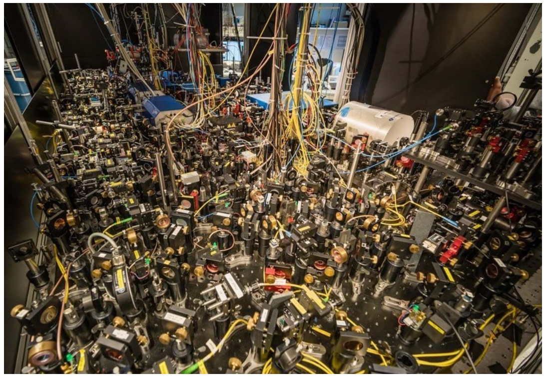

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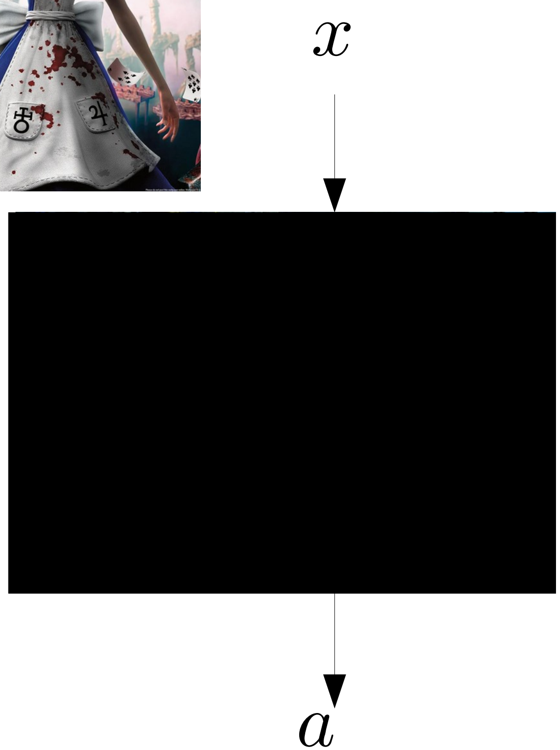

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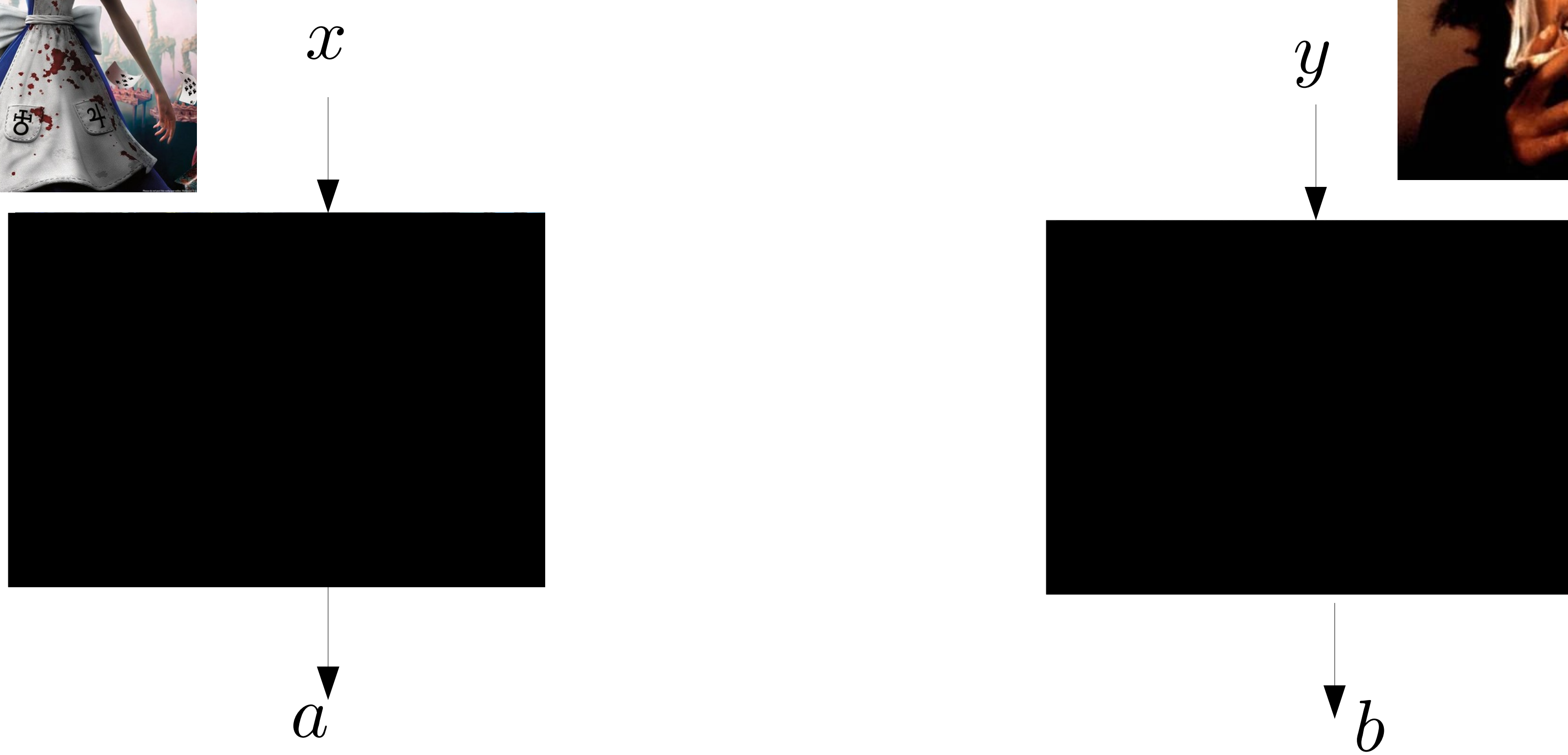

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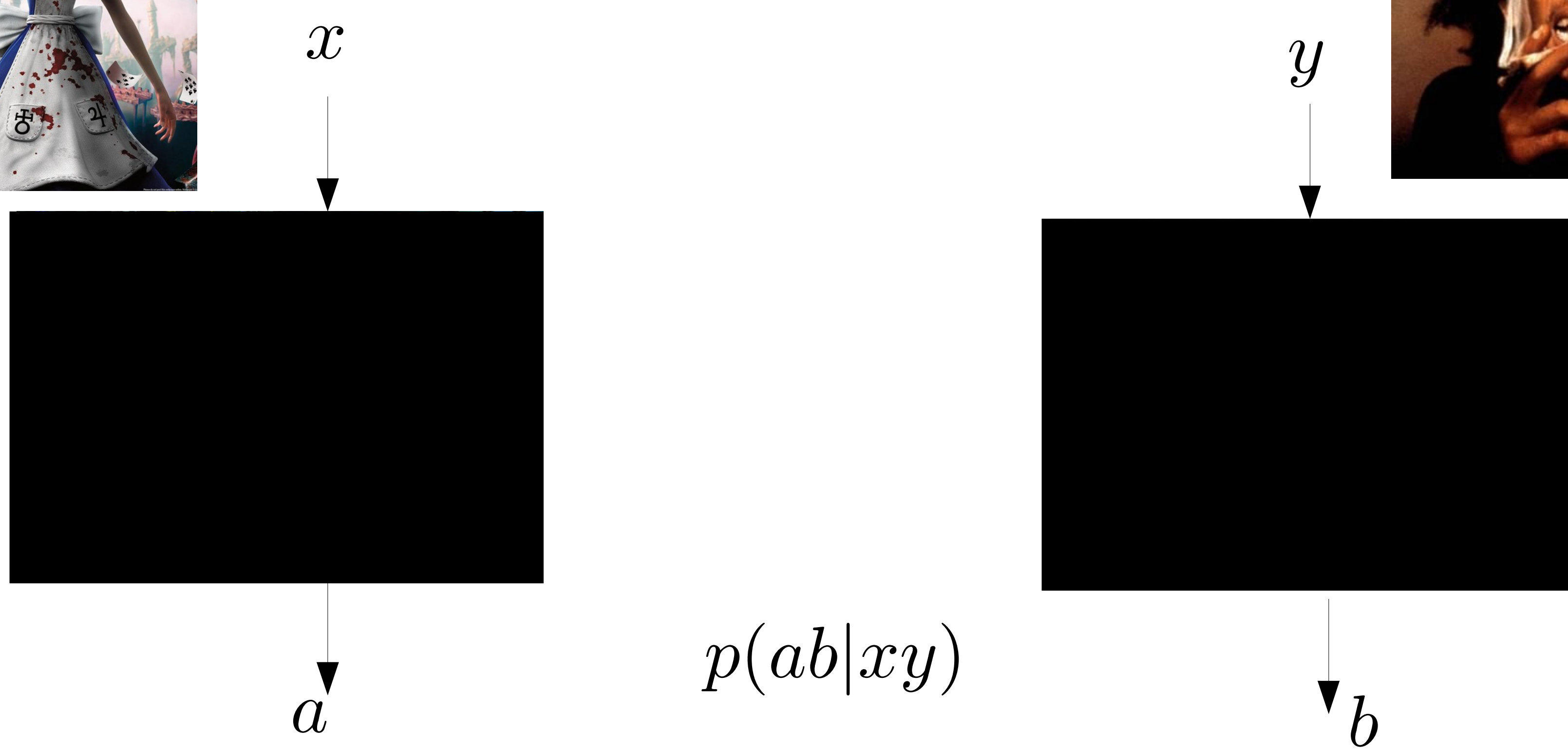

Nonlocality in many-body quantum systems

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- The complexity of the problem

Nonlocality in many-body quantum systems

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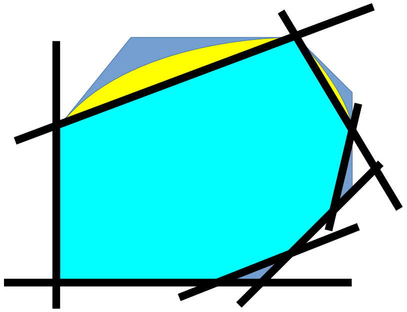
Finding all Bell inequalities ⟷ Convex Hull problem



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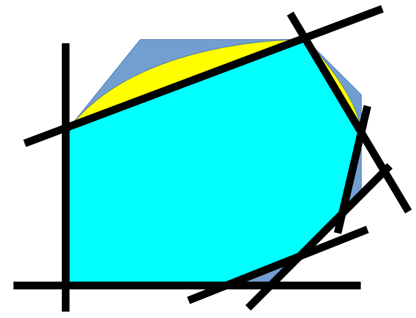
(n, m, d) scenario



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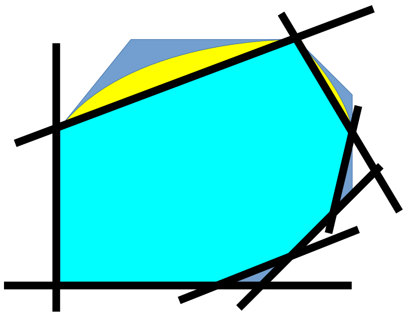

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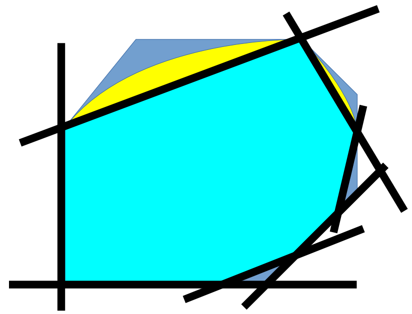

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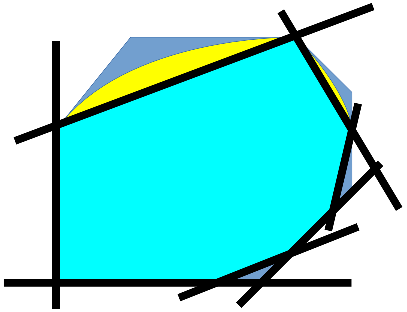

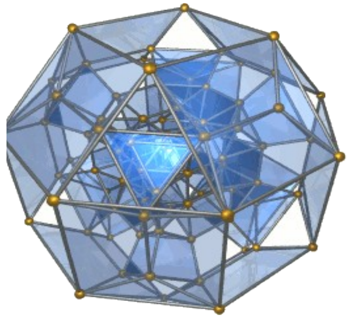

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Dimension of the Local Polytope $D \approx (md)^n$

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[B. Chazelle, *An optimal convex hull algorithm in any fixed dimension, Discrete Comput. Geom.* **10** 377409 (1993)]

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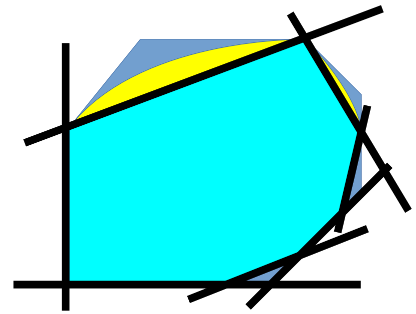

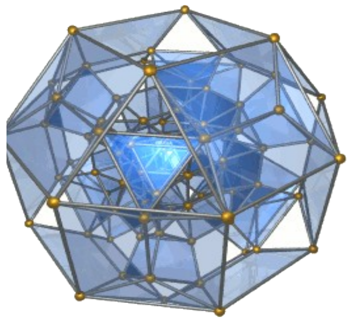

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Examples

[redacted]

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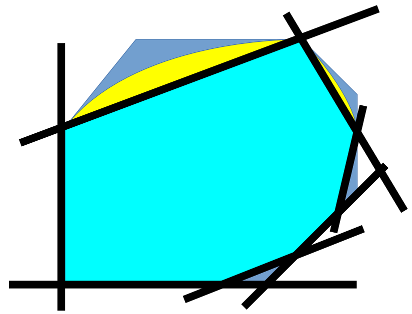

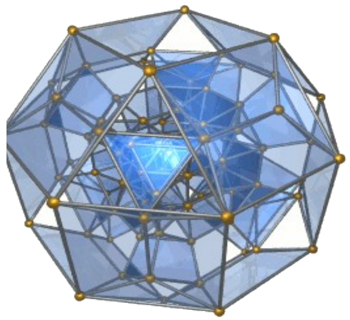

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$(2, 2, 2) \longrightarrow O(\text{ms})$

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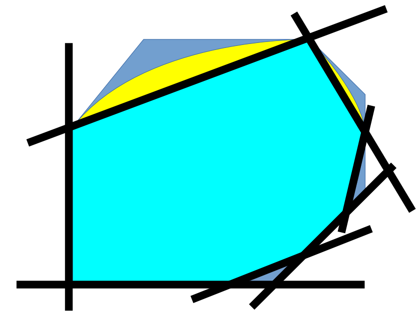

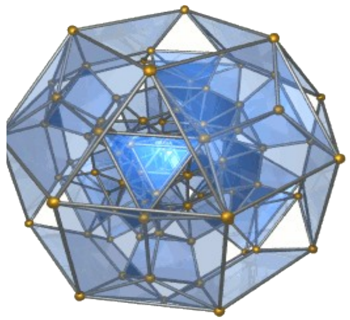

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$(2, 2, 2) \longrightarrow O(\mathrm{ms})$
$(3, 2, 2) \longrightarrow 5'$

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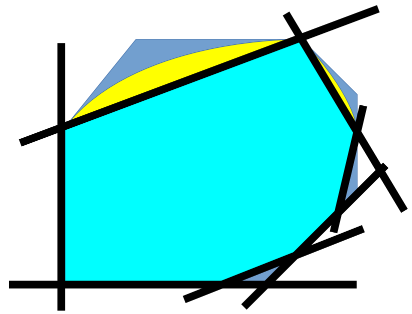
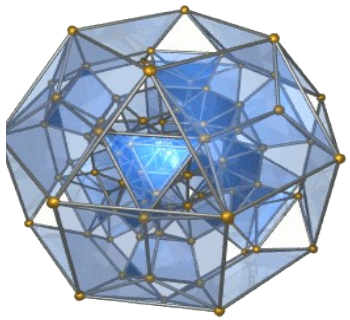
- *The complexity of the problem*

Finding all Bell inequalities ⟷ *Convex Hull problem*





(n, m, d) *scenario*

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Examples

$(2,2,2) \longrightarrow O(\mathrm{ms})$

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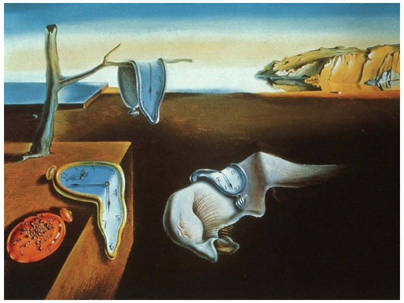
$(4,2,2) \longrightarrow 10^{67}$ years

$\vdots$



[S. Dalí *The persistence of memory (1931)*]

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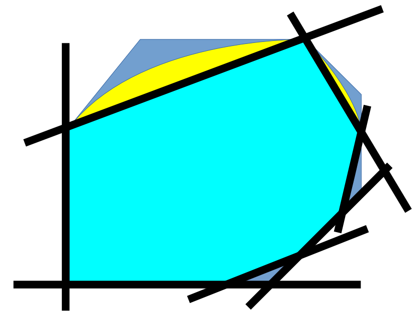

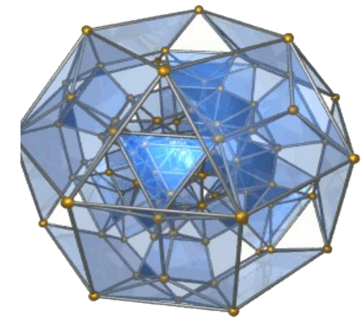

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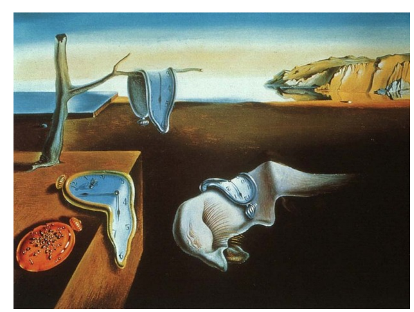
$\vdots$

$(10^4,2,2) \longrightarrow 10^{10^{10^{4.67867\ldots}}}$ basically any timescale you want



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- Reducing the mathematical complexity

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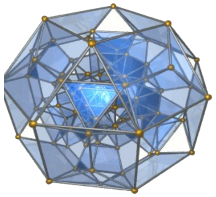
Polytope				
Dimension				
Vertices				

Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

Polytope	$\mathbb{P}_n$			
Dimension	$3^n - 1$			
Vertices	2^{2n}			



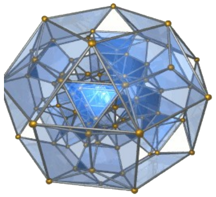
Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

Lower order correlators

Polytope	$\mathbb{P}_n$			
Dimension	$3^n - 1$			
Vertices	2^{2n}			



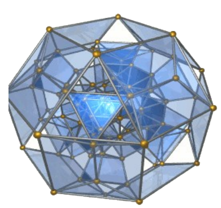
Nonlocality in many-body quantum systems

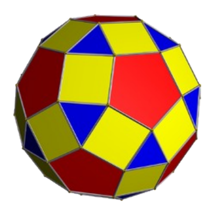
- Reducing the mathematical complexity

2-body

Lower order correlators

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





Nonlocality in many-body quantum systems

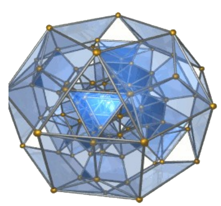
- Reducing the mathematical complexity

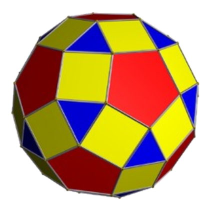
2-body

Lower order correlators

Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





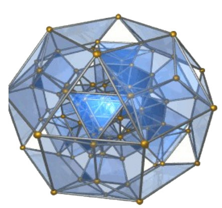
Nonlocality in many-body quantum systems

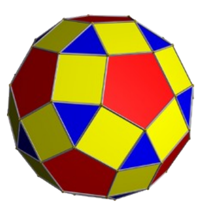
- *Reducing the mathematical complexity*

2-body ← Lower order correlators

Cyclic group ← Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak
J. Phys. A: Math. Theor. **47** 424024 (2014)]

Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

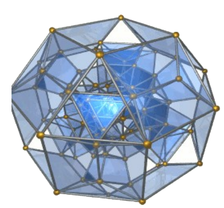
Cyclic group

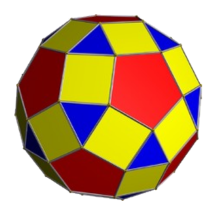
Symmetric group

Lower order correlators

Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak *J. Phys. A: Math. Theor.* **47** 424024 (2014)]

[JT, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín *Science* **344** 1256 (2014)]

Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

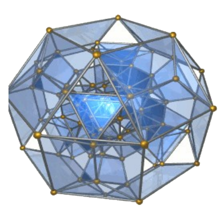
Cyclic group

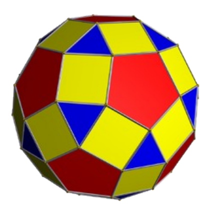
Symmetric group

Lower order correlators

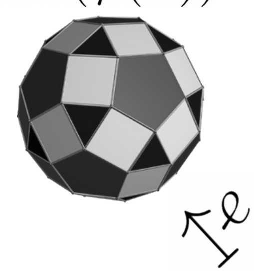
Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$	$\mathbb{P}_2^S$	
Dimension	$3^n - 1$	$2n^2$	5	
Vertices	2^{2n}	2^{2n}	$\binom{n+3}{3}$	





$\mathrm{CH}(\varphi(\mathbb{T}))$



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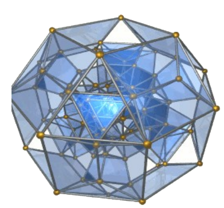
Science **344** 1256 (2014)]

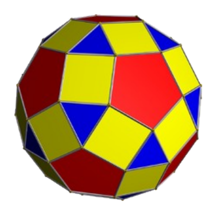
Nonlocality in many-body quantum systems

- *Reducing the mathematical complexity*

2-body ↗ *Cyclic group* ↗ *Symmetric group* ↗ *Deterministic Local Strategies Parametrization*

Polytope	$\mathbb{P}_n$ → *Lower order correlators*	$\mathbb{P}_2$ → Action of a symmetry group	$\mathbb{P}_2^{\mathcal{S}}$ → Extremality analysis	
Dimension	$3^n - 1$	$2n^2$	5	
Vertices	2^{2n}	2^{2n}	$\binom{n+3}{3}$	





$\mathrm{CH}(\varphi(\mathbb{T}))$

$\uparrow \varphi$

Parameter space $\mathbb{T}$

[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak *J. Phys. A: Math. Theor.* **47** 424024 (2014)]

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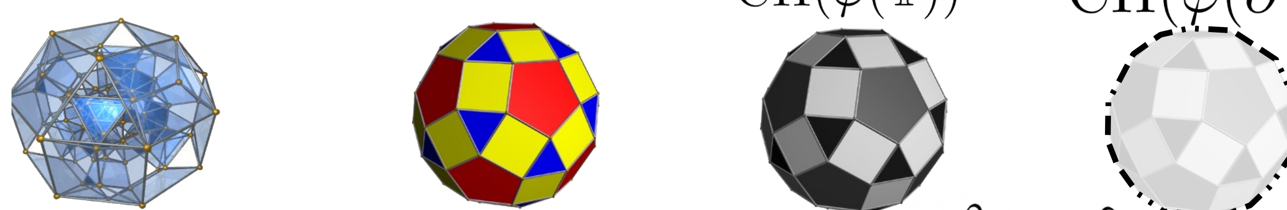
Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body
Cyclic group
Symmetric group
Deterministic Local Strategies Parametrization

Polytope	$\mathbb{P}_n$	Lower order correlators	$\mathbb{P}_2$	Action of a symmetry group	$\mathbb{P}_2^S$	Extremality analysis	$\mathbb{P}_2^S$
Dimension	$3^n - 1$		$2n^2$		5		5
Vertices	2^{2n}		2^{2n}		$\binom{n+3}{3}$		$2(n^2+1)$

$\mathrm{CH}(\varphi(\mathbb{T}))$ $\mathrm{CH}(\varphi(\partial\mathbb{T}))$



Parameter space $\mathbb{T}$

[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak *J. Phys. A: Math. Theor.* **47** 424024 (2014)]

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Some applications

Some applications

PRL **119,** 230402 (2017) PHYSICAL REVIEW LETTERS week ending 8 DECEMBER 2017

Bounding the Set of Classical Correlations of a Many-Body System

Matteo Fadel[1,*] and Jordi Tura[2,3,†]

We present a method to certify the presence of Bell correlations in experimentally observed statistics, and to obtain new Bell inequalities. Our approach is based on relaxing the conditions defining the set of correlations obeying a local hidden variable model, yielding a convergent hierarchy of semidefinite programs (SDP's). Because the size of these SDP's is independent of the number of parties involved, this technique allows us to characterize correlations in many-body systems. As an example, we illustrate our method with the experimental data presented in Science **352**, 441 (2016).

DOI: 10.1103/PhysRevLett.119.230402

Some applications

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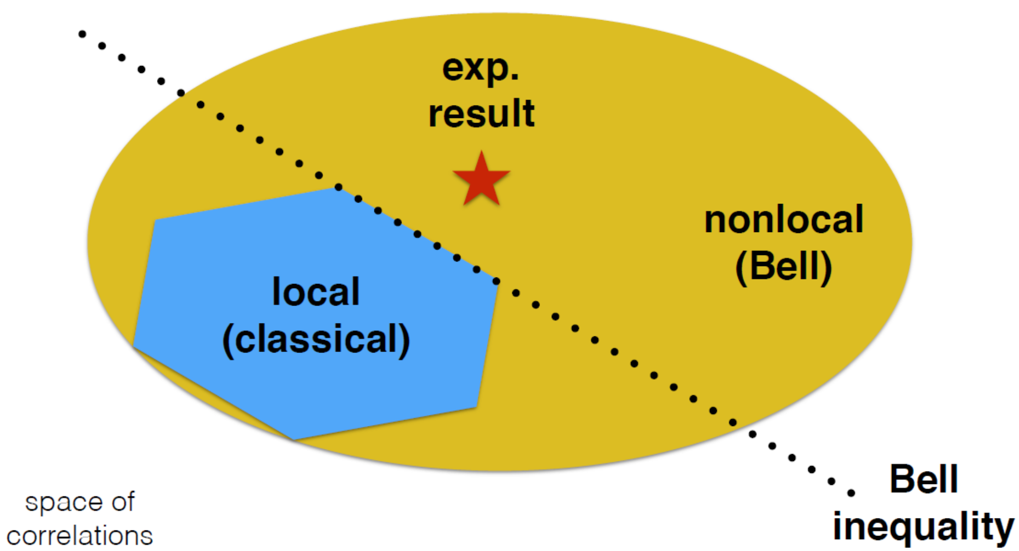

Some applications

PRL **119**, 230402 (2017) PHYSICAL REVIEW LETTERS week ending 8 DECEMBER 2017

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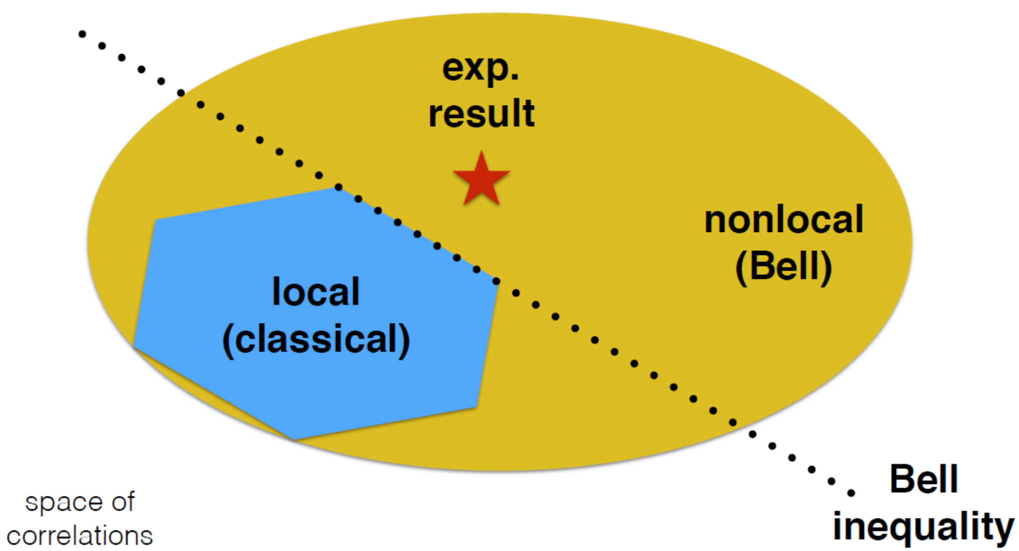

- *Certify Bell correlations from experiments*

Some applications

PRL 119, 230402 (2017) PHYSICAL REVIEW LETTERS week ending 8 DECEMBER 2017

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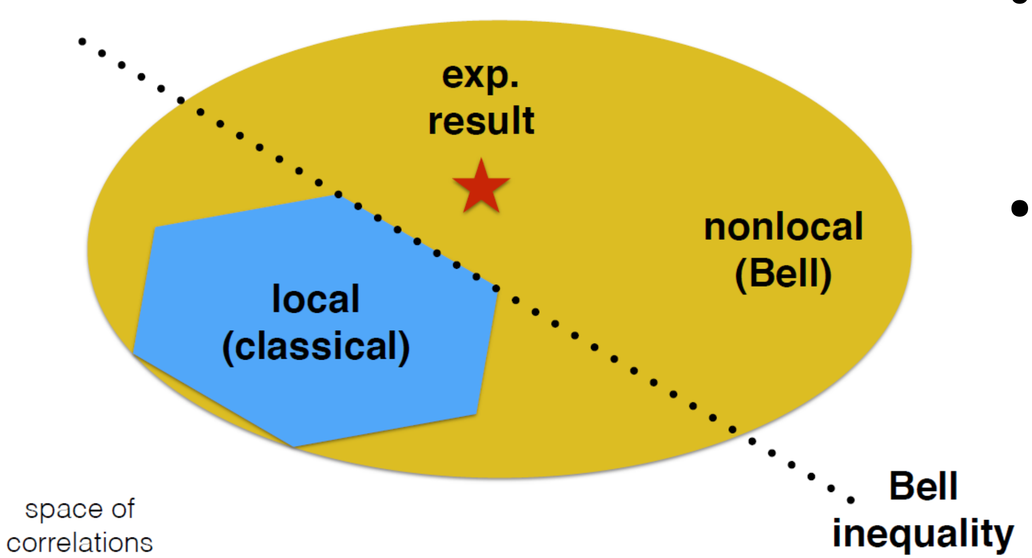

- *Certify Bell correlations from experiments*
- *Find new Bell Inequalities*

Some applications

Some applications

We consider Bell inequalities of the form

$$\sum_{k} \sum_{j_1 \leq \ldots \leq j_k} \alpha_{j_1 \ldots j_k} \mathcal{S}_{j_1 \ldots j_k} + \beta_C \geq 0$$

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with $$\mathcal{S}_{j_1 \ldots j_k} = \sum_{\substack{i_1, \ldots, i_k = 1 \\ \text{all } i\text{'s different}}}^{N} \langle \mathcal{M}_{j_1}^{(i_1)} \ldots \mathcal{M}_{j_k}^{(i_k)} \rangle$$

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Dimension depends on

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Dimension depends on

- Order of the correlators

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Dimension depends on

- Order of the correlators
- #Measurements

Some applications

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Dimension depends on

- Order of the correlators
- #Measurements
- #outcomes

Some applications

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Dimension depends on

- Order of the correlators
- #Measurements
- #outcomes

Does NOT depend on

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Dimension depends on

- Order of the correlators
- #Measurements
- #outcomes

Does NOT depend on

- #Parties

Previous results

Previous results

Science **344**, 1256 (2014)

Detecting nonlocality in many-body quantum states

J. Tura,[1] R. Augusiak,[1*] A. B. Sainz,[1] T. Vértesi,[2] M. Lewenstein,[1,3] A. Acín[1,3]

Previous results

Science **344**, 1256 (2014)

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Example $-2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} + 2N \geq 0$

Previous results

Science **344**, 1256 (2014)

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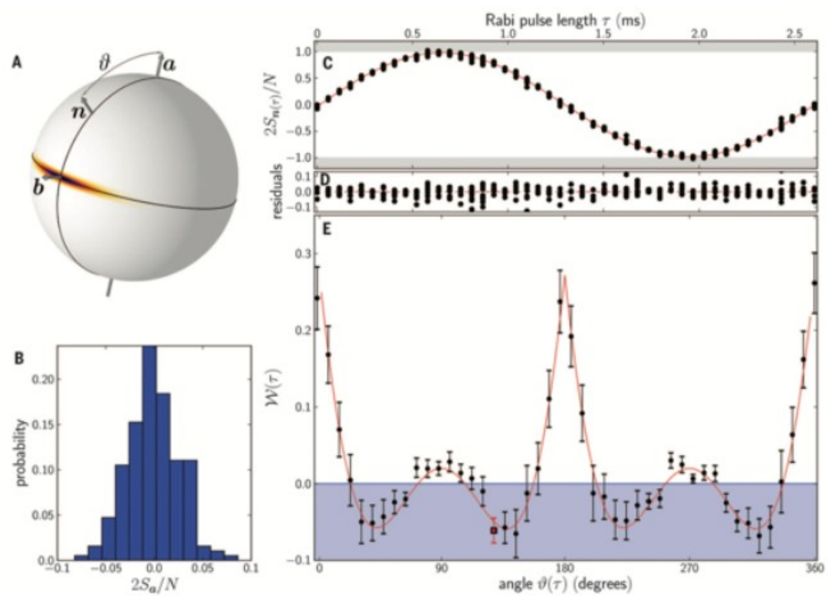

Science **352**, 441 (2016)

Bell correlations in a Bose-Einstein condensate

Roman Schmied,[1*] Jean-Daniel Bancal,[2,4*] Baptiste Allard,[1*] Matteo Fadel,[1]
Valerio Scarani,[2,3] Philipp Treutlein,[1†] Nicolas Sangouard[4†]

$$\hat{W} = -\left|\frac{\hat{S}_n}{N/2}\right| + (a\cdot n)^2\frac{\hat{S}_a^2}{N/4} + 1 - (a\cdot n)^2$$

Previous results

Science **344**, 1256 (2014)

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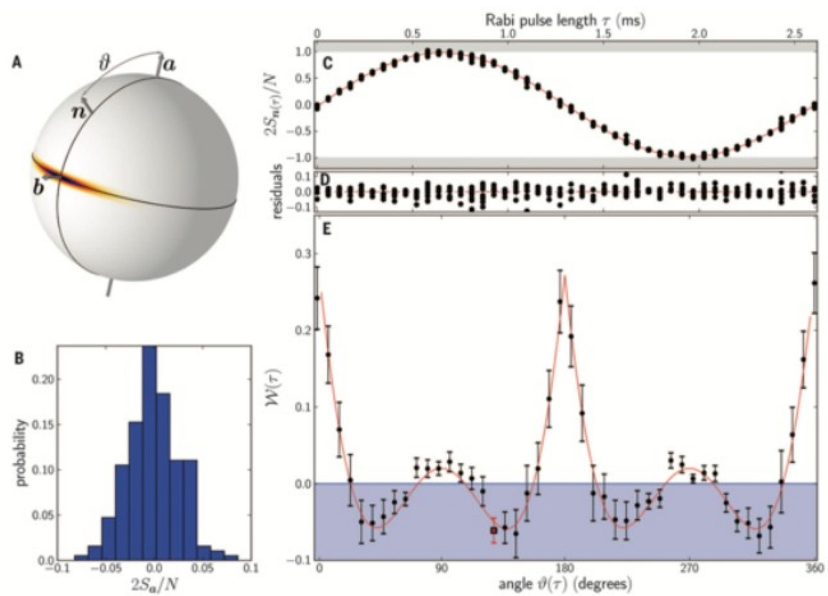

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PRL **118**, 140401 (2017) PHYSICAL REVIEW LETTERS week ending 7 APRIL 2017

Bell Correlations in Spin-Squeezed States of 500 000 Atoms

Nils J. Engelsen, Rajiv Krishnakumar, Onur Hosten, and Mark A. Kasevich*

The Local polytope

- Solving for a few values of N…

The Local polytope

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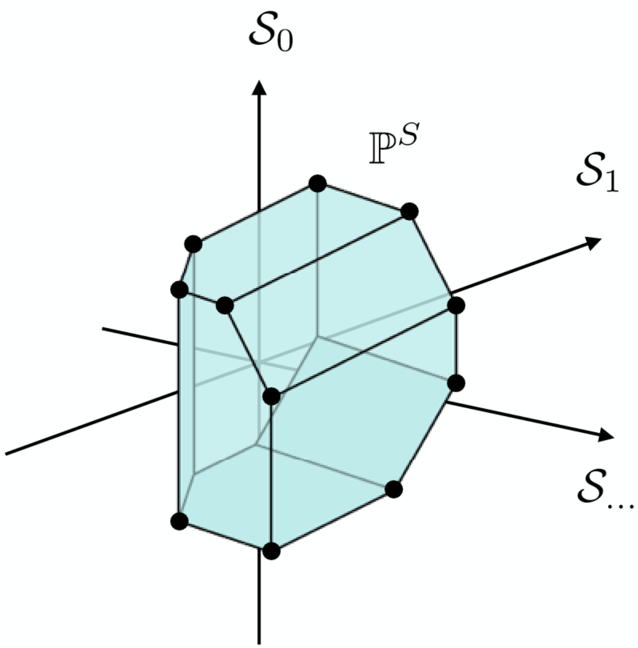
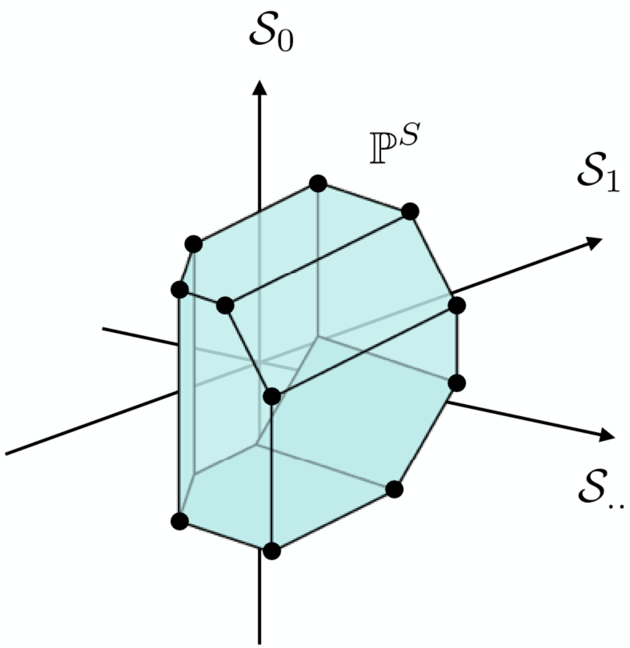


The Local polytope

- Solving for a few values of N…



25314	-2531	268	134	-15	1
26334	-2601	306	136	-17	1
25994	-2584	287	136	-16	1
13264	-1197	325	57	-18	1
12734	-1161	306	56	-17	1
9858	-981	270	53	-16	2
31754	-3128	867	167	-51	6
11138	-1083	308	57	-18	2
29470	-3021	970	171	-60	10
73560	-8307	3465	544	-245	70
125548	-14079	5791	912	-405	112
7830	-941	486	70	-39	15
53590	-6422	3297	475	-263	100
13808	-1473	213	84	-13	2
8712	-916	272	53	-17	3
7294	-737	154	40	-9	1
29820	-3667	3011	456	-409	280
68590	-1710	1900	1359	-1800	1900
34166	-4313	2892	399	-290	154
16086	430	-2123	3	-35	164
76470	1886	-10051	15	-151	760
7980	627	-1140	46	-110	190
53706	1418	-7063	9	-115	544
5674	138	-743	1	-11	56
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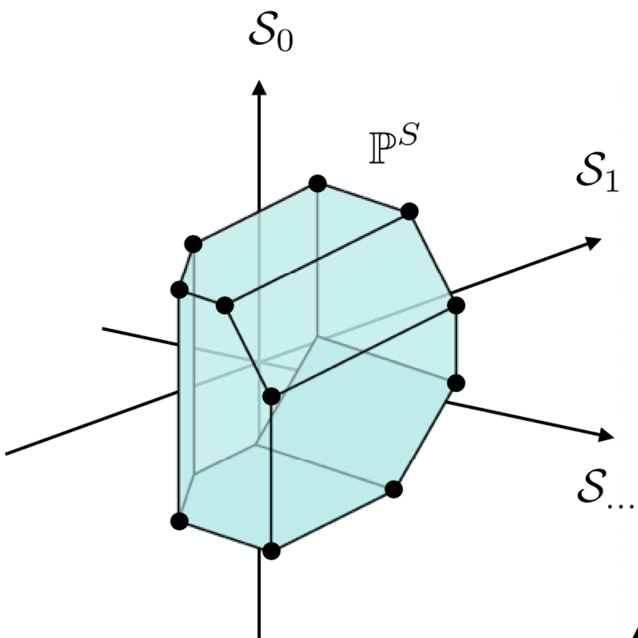
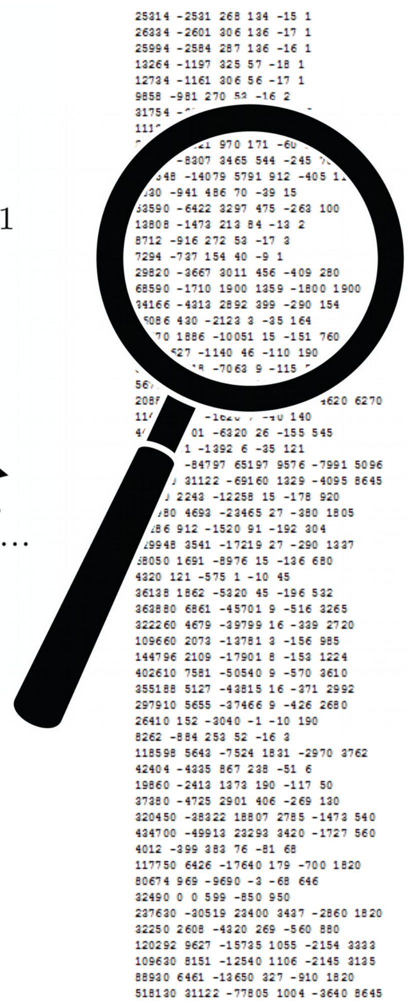
11490	437	-1620	7	-40	140
44970	1701	-6320	26	-155	545
9874	381	-1392	6	-35	121
661284	-84797	65197	9576	-7991	5096
456380	31122	-69160	1329	-4095	8645
93970	2243	-12258	15	-178	920
178980	4693	-23465	27	-380	1805
11286	912	-1520	91	-192	304
129948	3541	-17219	27	-290	1337
68050	1691	-8976	15	-136	680
4320	121	-575	1	-10	45
36138	1862	-5320	45	-196	532
363880	6861	-45701	9	-516	3265
322260	4679	-39799	16	-339	2720
109660	2073	-13781	3	-156	985
144796	2109	-17901	8	-153	1224
402610	7581	-50540	9	-570	3610
355188	5127	-43815	16	-371	2992
297910	5655	-37466	9	-426	2680
26410	152	-3040	-1	-10	190
8262	-884	253	52	-16	3
118598	5643	-7524	1831	-2970	3762
42404	-4335	867	238	-51	6
19860	-2413	1373	190	-117	50
37380	-4725	2901	406	-269	130
320450	-38322	18807	2785	-1473	540
434700	-49913	23293	3420	-1727	560
4012	-399	383	76	-81	68
117750	6426	-17640	179	-700	1820
80674	969	-9690	-3	-68	646
32490	0	0	599	-850	950
237630	-30519	23400	3437	-2860	1820
32250	2608	-4320	269	-560	880
120292	9627	-15735	1055	-2154	3333
109630	8151	-12540	1106	-2145	3135
88930	6461	-13650	327	-910	1820
518130	31122	-77805	1004	-3640	8645
232524	13293	-34965	401	-1498	3731

The Local polytope

- Solving for a few values of N…





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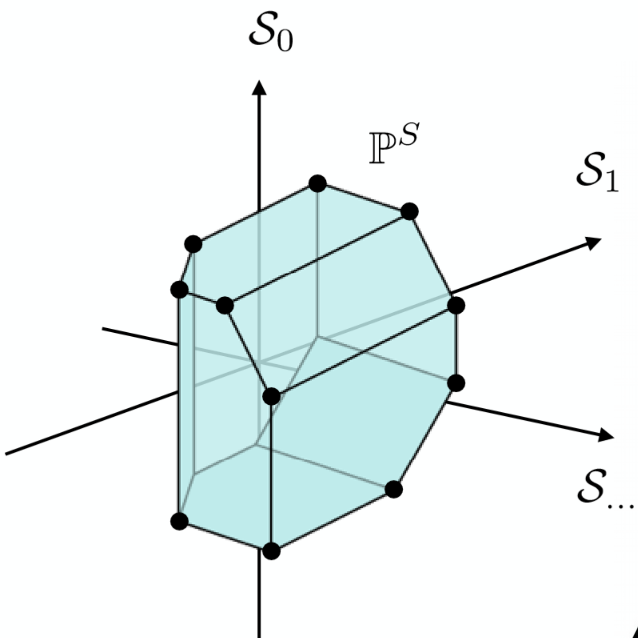

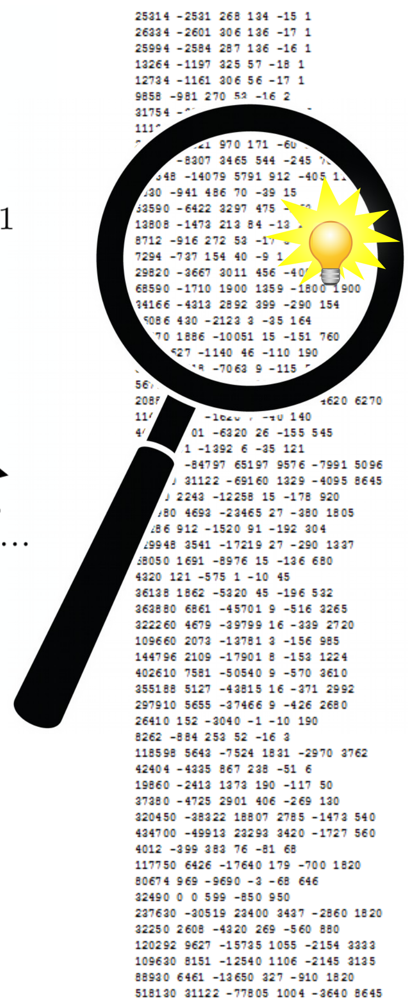

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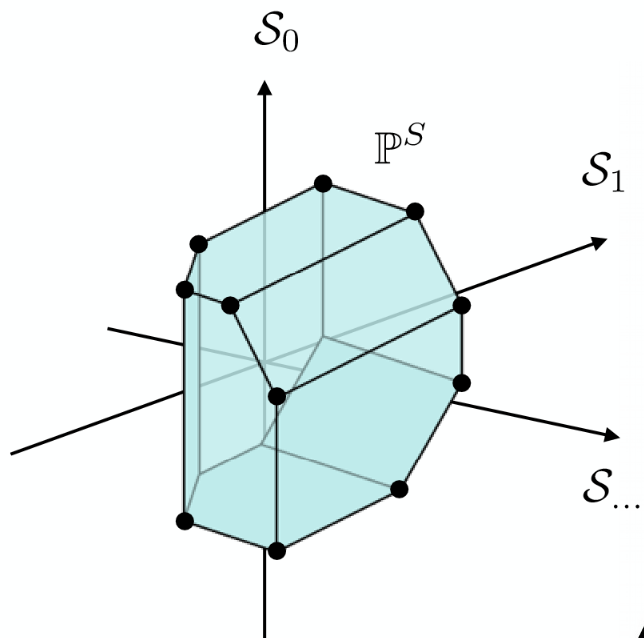

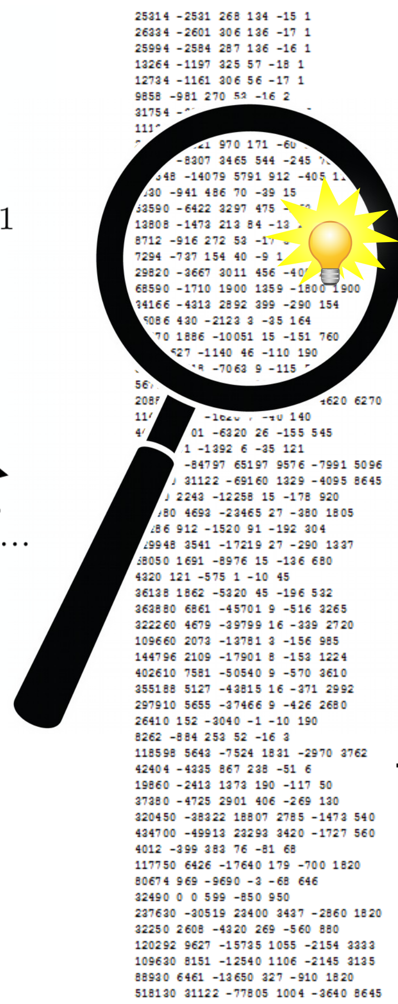

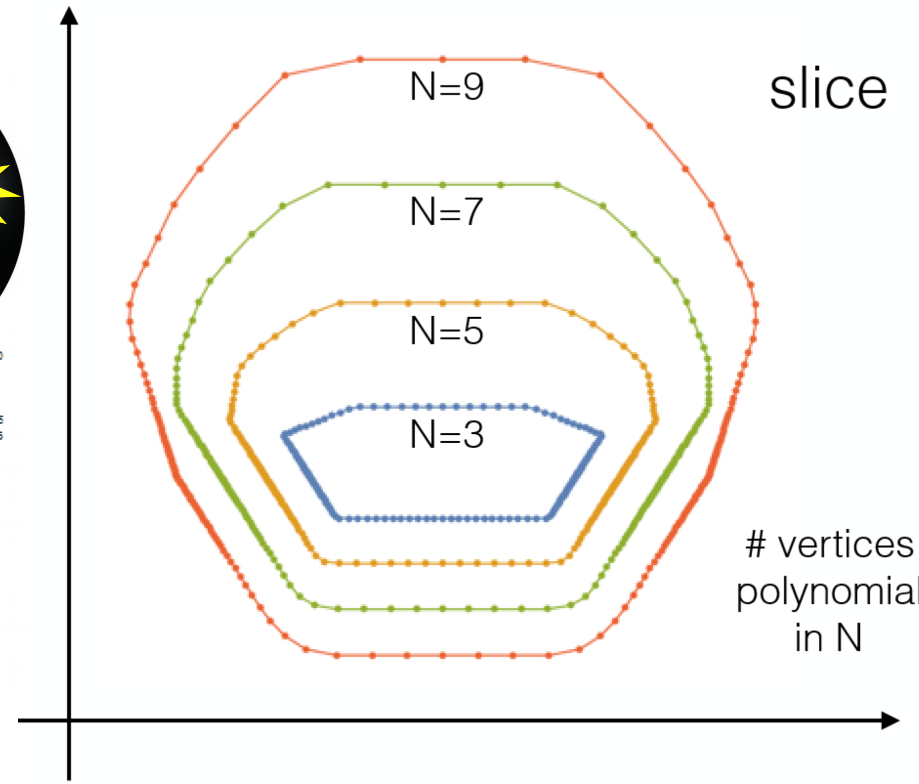

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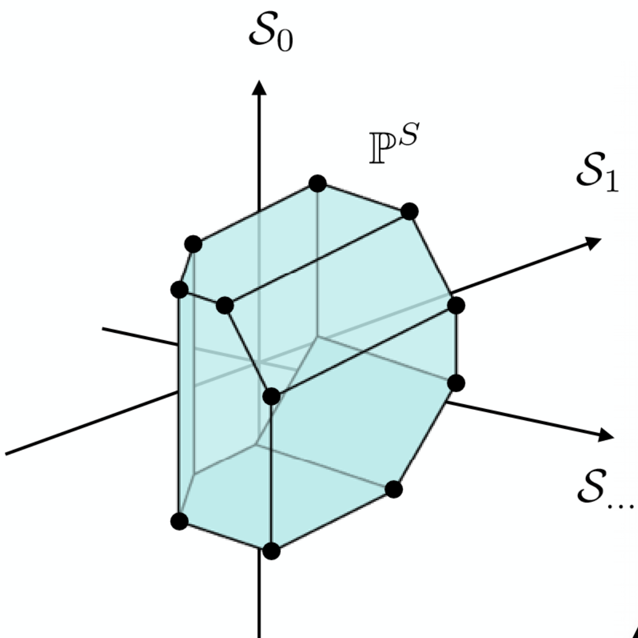
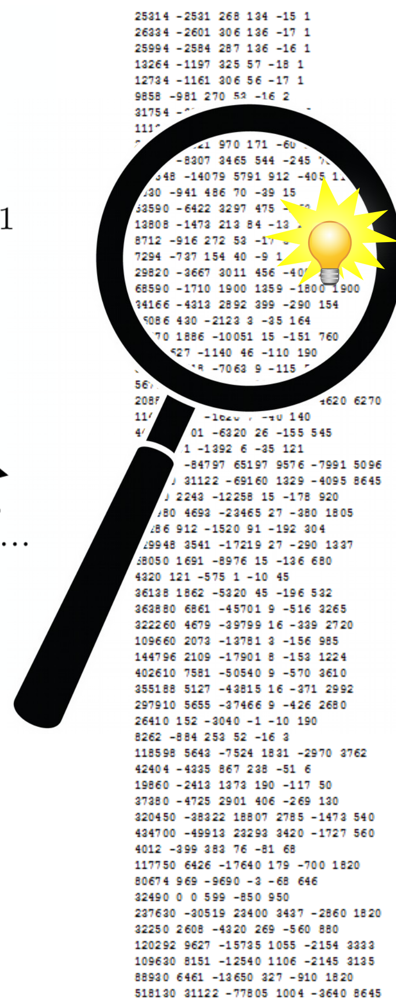
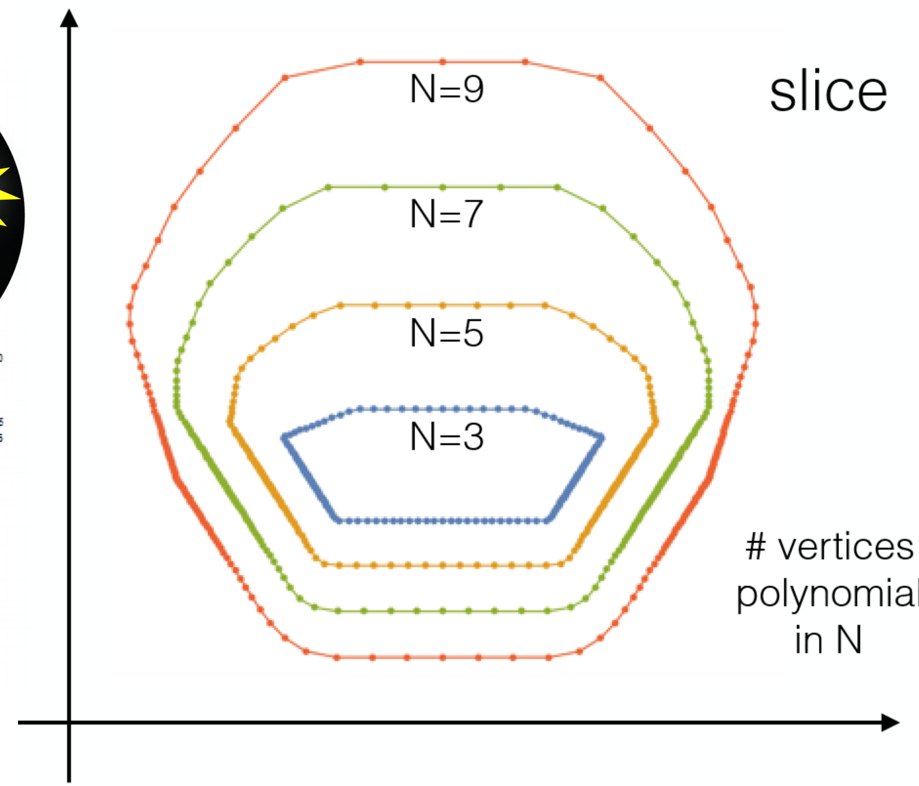


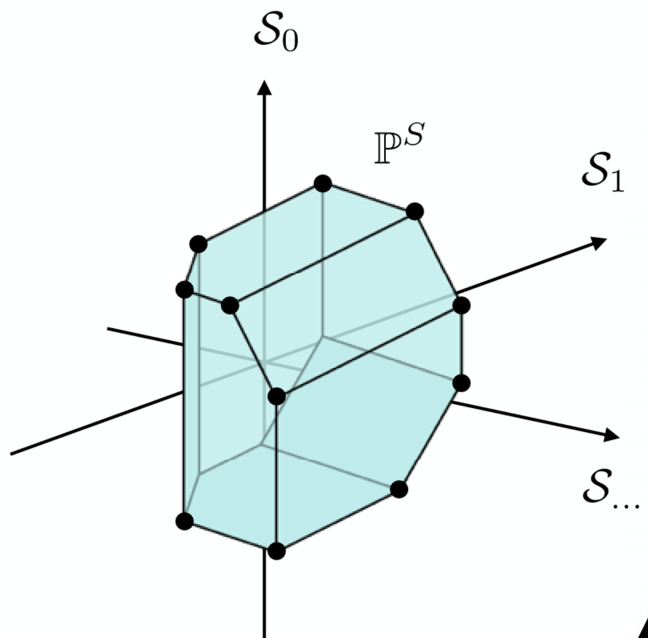


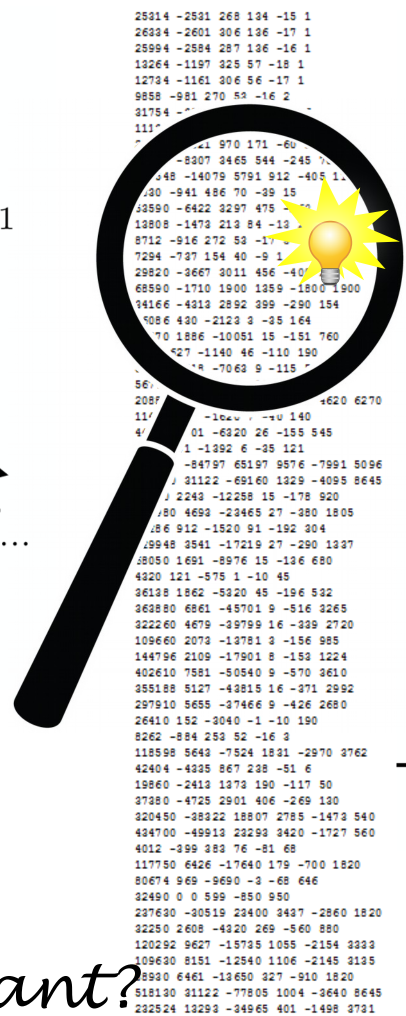


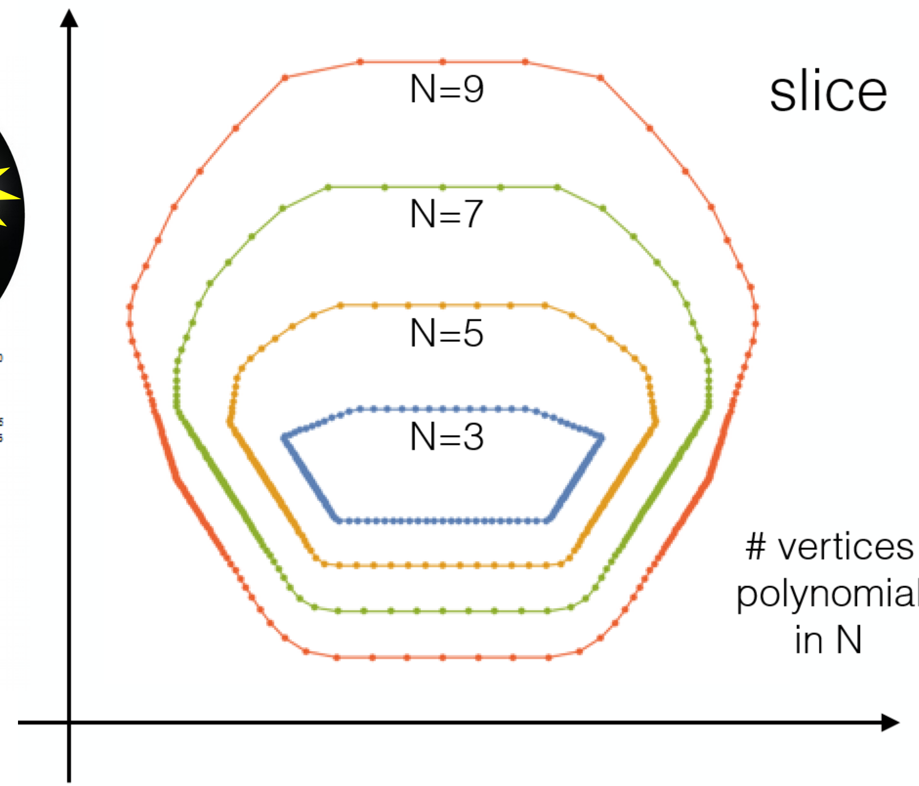
What's the physical significance of each inequality?

The Local polytope

- Solving for a few values of N…







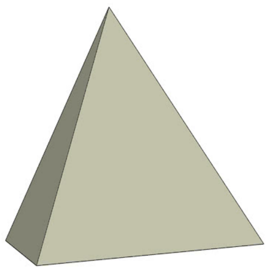
What's the physical significance of each inequality?
Are they equally relevant?

Bounding the LHVM set

- Main Observation

Bounding the LHVM set

- Main Observation



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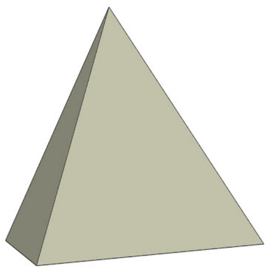

As the system becomes larger

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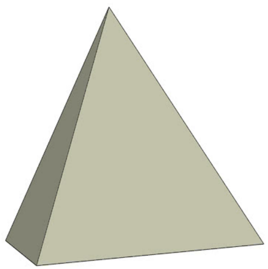

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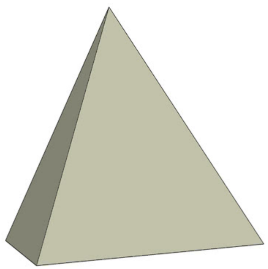

As the system becomes larger

Where does this extra structure come from?

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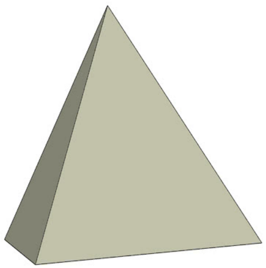

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Local
Deterministic
Strategy view:

Bounding the LHVM set

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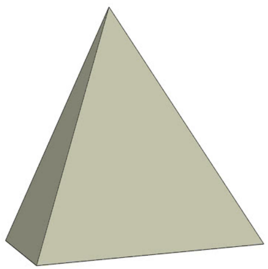

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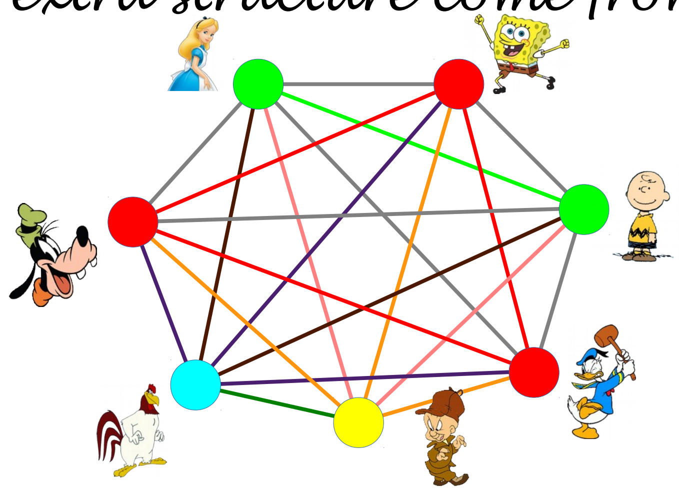

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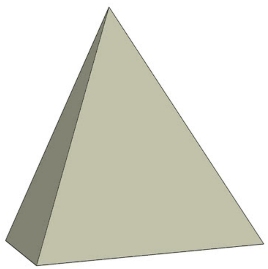

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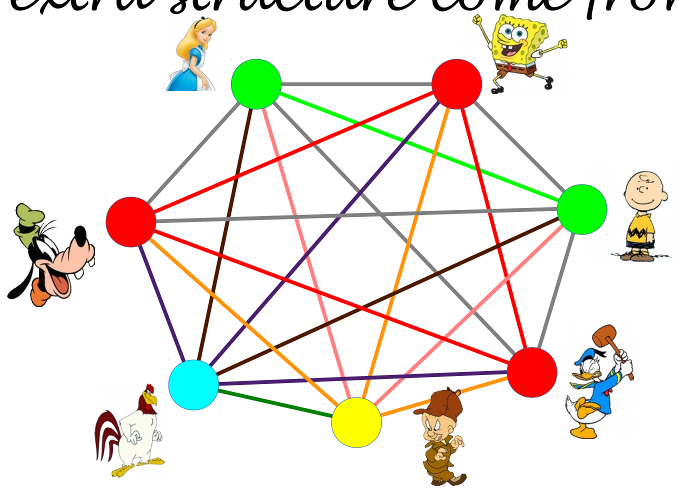
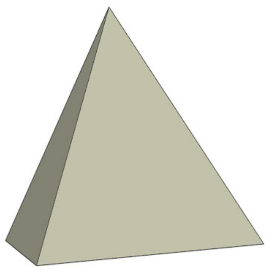


Permutational Invariance:

Bounding the LHVM set

- Main Observation



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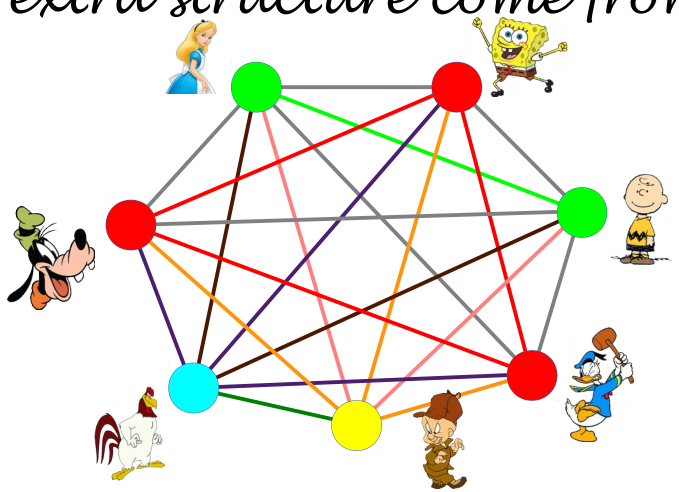
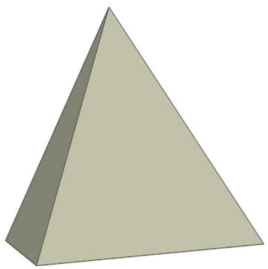


Permutational Invariance:

Only amount of each color becomes relevant

Bounding the LHVM set

- Main Observation



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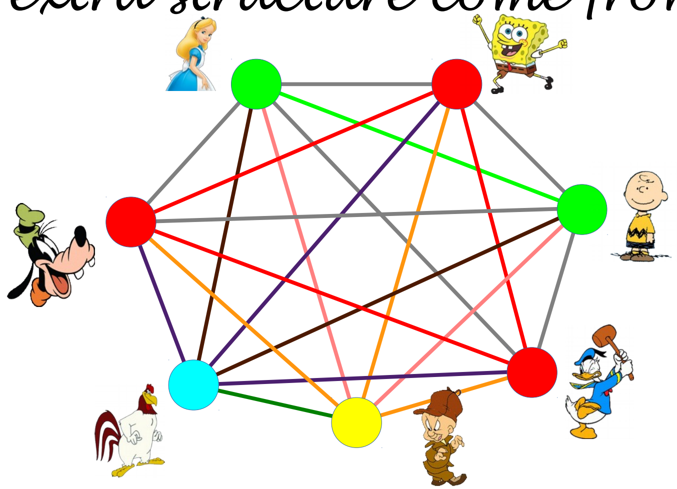

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$$x_i \geq 0$$

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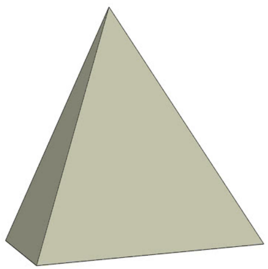

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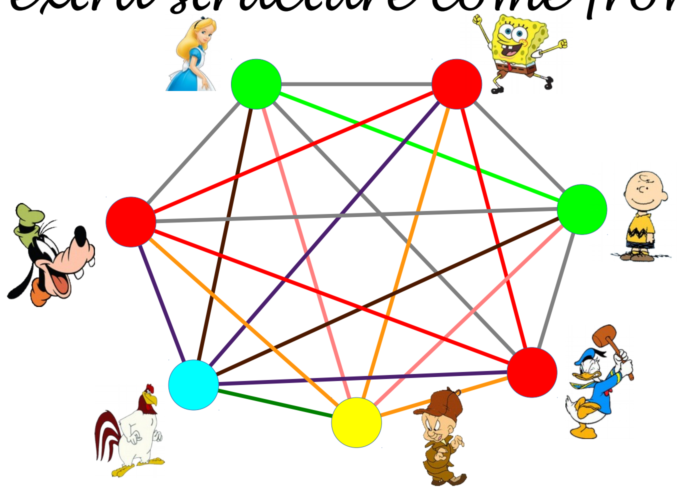

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$$x_i \geq 0 \quad \sum_i x_i = N$$

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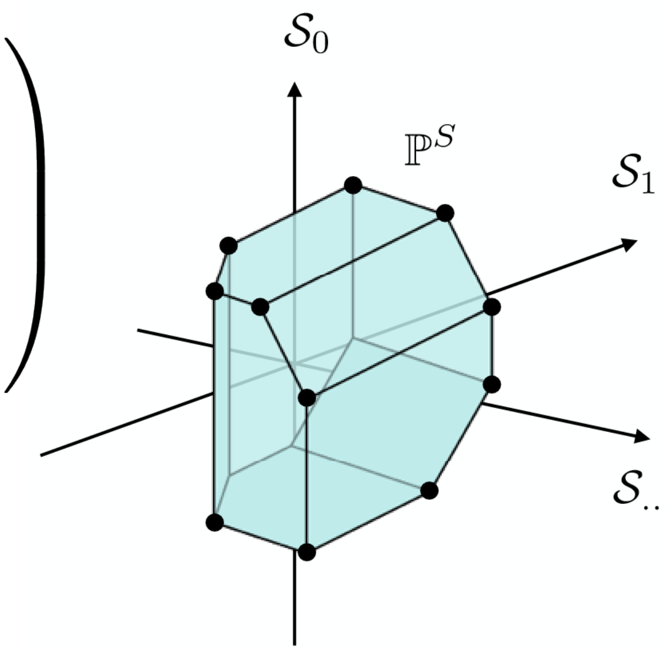

$$\mathbb{P}^{\mathrm{S}} = \mathrm{CH}\left\{\vec{\mathcal{S}}(\vec{x}) \ \text{ s.t. } \ \sum_i x_i = N, \ x_i \in \mathbb{Z}_{\geq 0}\right\}$$

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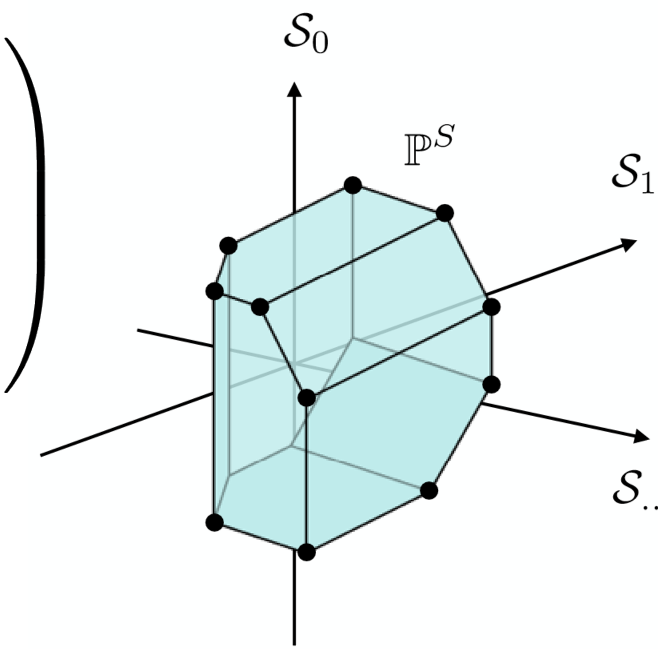

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Goal: Define a manifold interpolating the vertices of the symmetric 2-body polytope, and compute its convex hull

First relaxation

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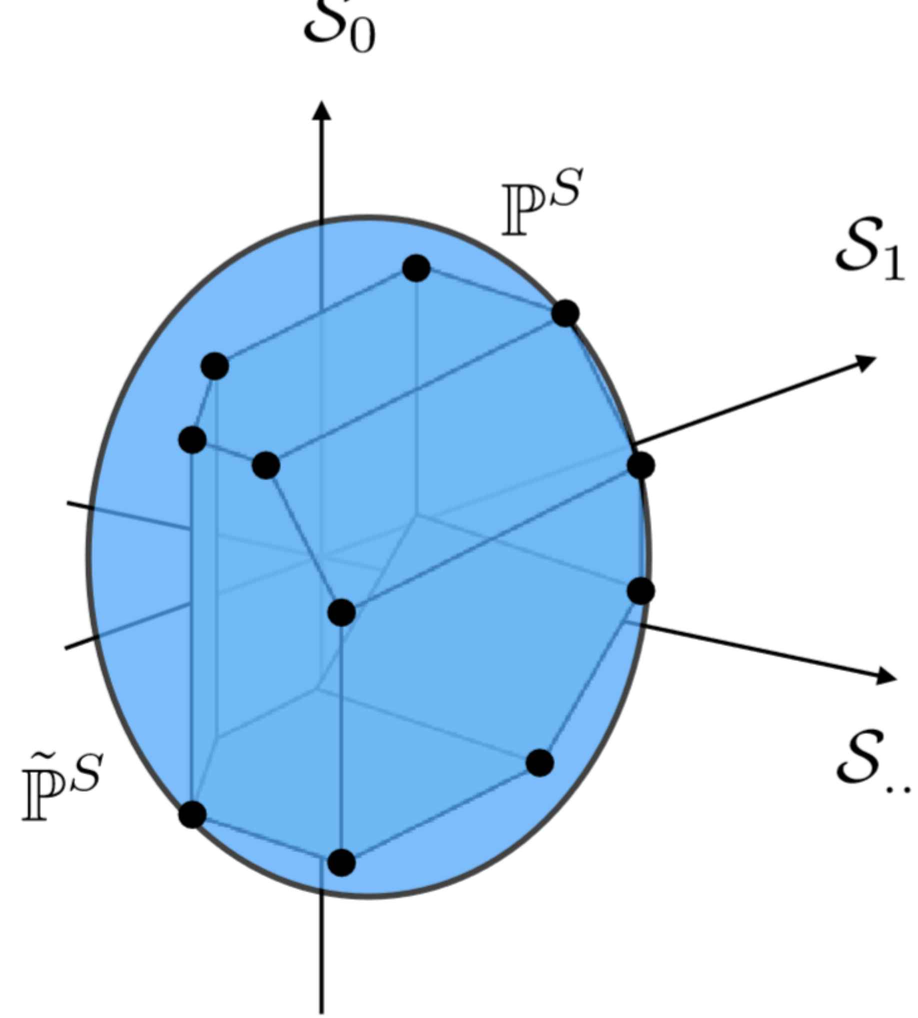

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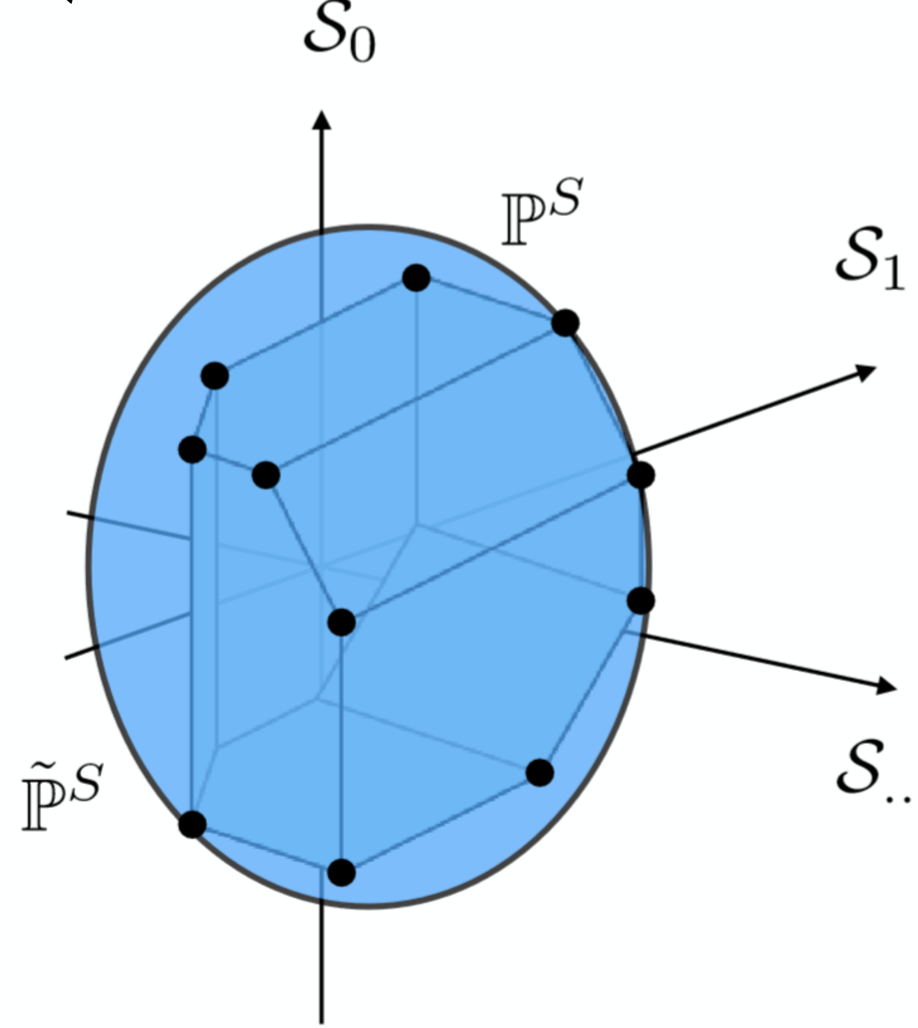
$$\begin{cases} f_i(\vec{\mathcal{S}}_K) = 0 \\ g_j(\vec{\mathcal{S}}_K) \geq 0 \end{cases}$$



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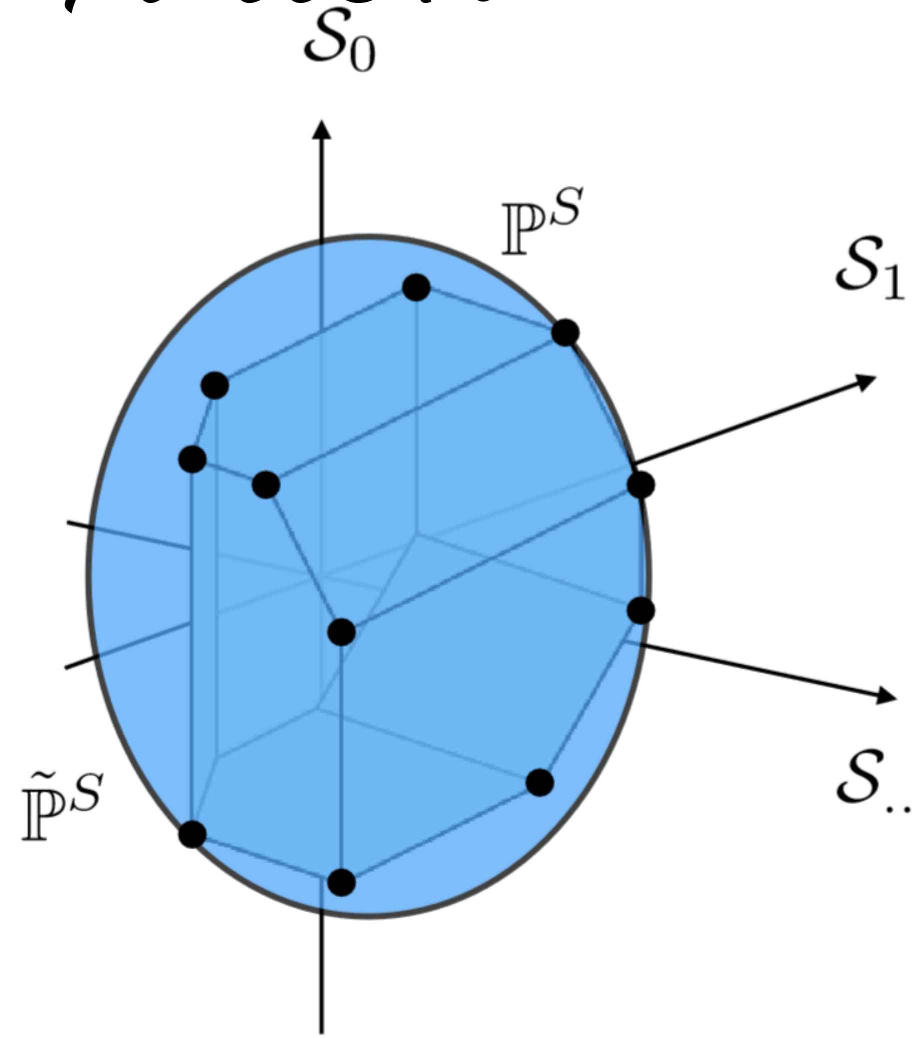


Computing the convex hull of a semialgebraic set is NP-hard

Second relaxation

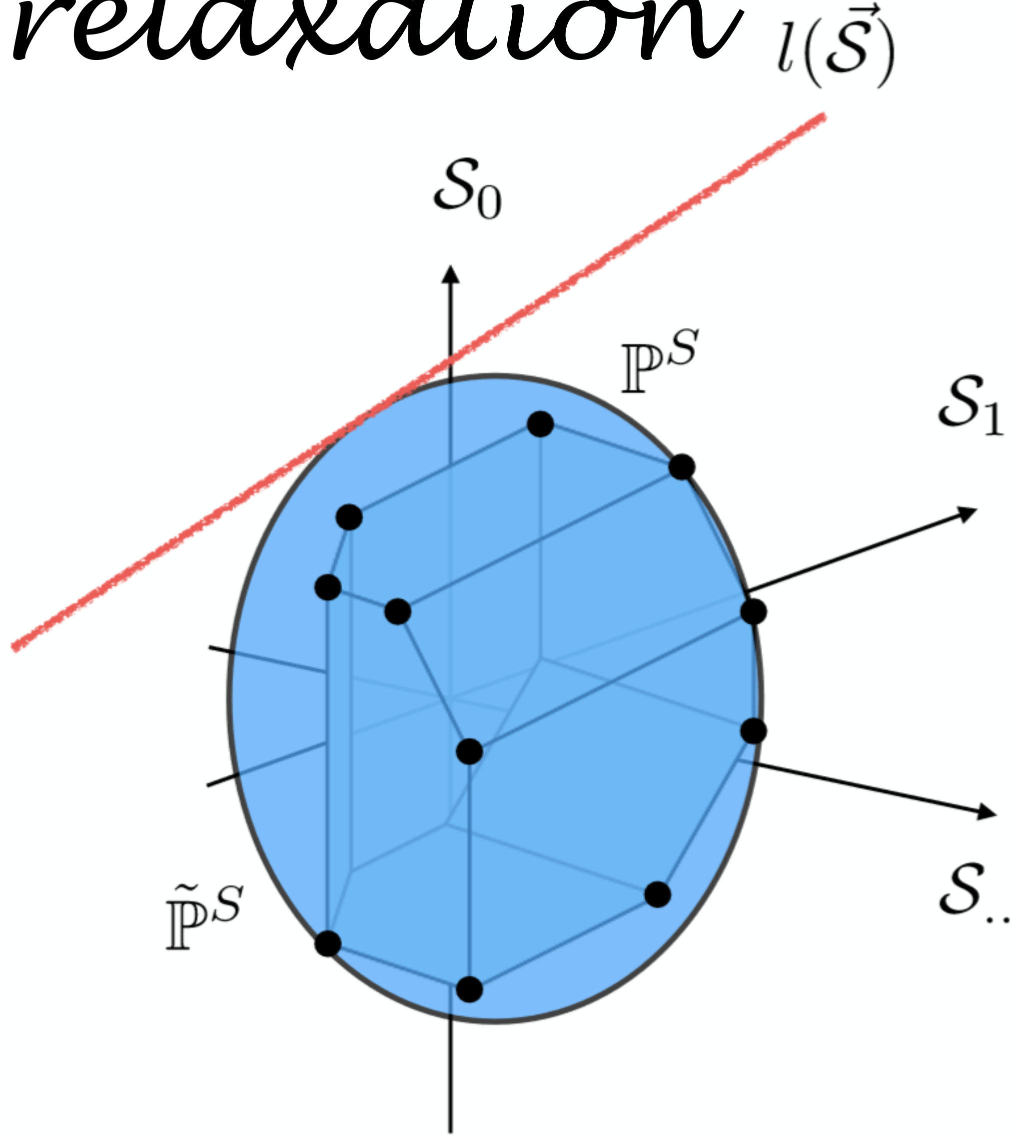
Second relaxation

$\tilde{\mathbb{P}}^S$ is defined by $\begin{cases} f_i(\vec{\mathcal{S}}_K) = 0 \\ g_j(\vec{\mathcal{S}}_K) \geq 0 \end{cases}$

ansatz: $l(\vec{\mathcal{S}}) = \sum_{i=0}^{m} g_i(\vec{\mathcal{S}})\, \sigma_i(\vec{\mathcal{S}})$

$\sigma_i(\vec{\mathcal{S}})$ sos polynomials

NOTE: l is positive in $\tilde{\mathbb{P}}^S$



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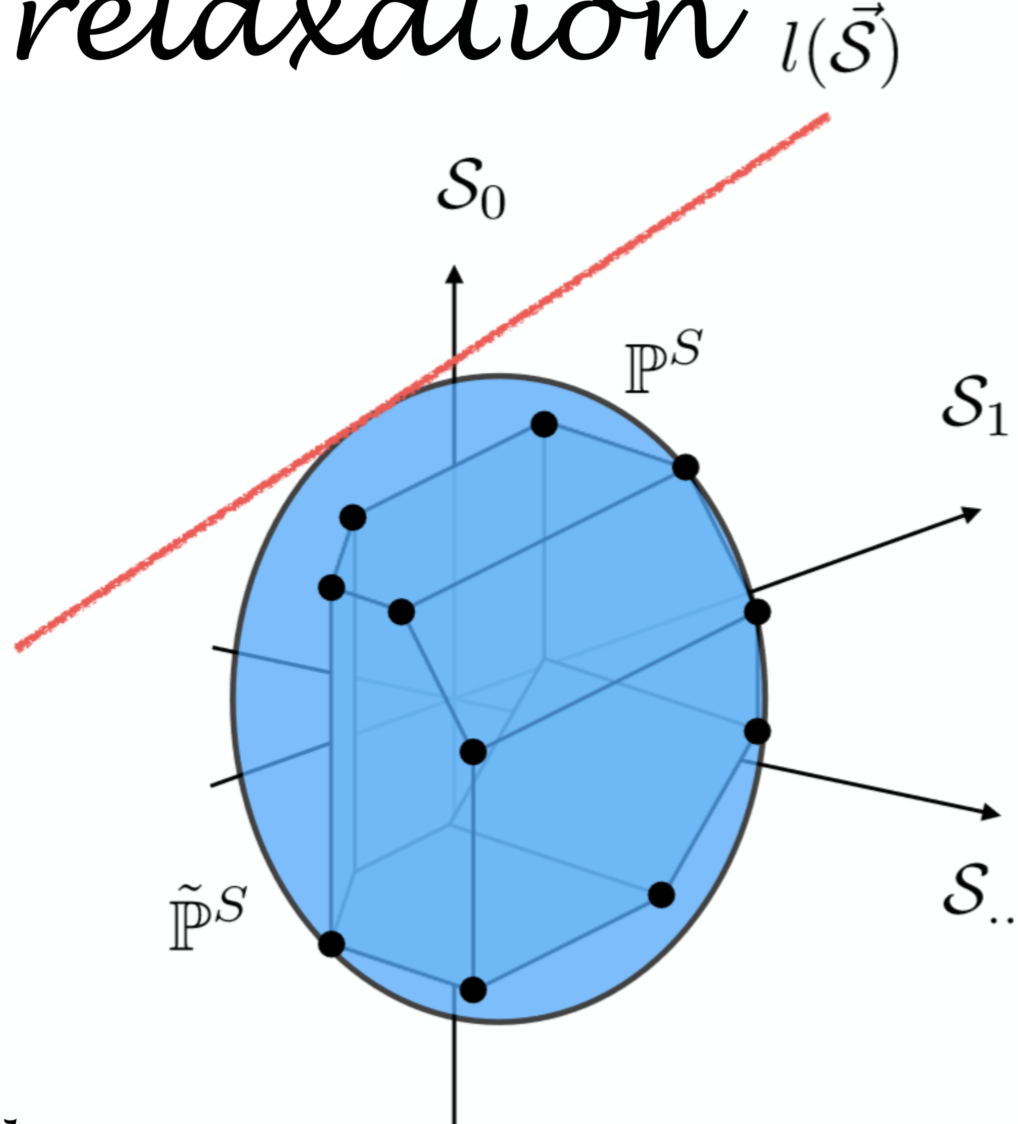


Solve the sos representation problem as an SDP!

The SDP

The SDP

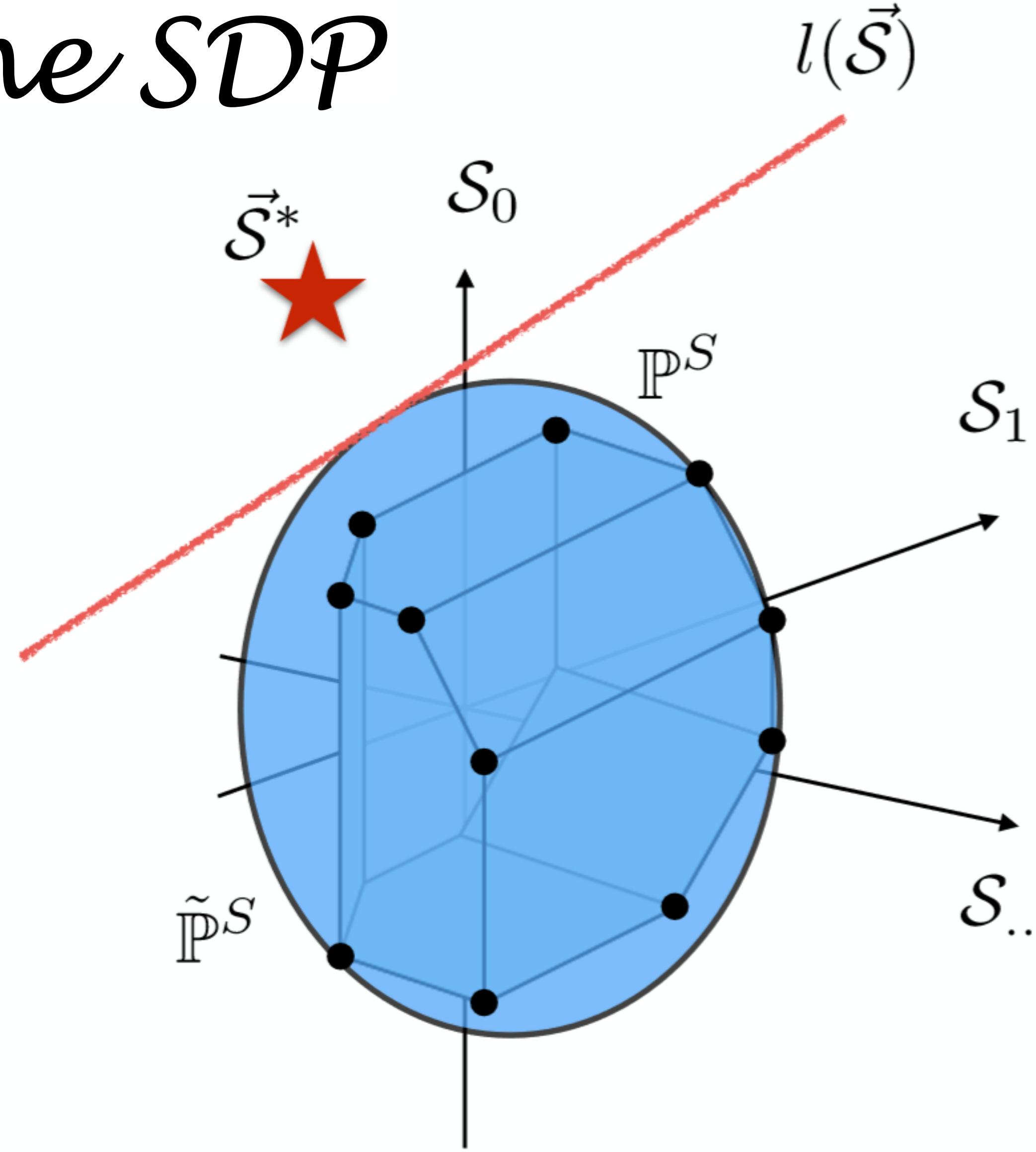
$$l(\vec{\mathcal{S}}) = \sum_{i=0}^{m} g_i(\vec{\mathcal{S}})\, \sigma_i(\vec{\mathcal{S}})$$
$$= \Gamma \cdot G$$

GOAL: prove $l(\vec{\mathcal{S}}^*) < 0$
for some $\sigma_i(\vec{\mathcal{S}})$

$G \succeq 0$ $\quad \Gamma = \begin{pmatrix} 1 & \mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_{00} & \dots \\ \mathcal{S}_0 & \mathcal{S}_0^2 & \mathcal{S}_0\mathcal{S}_1 & \mathcal{S}_0\mathcal{S}_{00} & \dots \\ \mathcal{S}_1 & \mathcal{S}_1\mathcal{S}_0 & \mathcal{S}_1^2 & \mathcal{S}_1\mathcal{S}_{00} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$

SDP: $\Gamma \succeq 0$

subject to: $\mathcal{S}_0 = \mathcal{S}_0^*\,,\ \mathcal{S}_1 = \mathcal{S}_1^*\,,\ \dots$



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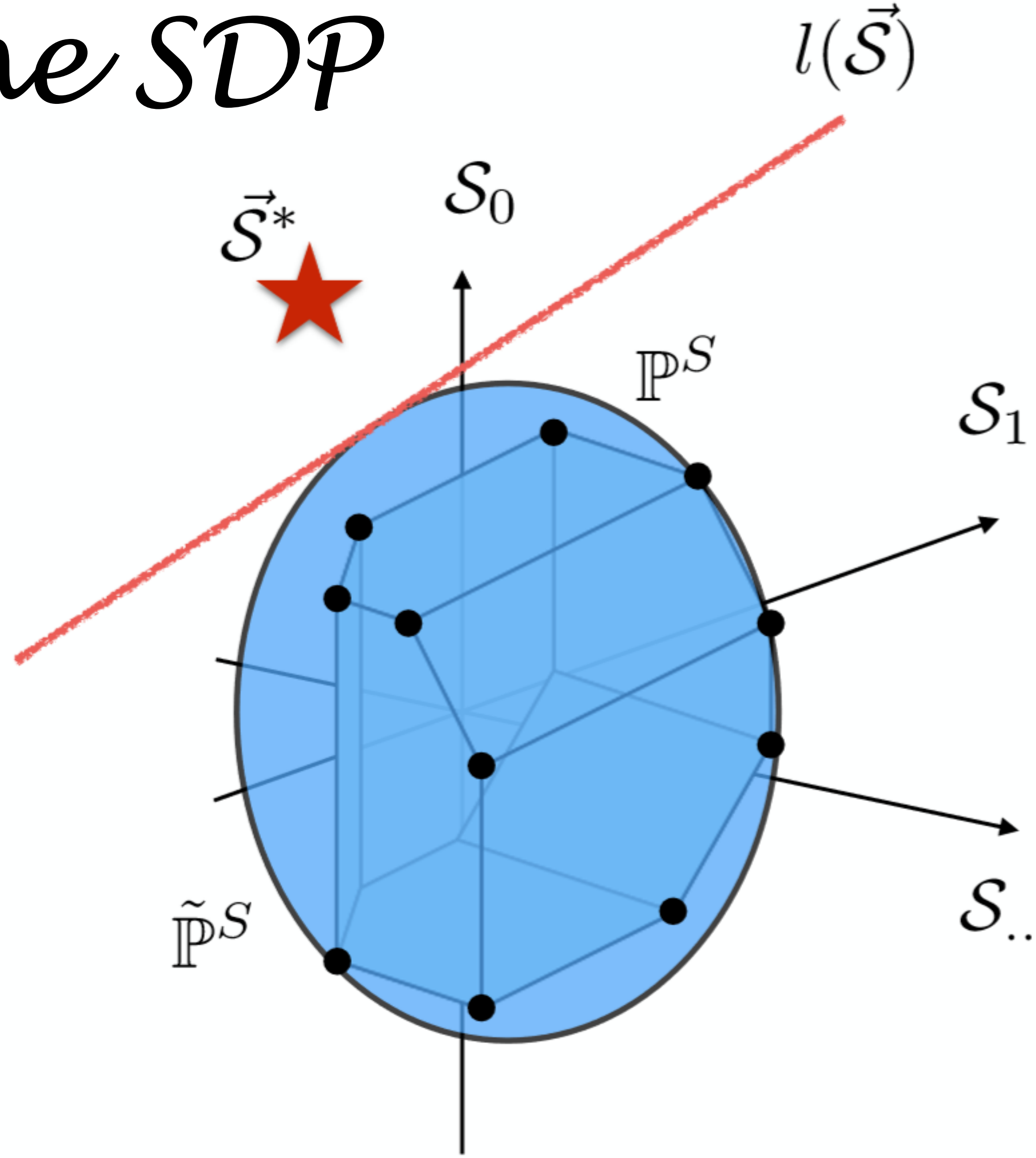

(Experimental data point)

Results

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- ***Testing all BI*** *of a certain form with a single SDP*

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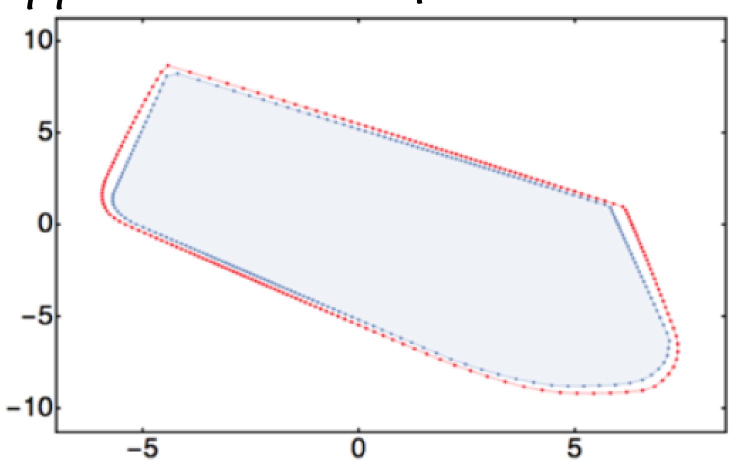
Approximation for N= 10



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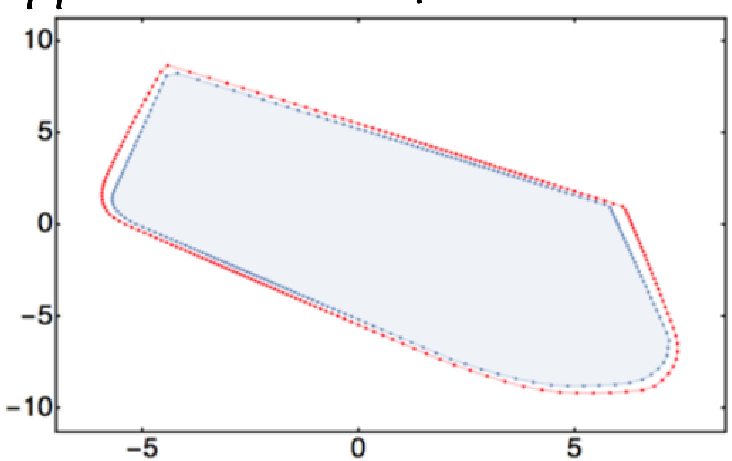
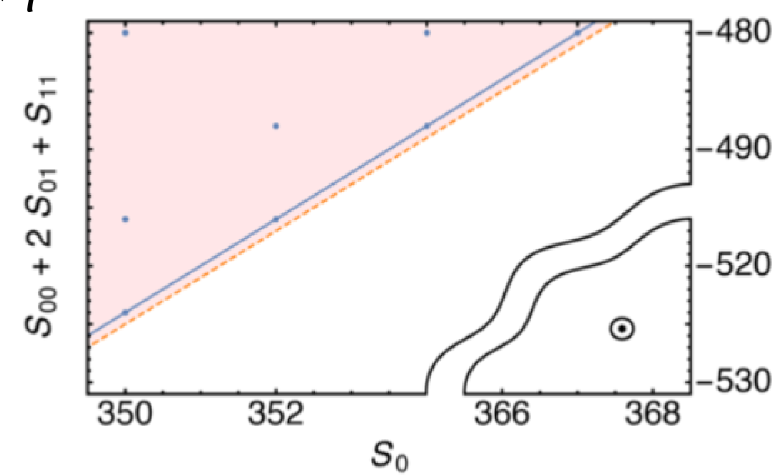


Approximation with actual experimental data and N=476



Outlook

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- Generalization to more outcomes, higher-order correlators…

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Spin-nematic squeezing

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[Ongoing work with A. Aloy and M. Fadel]

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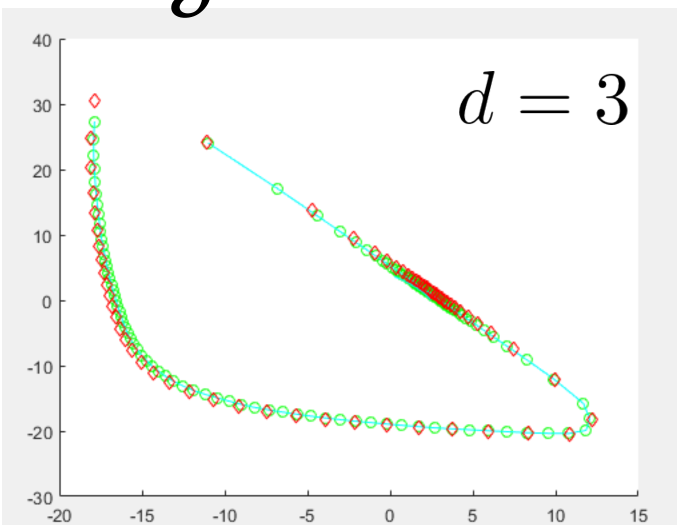

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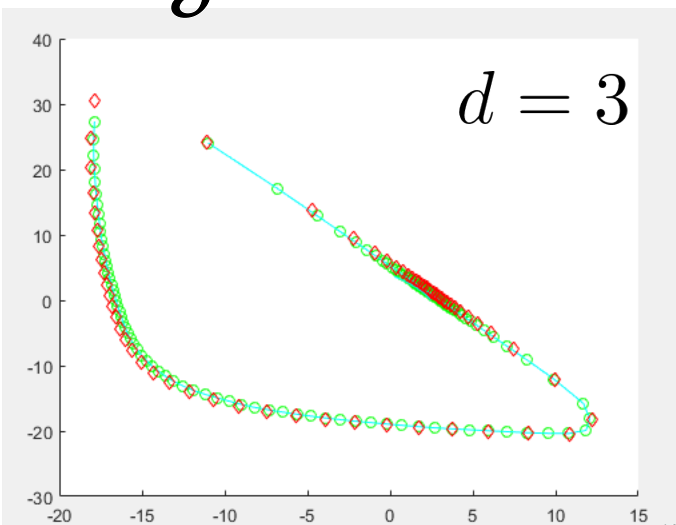

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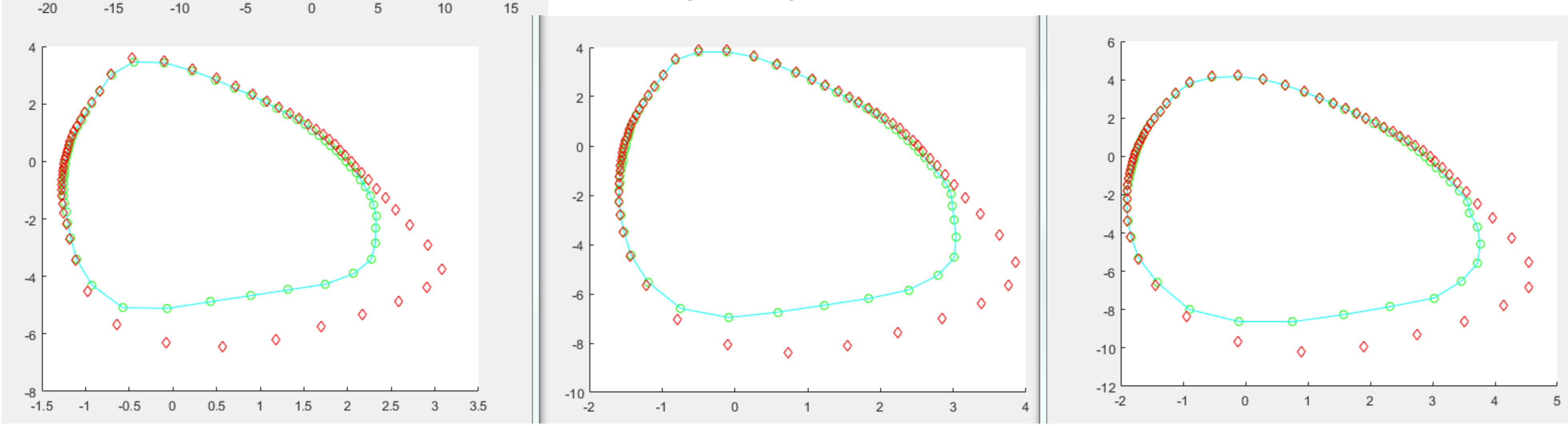

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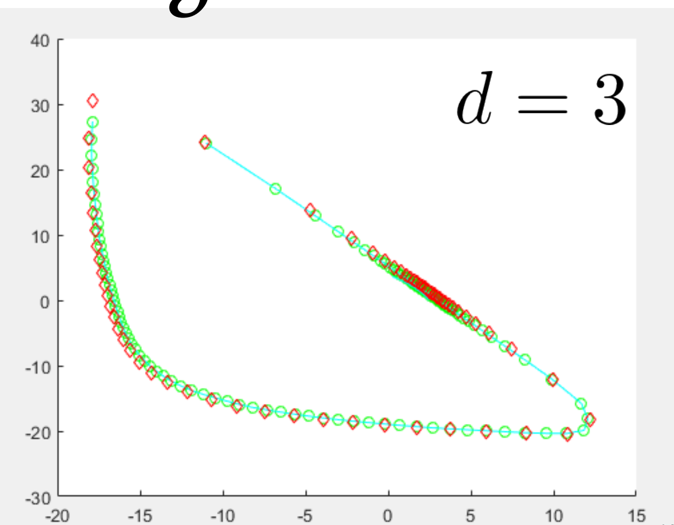

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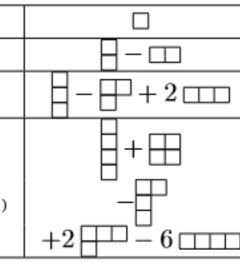
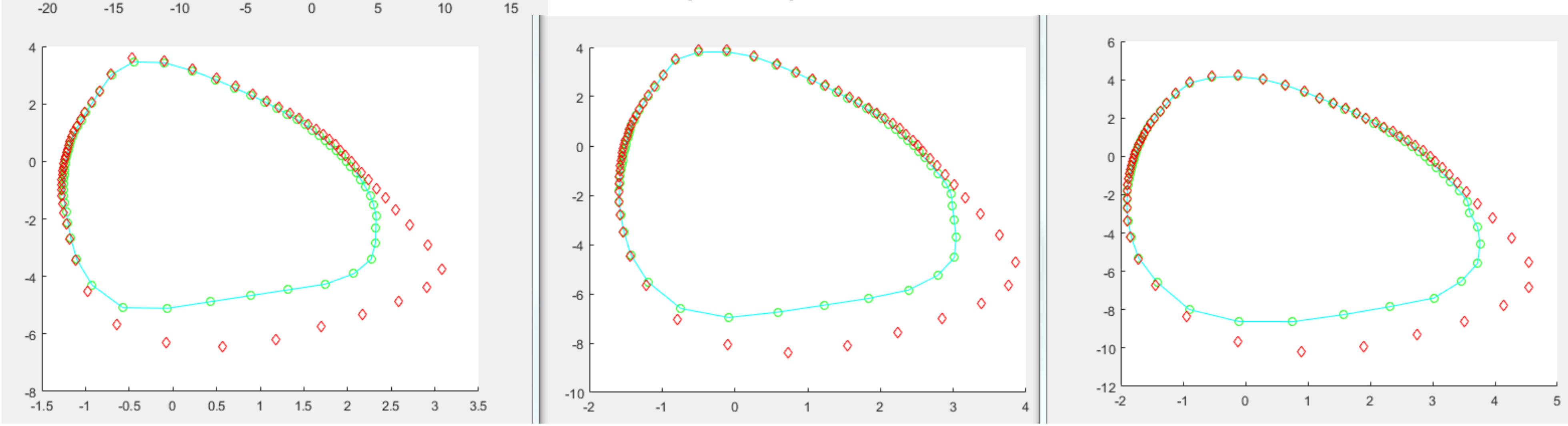
1	x_1	
2	$x_{01}x_{10} - x_{11}$	
3	$x_{001}x_{010}x_{100} - (x_{011}x_{100} + x_{010}x_{101} + x_{001}x_{110}) + 2x_{111}$	
4	$x_{0001}x_{0010}x_{0100}x_{1000} + (x_{0110}x_{1001} + x_{0101}x_{1010} + x_{0011}x_{1100})$ $-(x_{0011}x_{0100}x_{1000} + x_{0010}x_{0101}x_{1000} + x_{0001}x_{0110}x_{1000} + x_{0010}x_{0100}x_{1001} + x_{0001}x_{0100}x_{1010} + x_{0001}x_{0010}x_{1100})$ $+2(x_{0100}x_{1011} + x_{0010}x_{1101} + x_{0001}x_{1110} + x_{0111}x_{1000}) - 6x_{1111}$	

[Ongoing work with A. Aloy and M. Fadel]



Outlook

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- Bell inequalities for Nonlocality depth

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arXiv.org > quant-ph > arXiv:1802.09516

Quantum Physics

Bell correlations depth in many-body systems

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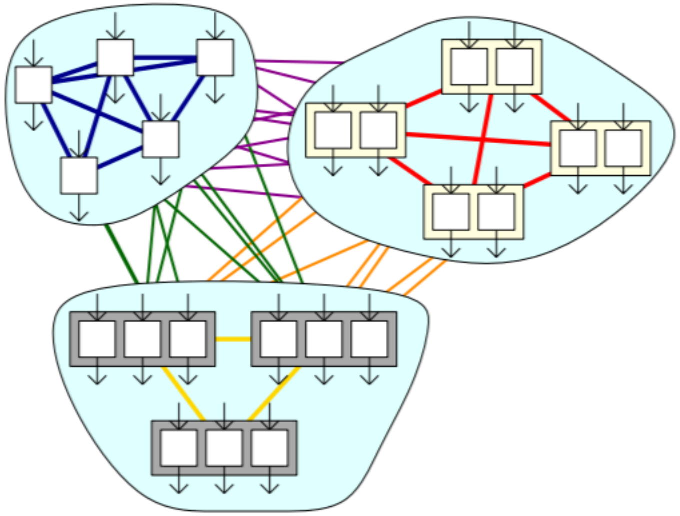

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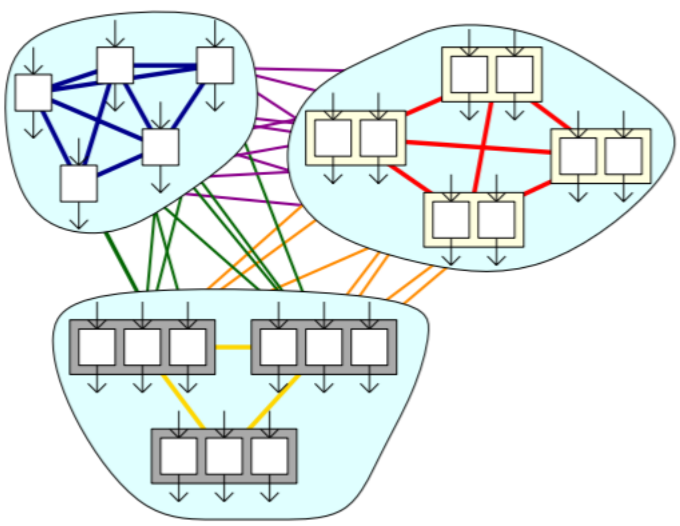

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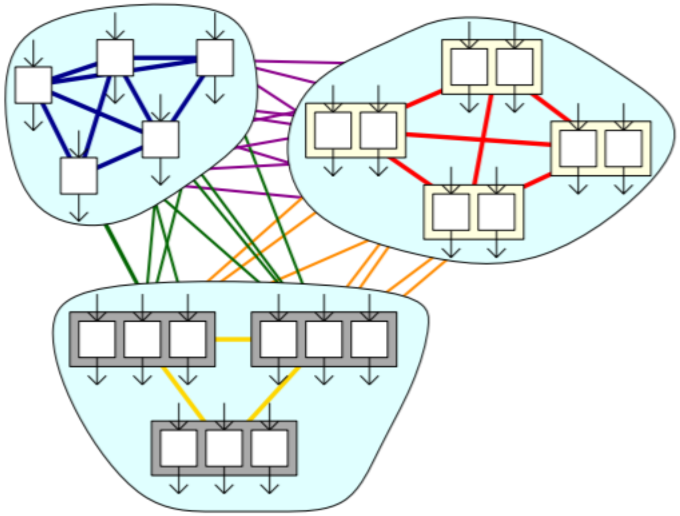
- Convergence analysis
- Self-testing of spin-squeezed states



Thanks for your attention!

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