Useful correlations from bound entangled states

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Abstract

- **Bound entanglement** is a curious form of quantum correlation.
- Many tasks in quantum information require pure state entanglement. However, bound entangled states are so weakly entangled that no pure state entanglement can be distilled from them.
- It became an open problem for which tasks bound entanglement can be a useful quantum resource. In this talk we show two such applications: **quantum metrology** and **Bell nonlocality**.
- In particular, it is shown that bound entangled states can outperform separable states in linear interferometers, and they can give rise to Bell nonlocality from which true randomness can be certified.
- Powerful iterative methods are presented to find such bound entangled states.

Outline

1. Motivation
2. Bound entanglement
3. Quantum metrology
4. Metrologically useful bound entangled states
5. Bell nonlocality
6. Nonlocal bound entangled states

Motivation

The (bipartite) state is separable:

$$\rho_{AB} = \sum_{\lambda} p_{\lambda} \rho_{A}^{\lambda} \otimes \rho_{B}^{\lambda}$$

The state is entangled iff it cannot be written in the above form.

Motivation

The singlet state is a pure state, and it is a maximally entangled state of two quantum bits (qubits):

$$\left|\Psi^{-}\right\rangle = \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

The above singlet state is a key resource for many quantum applications (e.g. teleportation, entanglement based cryptography).

Motivation

However, in the real world noise is unavoidable: we have entangled mixed states instead of entangled pure states.

How to solve this problem? Is it possible to find applications which use entangled mixed states as a resource?

Especially: Can very weakly entangled mixed states be useful for applications?

We focus on two application areas:

Quantum metrology and Bell nonlocality.

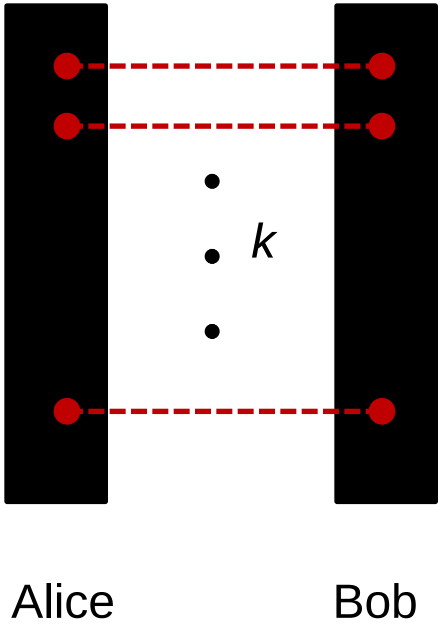
Bound entanglement

A mysteries family of weakly entangled mixed states is the so-called bound entangled states. These states require some initial amount of entanglement for their preparation. However, they contain this useful resource in such a noisy form that it is not possible to extract pure state entanglement out of them.

We introduce so-called PPT bound entangled states, and then show that some of them are useful in metrology and Bell nonlocality.

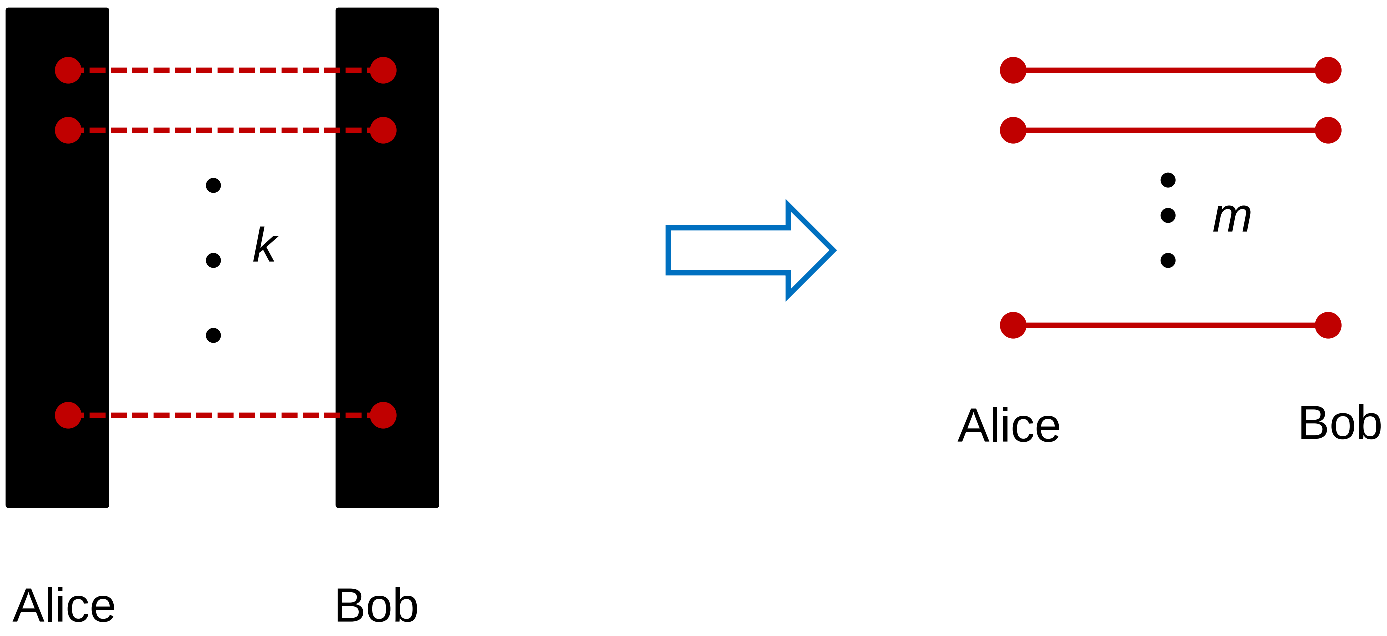
Bound entanglement

Suppose that Alice and Bob share *k* copies of a mixed entangled state ρ_{AB} :



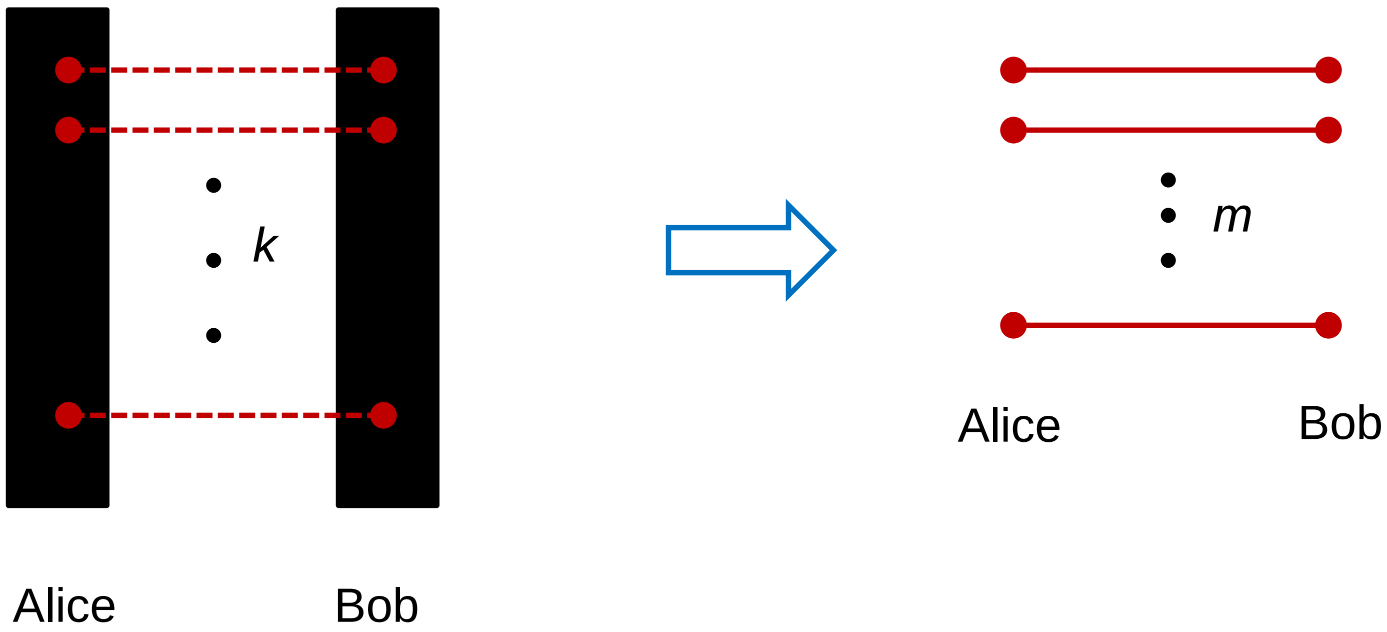
Bound entanglement

To extract singlet pairs, they run a distillation protocol (local quantum operations and classical communication -- LOCC):



Bound entanglement

As a result they end up with $m<k$ copies of a singlet state, which can be used for quantum information purposes.



Bound entanglement

Every two-qubit entangled state ρ_{AB} can be used to distill singlets with the above distillation protocol (Horodecki et al, 1997).

Is it also true for systems of higher dimension?

Bound entanglement

It turns out not to be the case: in higher dimensions there exist noisy entangled states that cannot be distilled by LOCC into the singlet state (Horodeckis, 1998).

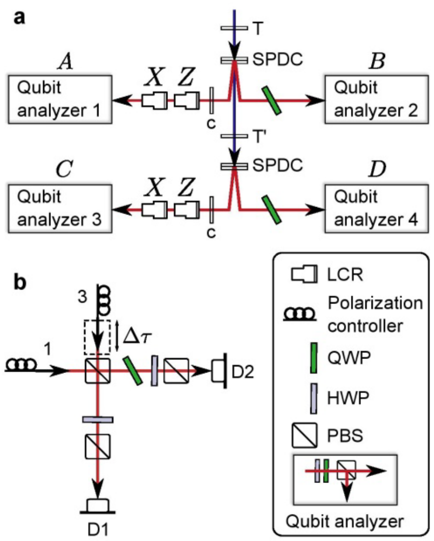
These states that are entangled yet not distillable are called bound entangled.

The smallest example is a 3x3 dimensional state.

Bound entanglement

Multipartite bound entangled states have been experimentally realized.

The 4-qubit Smolin state was encoded in the polarization of 4 photons:



Lavoire et al. 2010, PRL

Bound entanglement

Given a state ρ_{AB} . How to decide if it is undistillable?

It is a difficult question in general, since there is no restriction on the specific type of LOCC operations or on the number of copies used in the distillation protocol.

Bound entanglement

Still there is a sufficient condition to undistillability:

If a state is positive under the partial transposition (PT) map: $\mathrm{PT}(\rho_{AB}) = (\mathrm{I} \otimes T_B)(\rho_{AB})$

then the state is undistillable. In this case we say that the state is PPT.

E.g. The following states are PPT:

$$\mathrm{PT}(\rho_{AB}) = \rho_{AB}$$

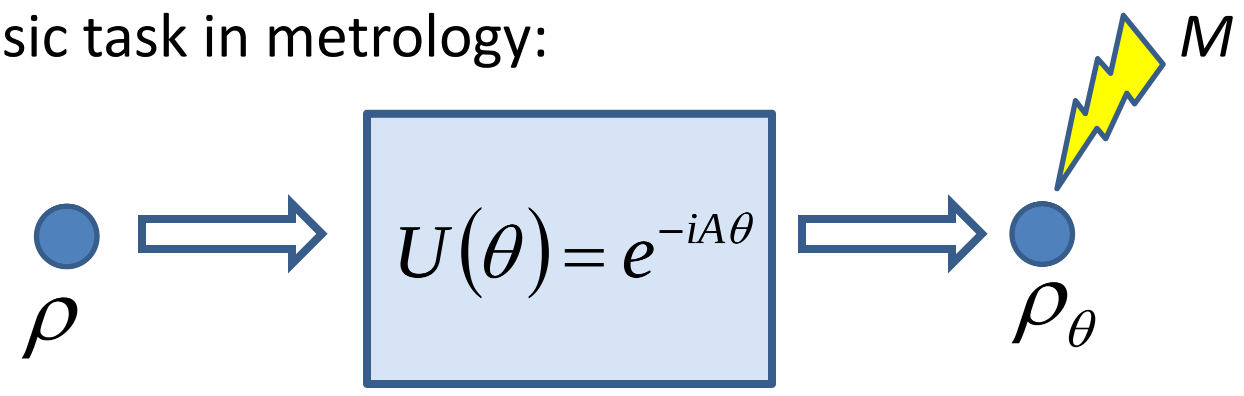
Hence, these states are undistillable.

Bound entanglement

Such PPT states, provided they are entangled, are called PPT bound entangled states. We will construct such states, and show that some of them are useful in Bell nonlocality and in metrology as well.

Metrology

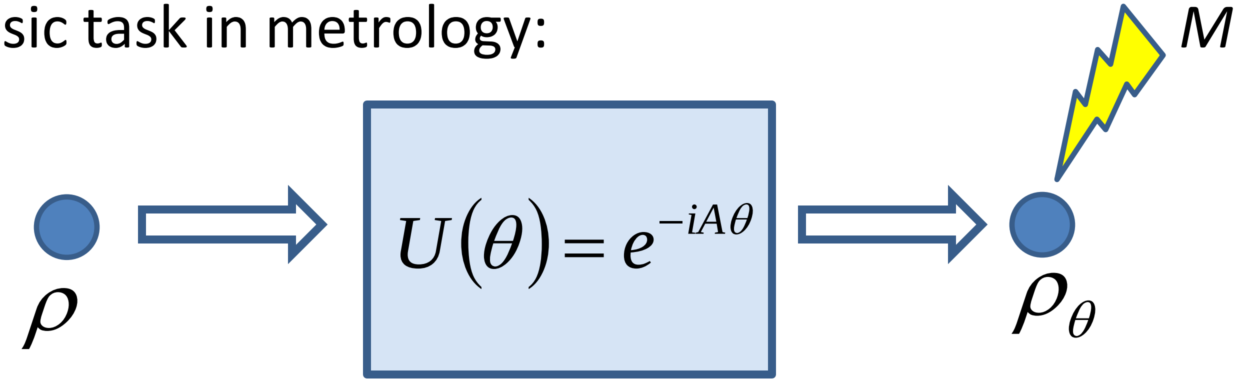
Basic task in metrology:



Estimate the parameter θ in the dynamics. In order to do it we prepare a probe state ρ, let it evolve, and finally measure the evolved state ρ_θ with an operator *M*.

Metrology

Basic task in metrology:



ρ is said to be useful if it gives better precision for θ than any separable state.

Metrology

The precision of the estimation of θ is given by the formula:

$$\frac{1}{(\Delta\theta)^2} = \frac{\left|\partial_\theta \langle M \rangle\right|^2}{(\Delta M)^2}$$

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Accordingly, it depends on two things:

- The sensitivity of <*M*> to the change of θ
- The variance of *M*

Metrology

The precision of parameter θ is limited by the Cramér-Rao bound as

$$\frac{1}{(\Delta\theta)^2} \leq F_Q(\rho, A)$$

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$$\frac{1}{(\Delta\theta)^2} \leq F_Q(\rho, A)$$

Above $F_Q(\rho, A)$ is the quantum Fisher information:

$$F_Q(\rho, A) = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \left|\langle k | A | l \rangle\right|^2$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|$. In linear interferometers *A* are collective operators:

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)} \qquad l \in \{x, y, z\}$$

Metrology

Metrology is linked to the entanglement of ρ :

Shot noise limit: $F_Q(\rho, J_l) \leq N \qquad l = x, y, z$

The above inequality holds for *N*-qubit separable states. A quantum state is useful in metrology if it violates the above inequality.

Metrology

Metrology is linked to the entanglement of ρ :

Shot noise limit: $F_Q(\rho, J_l) \leq N$ $\quad l = x, y, z$

The above inequality holds for *N*-qubit separable states. A quantum state is useful in metrology if it violates the above inequality.

Heisenberg limit: $F_Q(\rho, J_l) \leq N^2$ $\quad l = x, y, z$

The maximum attainable limit, where the N^2 bound can be saturated with certain multipartite states (e.g. with GHZ or Dicke states).

Metrologically useful bound entangled states

Is it possible to find multipartite fully bound entangled states which beat the shot noise limit? These states are PPT for all bipartitions.
This question was raised by Czekaj et al. in [*Phys. Rev. A* 92, 062303 (2015)].

The answer is yes. Results presented:

G. Tóth & T. Vértesi (2018). Quantum states with a positive partial transpose are useful for metrology. *Physical Review Letters* 120, 020506.

Metrologically useful bound entangled states

$F_Q > F_{Q,sep}$ for the following fully bound entangled states:

System	A	FQ	FQ_sep	p_white
four qubits	Jz	4.0088	4	0.0011
three qubits	jz(1) + jz(2)	2.0021	2	0.0005
2 x 4	jz(1) + jz(2)	2.0033	2	0.0008

Metrologically useful bound entangled states

$F_Q > F_{Q,sep}$ for the following bipartite bound entangled states:

System	FQ	FQ_sep	p_white
3 x 3	8.0085	8	0.0003
4 x 4	9.3726	8	0.0382
12 x 12	11.3618	8	0.0808

Here A is not the usual J_z operator. It has the form:

$$A = H \otimes I + I \otimes H \,, \quad \text{with} \quad H = \mathrm{diag}(1,1,\ldots,-1,-1,\ldots)$$

Metrologically useful bound entangled states

In the case of the 4x4 system, the bound entangled state is

$$\rho_{\mathrm{PPT}} = p\sum_{n=1}^{4} |\psi_n\rangle\langle\psi_n| + q\sum_{n=5}^{6} |\psi_n\rangle\langle\psi_n|$$

where the states $|\psi_n\rangle$ are rank-2 and rank-3 respectively in tha above sum.

Metrologically useful bound entangled states

How did we find the bound entangled states above?

Basically, there are two different approaches (and we used the second one).

Metrologically useful bound entangled states

1) „Brute force” approach. Let us recall the definition for the quantum Fisher information:

$$F_Q(\rho, A) = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \left|\langle k | A | l \rangle\right|^2$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|$ and let A be some fixed collective operator.

Let us optimize F_Q over $\rho \in \mathrm{PPT}$ states.
However, it is extremely hard to maximize a convex function over a convex set.

Metrologically useful bound entangled states

2) Iterative approach. Note that the maximum for PPT bound entangled states can alternatively be written as a double optimization:

$$\max_{\rho\in\mathrm{PPT}} F_Q(\rho, A) = \max_{\rho\in\mathrm{PPT}} \max_{M} \frac{\langle i[M, A]\rangle_\rho^2}{(\Delta M)^2}$$

This can be optimized in a see-saw manner for a fixed *A*.

Metrologically useful bound entangled states

Goal: $$\max_{\rho \in \mathrm{PPT}} \max_{M} \frac{\langle i[M, A] \rangle_\rho^2}{(\Delta M)^2}$$

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Metrologically useful bound entangled states

Goal:

$$\max_{\rho \in \mathrm{PPT}} \max_{M} \frac{\langle i[M, A] \rangle_\rho^2}{(\Delta M)^2}$$

step 0: pick random operators *M*

step 1: maximize over PPT states ρ_PPT for a given *M*

This casts as an SDP

Metrologically useful bound entangled states

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→ This casts as an SDP

→ M is given by the SLD (e.g. M.G.A. Paris, 2009)

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step 3: back to step 1 until convergence is achieved

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Metrologically useful bound entangled states

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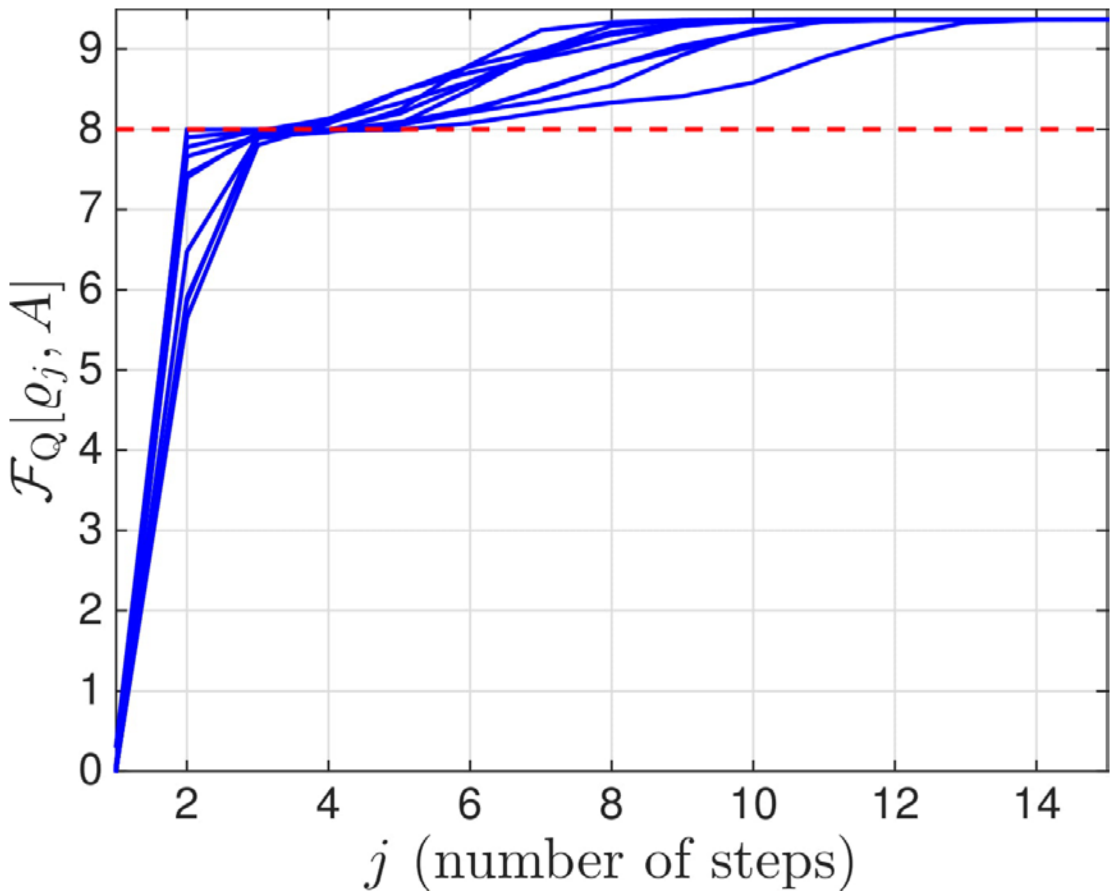
step 1: maximize over PPT states ρ_PPT for a given *M* → This casts as an SDP

step 2: maximize over *M* for a given PPT state → M is given by the SLD (e.g. M.G.A. Paris, 2009)

step 3: back to step 1 until convergence is achieved

The FQ value cannot get worse with the number of iterations!

Metrologically useful bound entangled states



Plot: convergence to the optimal FQ value during the generation of the 4x4 bound entangled state. Dashed line stands for FQ_{sep} .

Open problem

Are there fully bound entangled states which can be used to attain the Heisenberg scaling $F_Q(\rho, J_z) \sim N^2$?

Open problem

Only partial results in the literature.

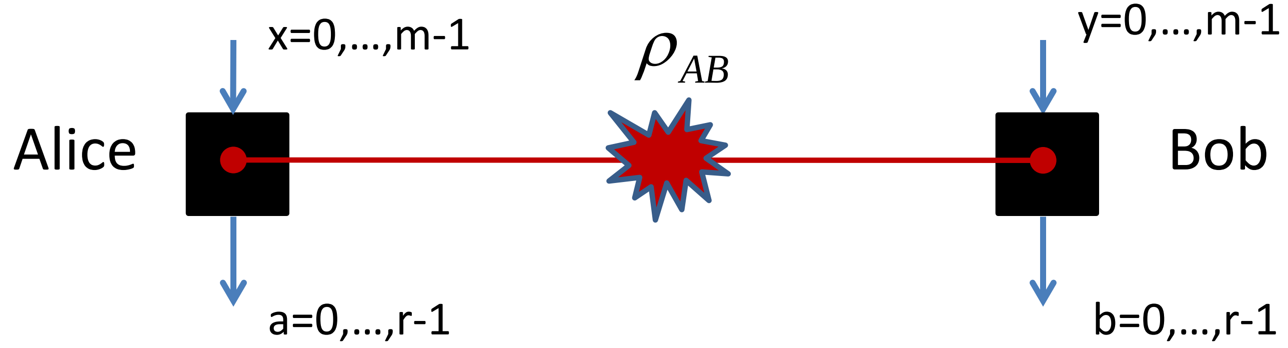
Bound entangled states with PPT and some non-PPT partitions:

o Violates an entanglement criterion with three quantum Fisher information terms [P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012).]

o Violates an entanglement criterion with a single quantum Fisher information term better than shot-noise limit [Ł. Czekaj, A. Przysiezna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015).]

Bell nonlocality

Bell scenario: distant parties (Alice and Bob) choose between *m* different measurements of *r* outcomes.

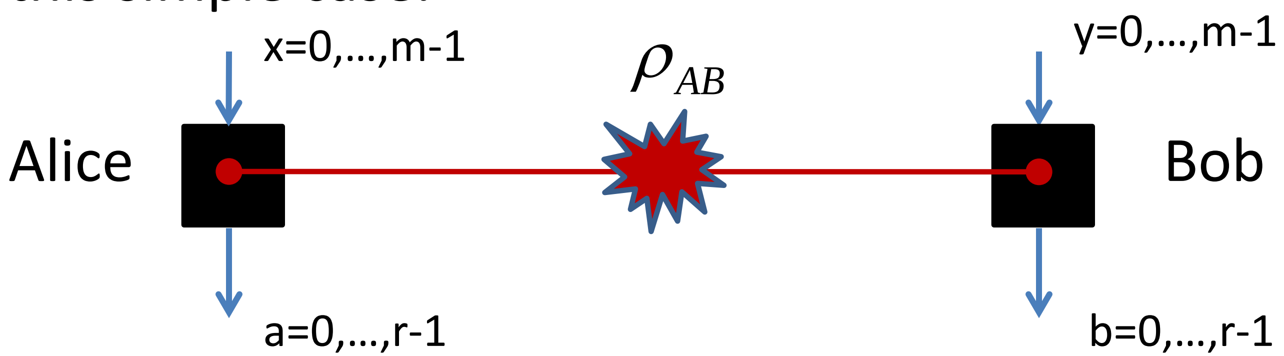


Experimental data: P(a,b|x,y)

J.S. Bell: On the einstein-podolsky-rosen paradox, 1964

Bell nonlocality

The simplest Bell scenario involves *m*=2 measurements with *r*=2 outcomes (CHSH scenario). However there exist implementations beyond this simple case.



Experimental data: P(a,b|x,y)

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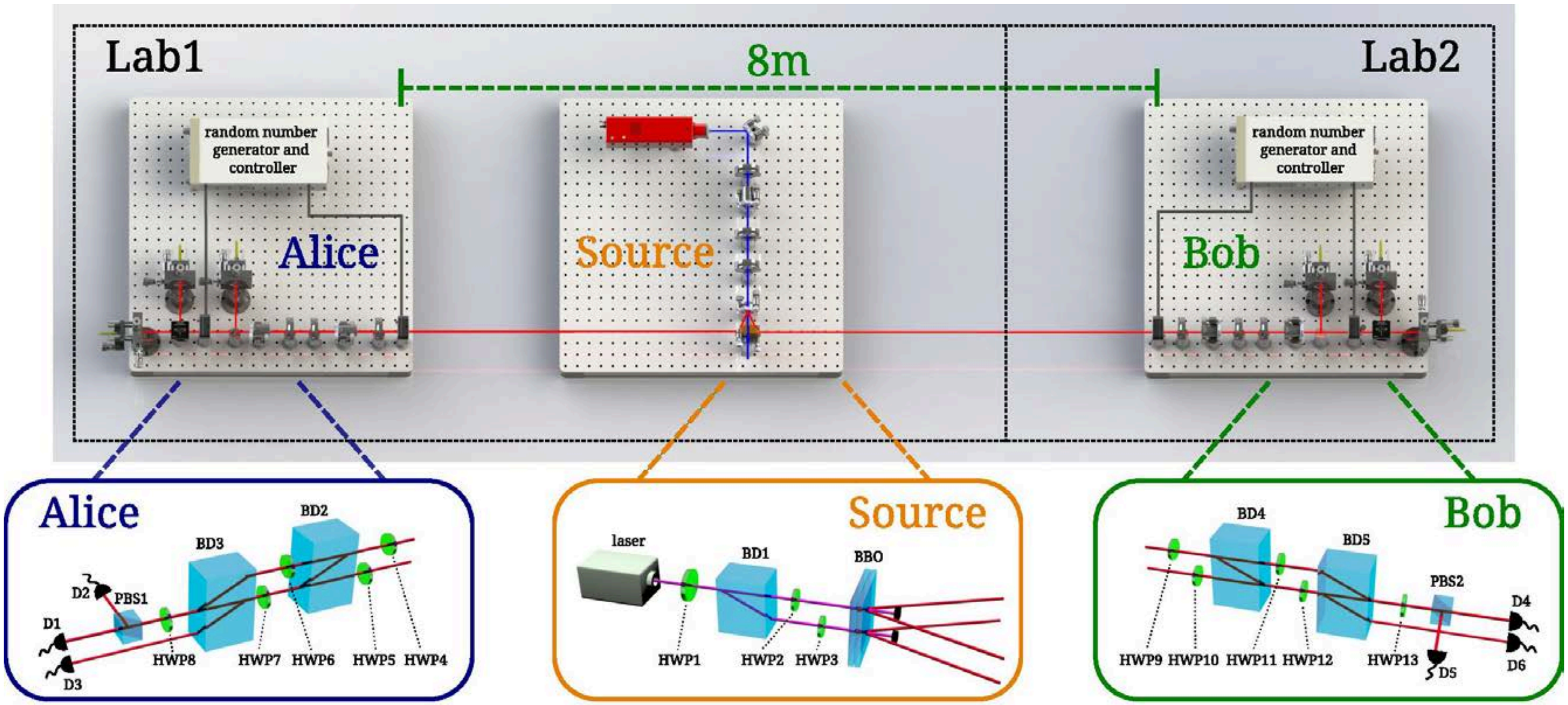
Bell nonlocality

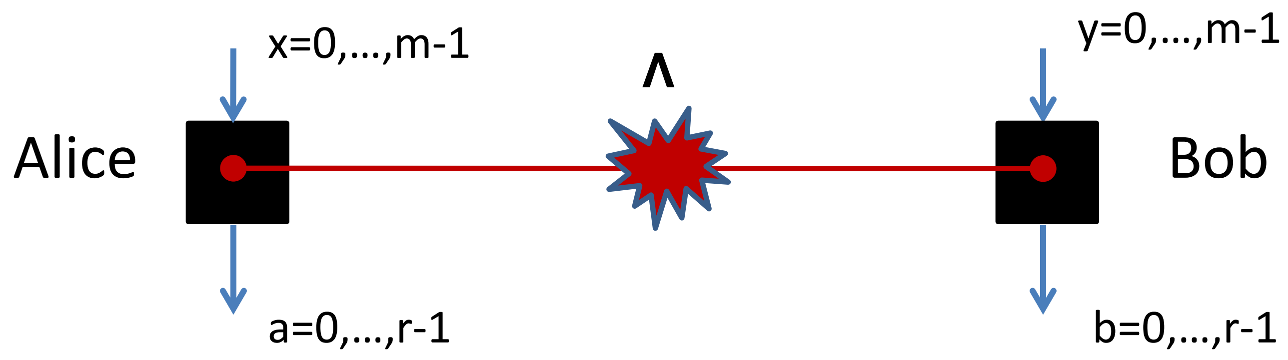
A photonic Bell setup (r=3 outcomes, two-qutrit systems):



X.M. Hu, B.H. Liu, Y. Guo, G.Y. Xiang, Y.F. Huang, C.F. Li, G.C. Guo, M. Kleinmann, T. Vértesi, A. Cabello: *Observation of stronger-than-binary correlations with entangled photonic qutrits,* PRL, 2018

Bell nonlocality

Local correlations: **Λ** defines a classical, random source.



P(a,b|x,y)

The set of local correlations is defined by

$$P(a,b\,|\,x,y)=\sum_{\lambda} p_{\lambda} P(a\,|\,x,\lambda) P(b\,|\,y,\lambda)$$

Bell nonlocality

If the distribution P(a,b|x,y) cannot be written in the local form

$$P(a,b\,|\,x,y)=\sum_{\lambda}p_{\lambda}P(a\,|\,x,\lambda)P(b\,|\,y,\lambda)$$

then it is said to be nonlocal.

Gisin's theorem:
Every pure entangled state (with some suitably chosen measurements) give rise to nonlocal correlation.

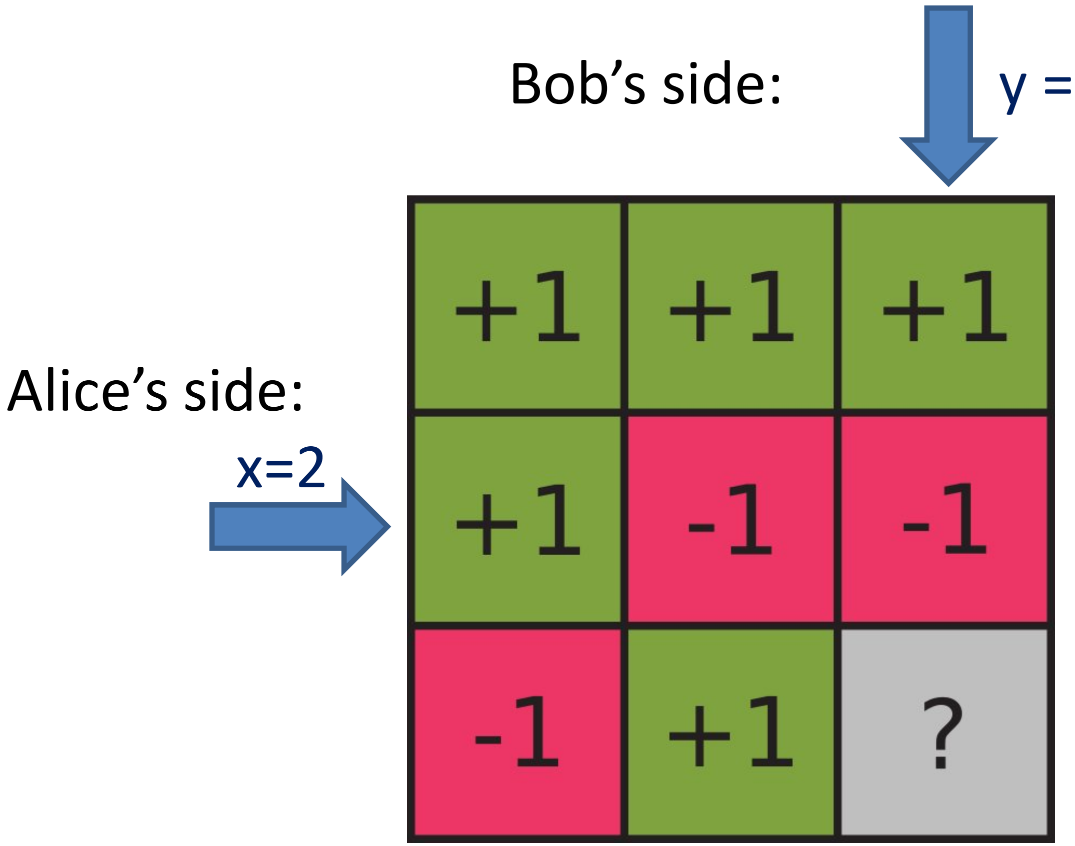
Bell nonlocality

Example: Let us consider the following game. The task is to fill a 3x3 table with the numbers +1 and -1 such that each row has an even number of negative entries and each column has an odd number of negative entries. Is such a filling possible?

+1	+1	+1
+1	-1	-1
-1	+1	?

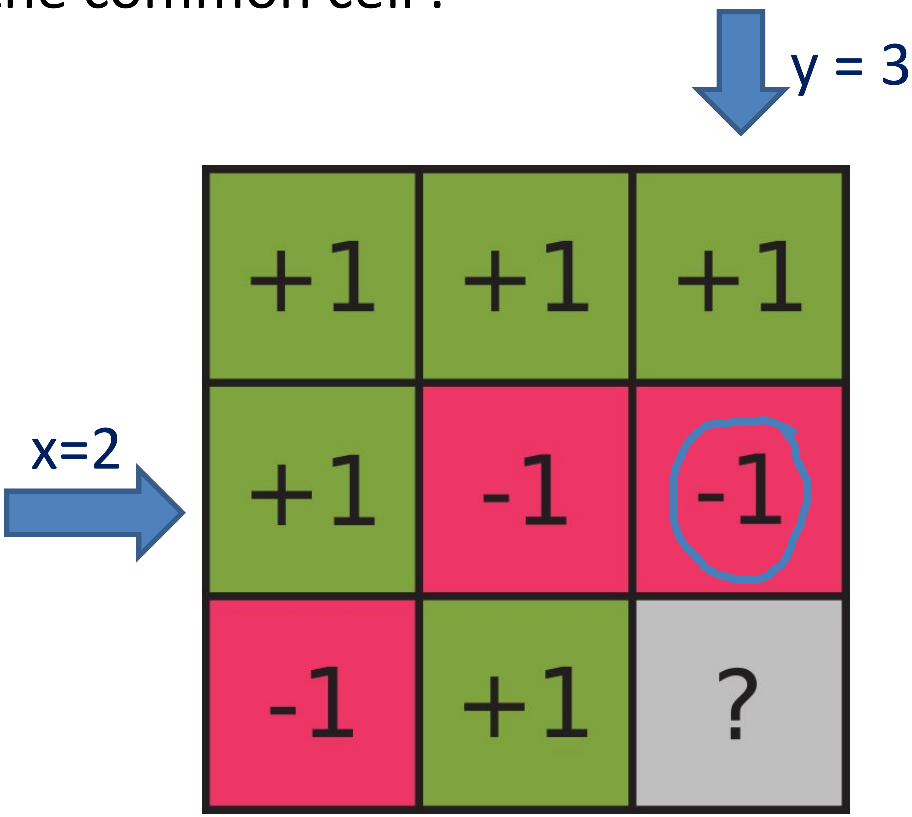
Bell nonlocality

Let's turn it into a nonlocality game (so-called Peres-Mermin game): Alice's random input labels the row. Bob's random input labels the column.



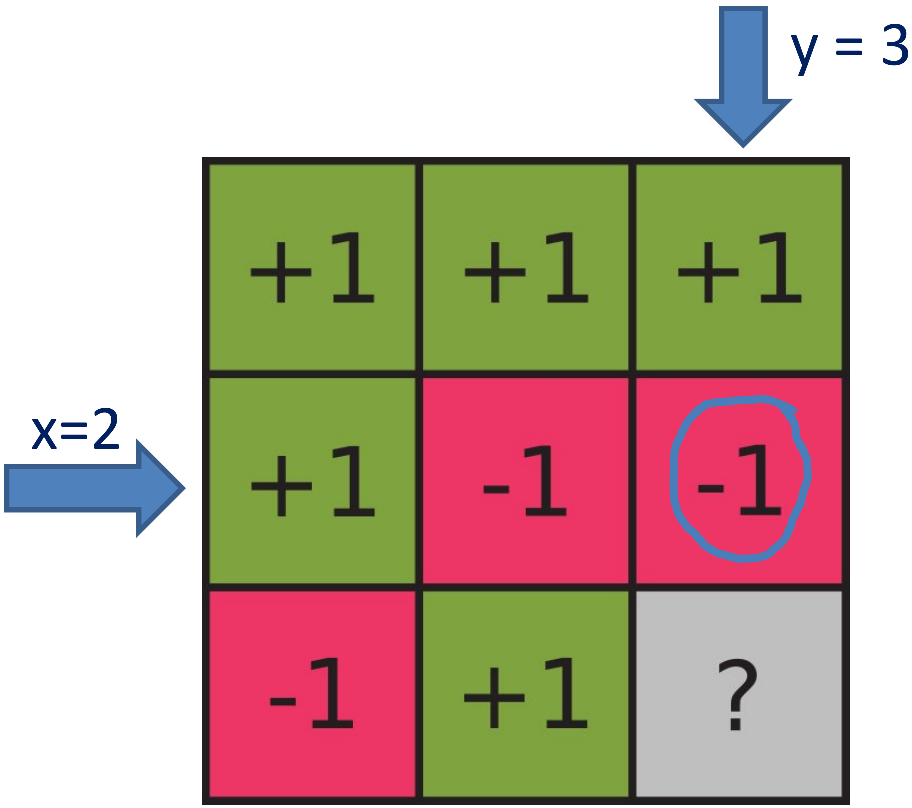
Bell nonlocality

Alice is asked to fill in her assigned row with ± signs (with even parity). Bob is asked to fill his column with ± signs (with odd parity).
They win the game if the two players always put the same sign in the common cell .



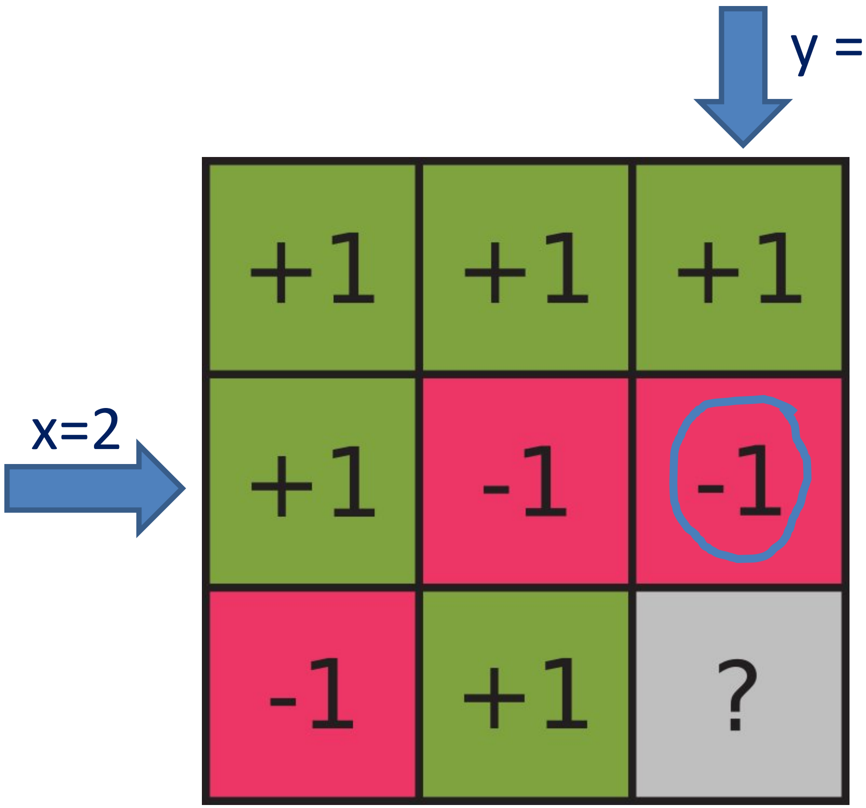
Bell nonlocality

It turns out that with classical strategies the best success probability is 8/9. This number corresponds to the local bound of the game.



Bell nonlocality

However, quantumly the players can win the game in 100% of the time (9/9). To this end, the players share two singlet pairs, and they perform special quantum measurements on their share of the state.



Bell nonlocality

In the above Bell nonlocality game the singlet state clearly gives advantage over local resources; it allows us to violate a Bell inequality associated with the game. Hence, we say that the singlet state is nonlocal.

Can a bound entangled state, a very weakly entangled state, be nonlocal as well? This is the question asked by Peres in 1999. He conjectured that such states always have a local model, so they cannot violate any Bell inequality. More formally he stated:

Nonlocal bound entangled states

Peres conjecture: Undistillable states admit a local model (A. Peres, Foundations of Physics 29, 589-614 (1999)):

”Note that there exist inseparable quantum states that cannot be distilled into singlets. In particular, quantum states whose partial transpose has no negative eigenvalue have that property. Thus, if the preceding conjectures are correct, it follows that these peculiar inseparable quantum states violate no Bell inequality, and therefore, owing to Farkas’s lemma, their statistical properties are compatible with the existence of local objective variables.”

Nonlocal bound entangled states

Until 2014 most of the works were in favour of Peres, e.g.:

- Violation of the CHSH inequality certifies that the underlying state can be distilled (Acin 2001, Masanes 2006). Hence it cannot be violated with PPT bound entangled states.

Nonlocal bound entangled states

Until 2014 most of the works are in favour of Peres, e.g.:

- A numerical method for upperbounding possible violation of a given Bell inequality for PPT bound entangled states was presented by Moroder et al. (PRL, 2013). This study presented more than 100 bipartite Bell inequalities, which cannot be violated using PPT bound entangled states (up to numerical precision).

Nonlocal bound entangled states

Hint that Peres conjecture maybe wrong:

- A PPT bound entangled state was presented by Moroder-Gittsovich-Huber-Gühne (PRL, 2014) which violates an EPR steering inequality. In this scenario Alice is an untrusted and Bob is a trusted party (Bob’s measurements are fully characterized).

Nonlocal bound entangled states

Then a counterexample to Peres conjecture appeared in 2014:

T. Vértesi & N. Brunner (2014). Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement. *Nature communications 5*, 5297.

We presented a 3x3 bipartite state, which is PPT bound entangled, and it violates a Bell inequality. So the state is Bell nonlocal.

Nonlocal bound entangled states

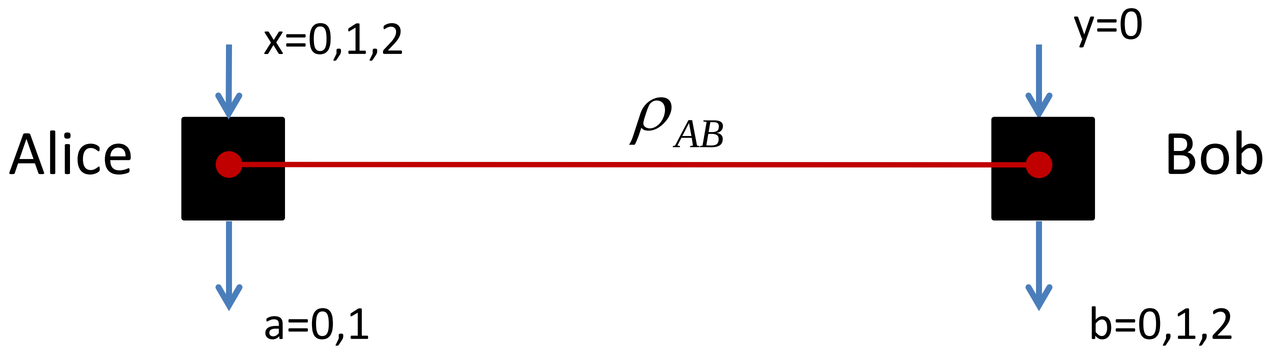
The state is of rank-4: $$\rho_{AB} = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i|$$

and it fulfills PT invariance: $$\mathrm{PT}(\rho_{AB}) = (\mathrm{I} \otimes T_B)\rho_{AB} = \rho_{AB}$$

This ensures that the state is PPT and therefore undistillable. This 3x3 state has minimal possible rank, and it is the smallest in terms of dimensions, since no PPT entangled state exists in smaller dimensional space.

Nonlocal bound entangled states

The Bell scenario (S. Pironio, J. Phys. A, 2014) is one of the simplest one beyond the CHSH (*m*=*r*=2) scenario:



P(a,b|x,y)

Nonlocal bound entangled states

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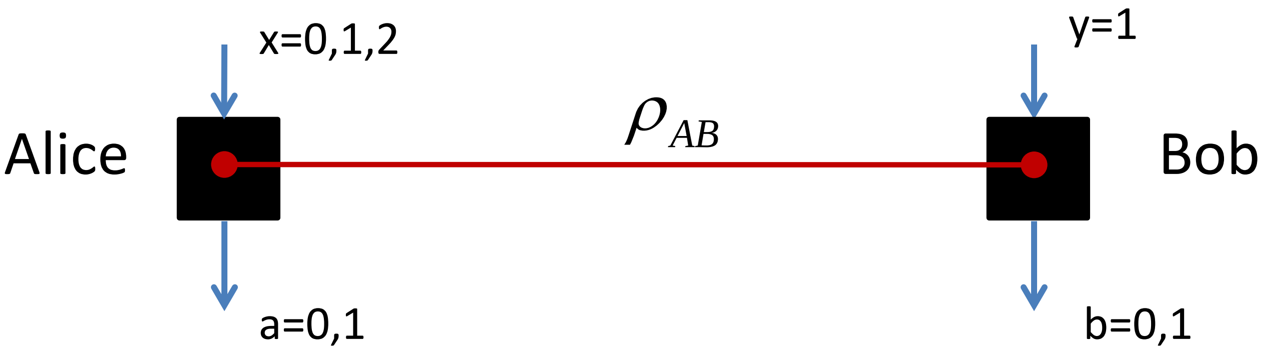
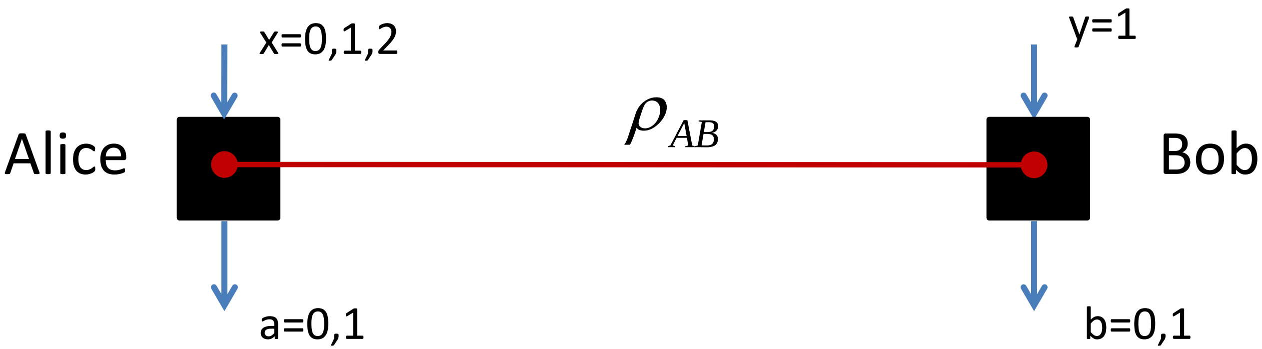


P(a,b|x,y)

Nonlocal bound entangled states

Bell inequality (S. Pironio, 2014):



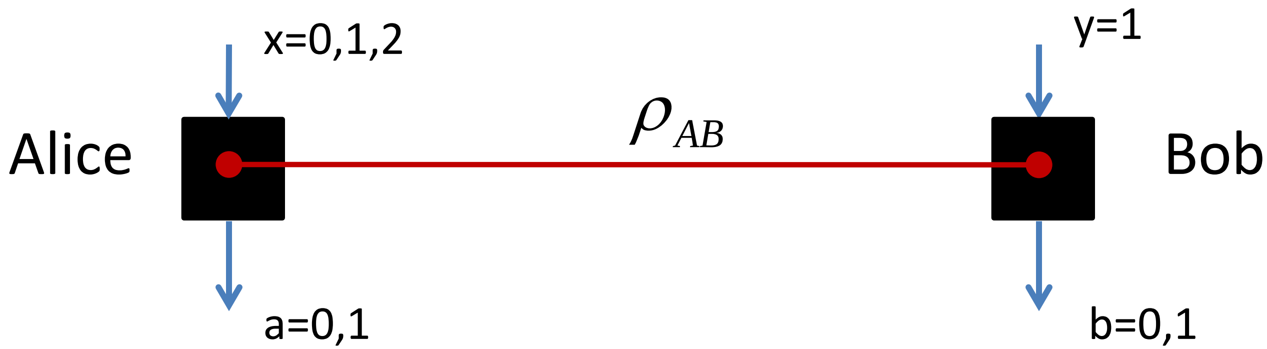
P(a,b|x,y) → Bell expression:

$$I = -p_A(0|2) - 2p_B(0|1) - p(01|00) - p(00|10) + p(00|20) + p(01|20) + p(00|01) + p(00|11) + p(00|21)$$

I <= 0 holds for all local P(a,b|x,y) distributions

Nonlocal bound entangled states

Bell inequality (S. Pironio, 2014):



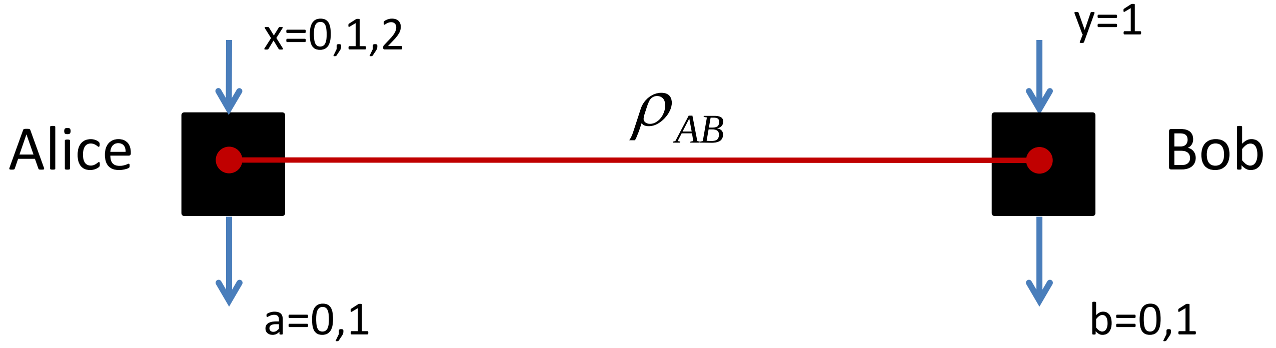
P(a,b|x,y) ⟹ Bell expression:

I <= 0 holds for all local P(a,b|x,y) distributions:

$$P(a,b\,|\,x,y) = \sum_{\lambda} p_{\lambda} P(a\,|\,x,\lambda) P(b\,|\,y,\lambda)$$

Nonlocal bound entangled states

Bell inequality (S. Pironio, 2014):

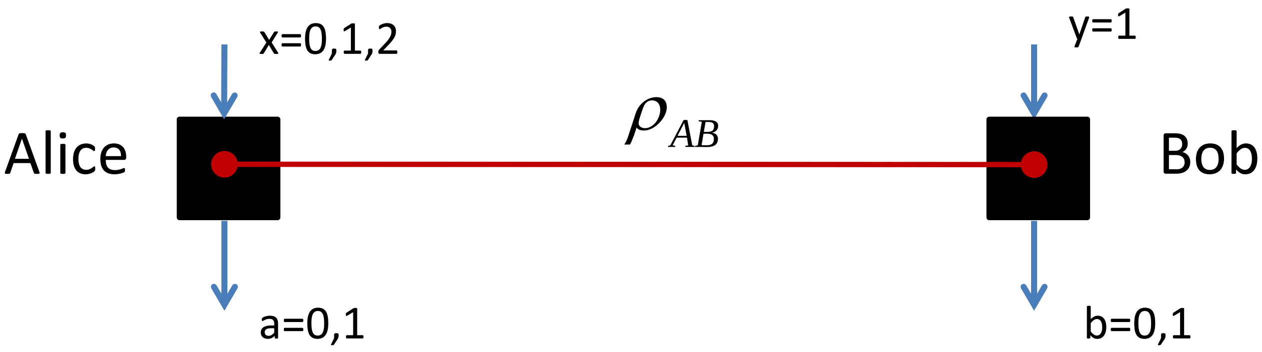


Quantumly:

$$P(a,b|x,y) = \mathrm{tr}\left(\rho\, M_{a|x} \otimes M_{b|y}\right)$$

Nonlocal bound entangled states

Bell inequality (S. Pironio, 2014):



Quantumly: with the use of the PPT state mentioned above

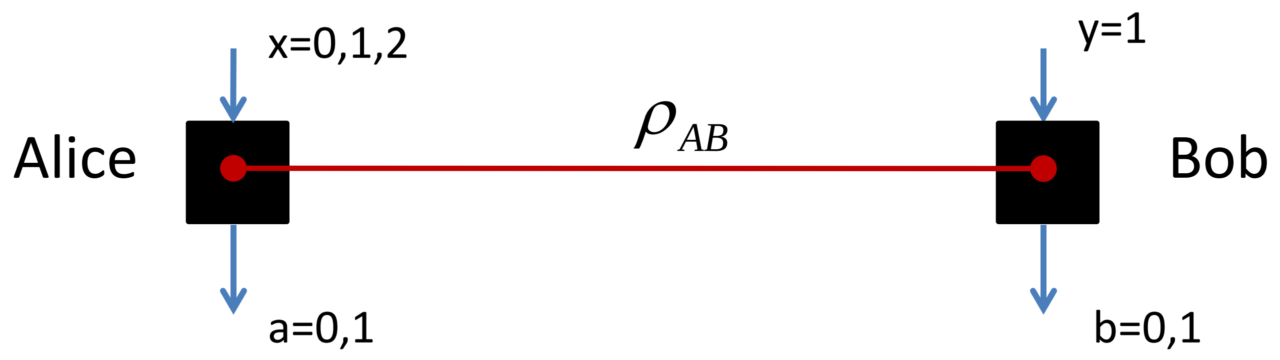
P(a,b|x,y) $= \mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$

we get a Bell value larger than zero:

$$I_{\mathrm{PPT}} = \frac{-3386 + 18\sqrt{42} - 5\sqrt{131} + 45\sqrt{5502}}{43025} \simeq 2.63144 \times 10^{-4}$$

Nonlocal bound entangled states

Bell inequality (S. Pironio, 2014):



Quantumly: with the use of the PPT state mentioned above

$$P(a,b|x,y) = \mathrm{tr}\left(\rho_{\mathrm{PPT}} M_{a|x} \otimes M_{b|y}\right)$$

we get a Bell value larger than zero.
To this end, we used an iterative method, similar in spirit to the one used to optimize FQ.

Nonlocal bound entangled states

An open problem: Do there exist PPT bound entangled states (possibly in high dimensions) which give rise to large violation of certain Bell inequalities? Possibly unbounded violation?

Nonlocal bound entangled states

Results so far:

o Bipartite d x d PPT states which violate families of Bell inequalities for any finite d.

Drawback: The Bell violation goes to zero as we increase the dimension d.

(i) S. Yu and C.H. Oh: A family of nonlocal bound entangled states, Phys. Rev. A 95, 032111 (2017).
(ii) K.F. Pál and T. Vértesi: Family of Bell inequalities violated by higher-dimensional bound entangled states, Phys. Rev. A 96, 022123 (2017).

Summary

We have shown that PPT bound entangled states are useful in overcoming the classical (shot noise) limit in quantum metrology and they can also be used to create Bell nonlocal correlations (thereby refuting Peres conjecture).

These results are based on the papers:

G. Tóth & T. Vértesi (2018). Quantum states with a positive partial transpose are useful for metrology. *Physical Review Letters 120*, 020506.

T. Vértesi & N. Brunner (2014). Disproving the Peres conjecture by showing Bell nonlocality from bound entanglement. *Nature communications 5*, 5297.

Thank you!