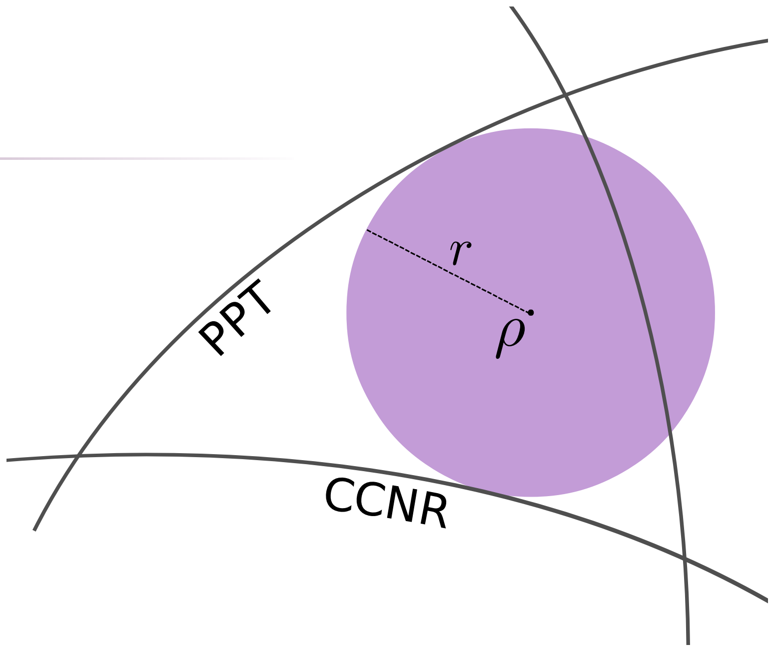
Bound entangled states fit for robust experimental verification

Gael Sentís | University of Siegen



joint work with

J.N. Greiner, J. Shang, J. Siewert, and M. Kleinmann

Entanglement Days 2018
Sep 28, 2018

arXiv:1804.07562

What is bound entanglement?

A bipartite entangled state ρ is **distillable**, if

$$\rho^{\otimes k} \xrightarrow{\text{LOCC}} |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle \quad \text{with finite probability.}$$

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Even though one cannot distil maximally entangled states, bound entanglement is still useful for…

- *Teleportation* — *Horodecki et al. PRL (1999)*; *Cavalcanti et al. PRL (2017)*
- *QKD* — *Horodecki et al. PRL (2005)*; *Horodecki et al. IEEE Trans. Inform. Theory (2008)*
- *Metrology* — *Czekaj et al. PRA (2015)*; *Tóth, Vértesi PRL (2018)*

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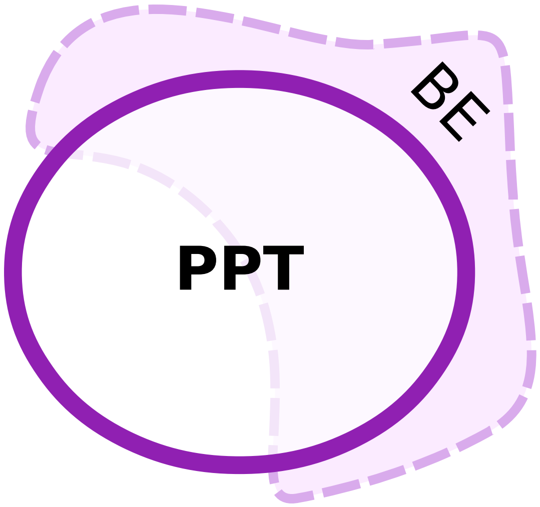
THE PPT CRITERION

[Horodecki[3]]

"Any state with positive partial transpose (PPT) is undistillable"

Smallest system with bound entanglement is two qutrits



What is *multipartite* bound ent.?

A somewhat different concept…

A multipartite state is **bound entangled**, if

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An example: **4-qubit Smolin state**

$$\rho_{ABCD} = \frac{1}{4}\left(\Phi^+ + \Phi^- + \Psi^+ + \Psi^-\right), \quad \text{with} \quad \Psi^- = |\psi^-\rangle\langle\psi^-|_{AB} \otimes |\psi^-\rangle\langle\psi^-|_{CD}, \ldots$$

This is globally entangled, but **separable** with respect to all bipartitions …

Feels like cheating!

Experiments

Multipartite BE

- Amselem & Bourneane, Nature Phys. (2009)
  Lavoie *et al.*, PRL (2010)
- Barreiro *et al.*, Nature Phys. (2010)
- Kampermann *et al.*, PRA (2010)
- Dobek *et al.*, PRL (2011), Laser Phys. (2013)
- …

Bipartite BE

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All these experiments use a **limited statistical analysis** and **symmetry assumptions**

Experiments | Certification protocol

The usual protocol in use:

I. **Perform state tomography**

II. **Reconstruct state** (maximum likelihood or least squares)

III. **Bootstrap**

IV. **Report fraction of bootstrapped states with bound entanglement**

Sounds reasonable…

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Sounds reasonable… but it cannot be trusted, **at all!**

- *“There cannot be an unbiased state reconstruction”* [Schwemmer et al., PRL 2015]

- Bound entangled states form non-convex sets and are high-dimensional (reconstructions prone to significant bias)

A proper statistical analysis

Noncentral χ^2 hypothesis test

If ρ_0 admits a bound entangled ball with radius r_0 around it, then we can compute the upper bound*

$$\mathbf{P}[\,\textbf{false positives}\,] \leq \mathbf{P}[\;\textbf{data looks good}\;|\;\|\rho_0 - \rho_{\mathrm{exp}}\|_2 \geq r_0]$$

*This yields a **p-value**.*

**assuming normal-distributed data*

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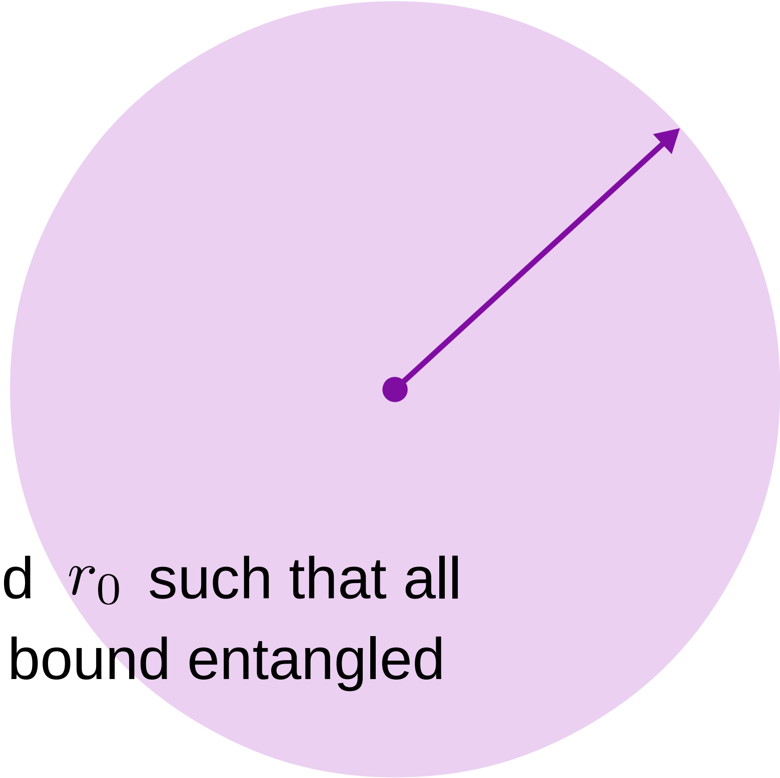
Advantages

- Easy to compute
- Correct!
- No computing cost

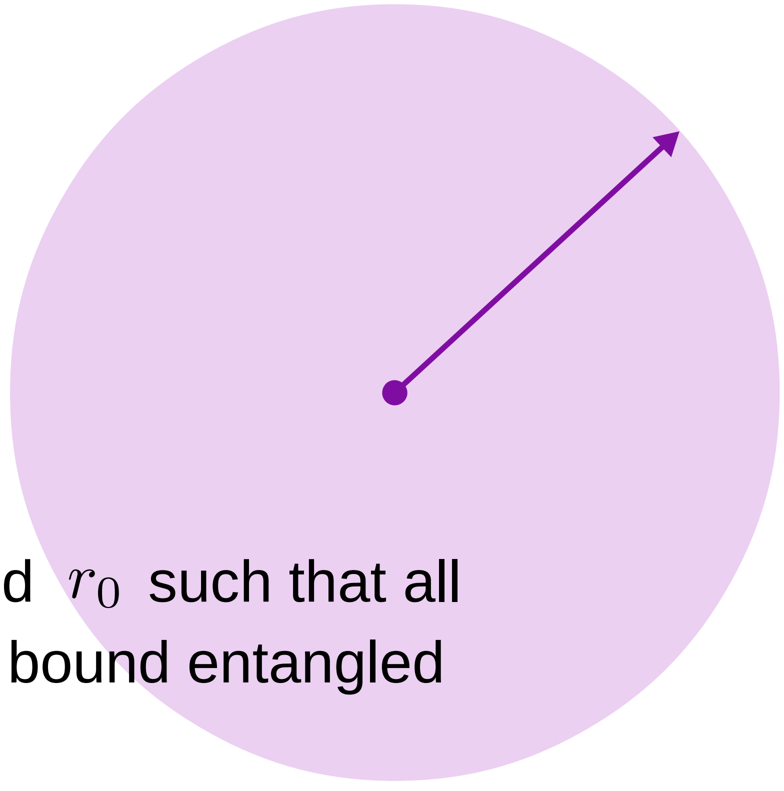
Disadvantages

- Conservative
- Assumes Gaussian regime



OUR TASK

For a bound entangled state ρ_0, find r_0 such that all states τ with $\|\rho_0 - \tau\|_2 \leq r_0$ are bound entangled



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***General idea**: test τ for undistillability (via PPT) and entanglement (via CCNR)*

Undistillability | Checking PPT

PPT criterion [HHH]: *a state* ρ *is undistillable if* $\Gamma(\rho) \geq 0$

|
partial transpose

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Lemma: *if* $\|\rho_0 - \tau\|_2 \leq r_0$ *, then*

$$\lambda_{\min}[\Gamma(\tau)] \geq \lambda_{\min}[\Gamma(\rho_0)] - r_0\sqrt{1 - 1/d}$$

dimension of the joint system

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dimension of the joint system

All states around ρ_0 **are undistillable if** $\lambda_{\min}[\Gamma(\rho_0)] \geq r_0\sqrt{1 - 1/d}$

Entanglement | Checking CCNR

Computable cross norm or Realignment (CCNR) criterion [Rudolph, Chen & Wu]:
Let $(g_k)_k$ *be an orthonormal basis of the Hermitian operators, and define* $R(\rho)_{k,\ell} = \mathrm{tr}\,(\rho\, g_k \otimes g_\ell)$*. Then, a state* ρ *is entangled if* $\|R(\rho)\|_1 > 1$ *.*

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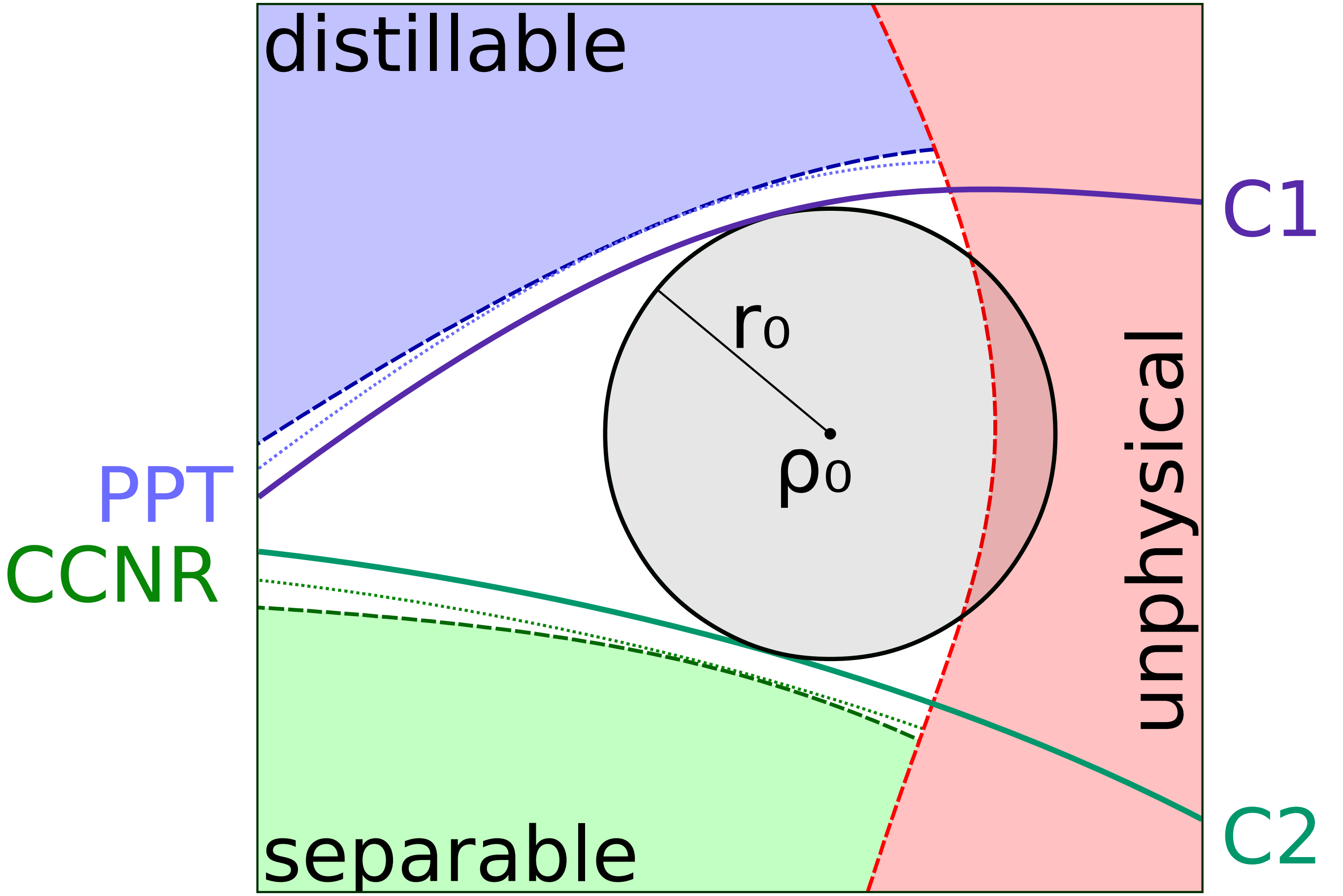
Lemma: if $\|\rho_0 - \tau\|_2 \leq r_0$, then

$$\|R(\tau)\|_1 \geq \|R(\rho_0)\|_1 - r_0\sqrt{d}$$

All states around ρ_0 are entangled if $\|R(\rho_0)\|_1 > 1 + r_0\sqrt{d}$

Undistillable & entangled states



C1: $\lambda_{\min}[\Gamma(\rho_0)] \geq r_0\sqrt{1-1/d}$

C2: $\|R(\rho_0)\|_1 > 1 + r_0\sqrt{d}$

Optimal bound entangled states

Given ρ_0 , the conditions put a bound on the largest r_0 .

We can as well search for the state ρ_0 with the largest r_0!

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Optimization problem

Find ρ_0 that realizes the optimal solution of

$$\begin{aligned} &\text{maximize} && r_0 \\ &\text{subject to} && \lambda_{\min}[\Gamma(\rho_0)] \geq r_0\sqrt{1-1/d} \\ & && \|R(\rho_0)\|_1 > 1 + r_0\sqrt{d} \end{aligned}$$

In principle, this yields the state with the largest ball of BE around it for a given dimension.

In practice, we need to optimize over families of states with few parameters.

Example 1: two qutrits

Family of states [Baumgartner *et al.*, PRA (2006)]

$$\rho = a\,|\phi_3\rangle\langle\phi_3| + b\sum_{k=0}^{2}|k,k\oplus 1\rangle\langle k,k\oplus 1| + c\sum_{k=0}^{2}|k,k\oplus 2\rangle\langle k,k\oplus 2|\ ,$$

(Horodecki states are in here)*

with $|\phi_3\rangle = \sum_i |ii\rangle/\sqrt{3}$

The optimization can be solved analytically!

*[HHH, PRL 80, 5239 (1998)]

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OPTIMAL STATES

$a \approx 0.21289,\ b \approx 0.04834,\ \text{and}\ c \approx 0.21403$

$r_0 \approx 0.02345$

$\text{rank}(\rho_0) = 7$

value of r_0 is (almost) tight with respect to CCNR and PPT

*[HHH, PRL 80, 5239 (1998)]

Example 1: two ququarts

Family of Bloch-diagonal states

$$\rho = \sum_k x_k g_k \otimes g_k \ , \quad \text{where} \qquad g_k = (\sigma_\mu \otimes \sigma_\nu)/2$$

(Smolin state is in here)*

The optimization can be turned into 32.768 linear programs

The feasibility polytope can be determined, it has 254.556 vertices

*[Smolin, PRA 63, 032306 (2001)]

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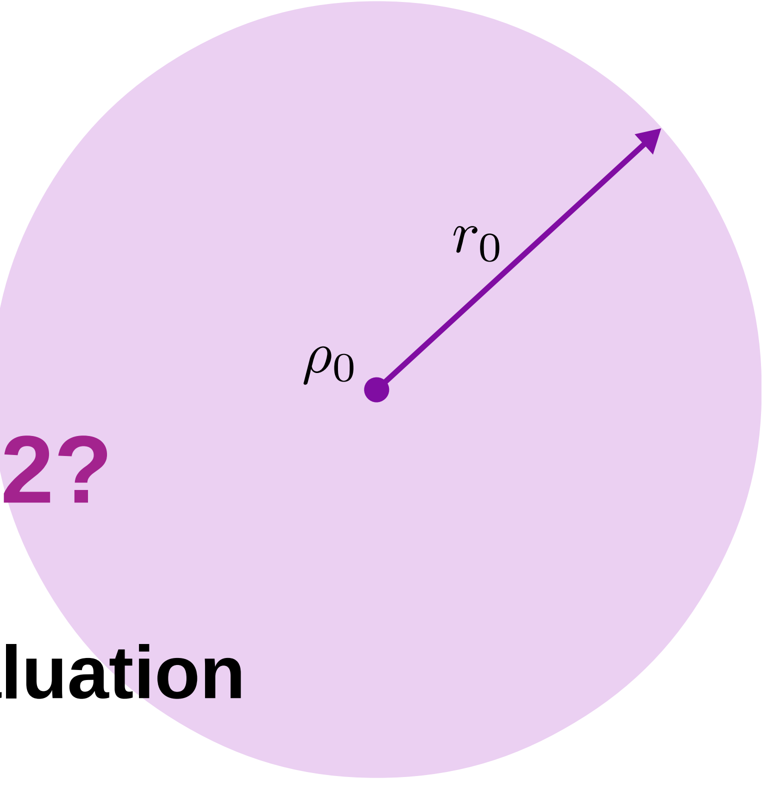
OPTIMAL STATES

$$\mathrm{rank}(\rho) < 9 \longrightarrow r_0 = 0$$

$$\mathrm{rank}(\rho) = 9 \longrightarrow r_0 \approx 0.0161$$

$$\mathrm{rank}(\rho) \geq 10 \longrightarrow r_0 \approx 0.0214$$

*[Smolin, PRA 63, 032306 (2001)]



HOW LARGE IS 0,02?

Some words on data evaluation

Data evaluation

Protocol

I. Characterize tomographic measurements with high precision

II. Decide critical statistical parameters

III. Perform state tomography

IV. Evaluate χ^2 test

Data evaluation

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I. **Characterize tomographic measurements with high precision**
II. **Decide critical statistical parameters**
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Statistical parameters

- *Distribution* of the data (Poissonian, multinormal,…)
- *Preprocessing method*: [raw data] $\longrightarrow \boldsymbol{x}$
- *Covariance matrix* Σ of $\boldsymbol{x}$
- *Test function* $\hat{t} : \boldsymbol{x} \mapsto t$
- *Significance threshold*, which determines critical value $t^*(r_0)$

Data evaluation

A good test function

$$\hat{t}(\boldsymbol{x}) = \|\Sigma^{-1/2}[\boldsymbol{x}_0 - \boldsymbol{x}]\|_2 \quad , \quad \text{where } \boldsymbol{x}_0 \text{ is the expected } \boldsymbol{x} \text{ for } \rho_0$$

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Roughly, proceed as follows:

1 *Compute a critical value t^* such that*

$\mathbf{P}[$ **false positives** $] \leq \mathbf{P}[\hat{t}(\boldsymbol{x}) \leq t^* \mid \|\rho_0 - \rho_{\text{exp}}\|_2 \geq r_0] \leq$ **significance threshold**

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2. *Certify bound entanglement with set significance if $\hat{t}(\boldsymbol{x}) \leq t^*$.*

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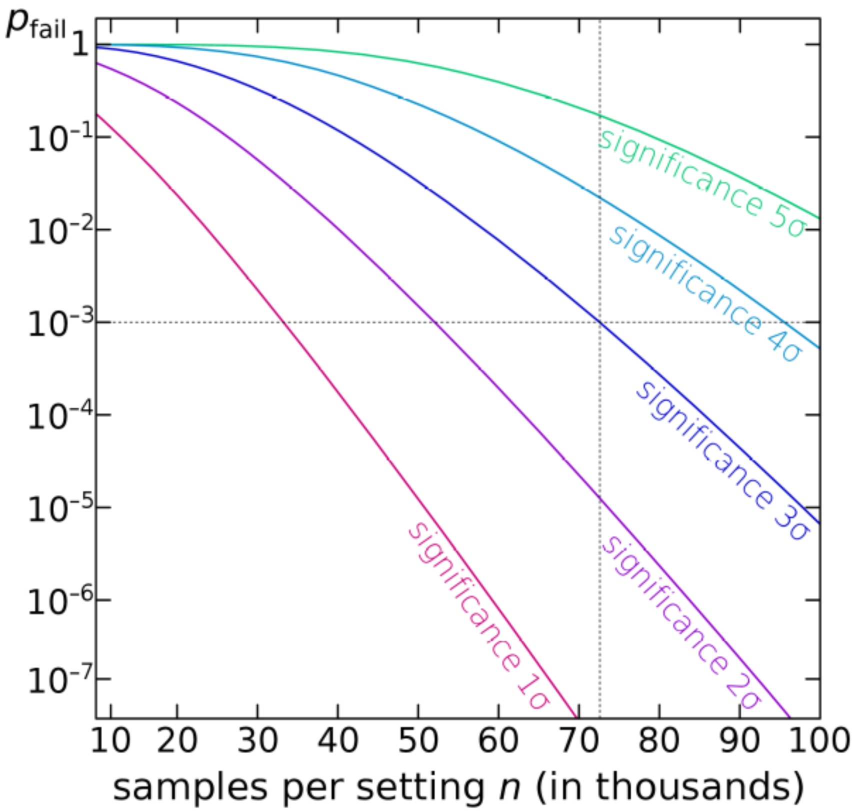
2. *Certify bound entanglement with set significance if $\hat{t}(\boldsymbol{x}) \leq t^*$.*

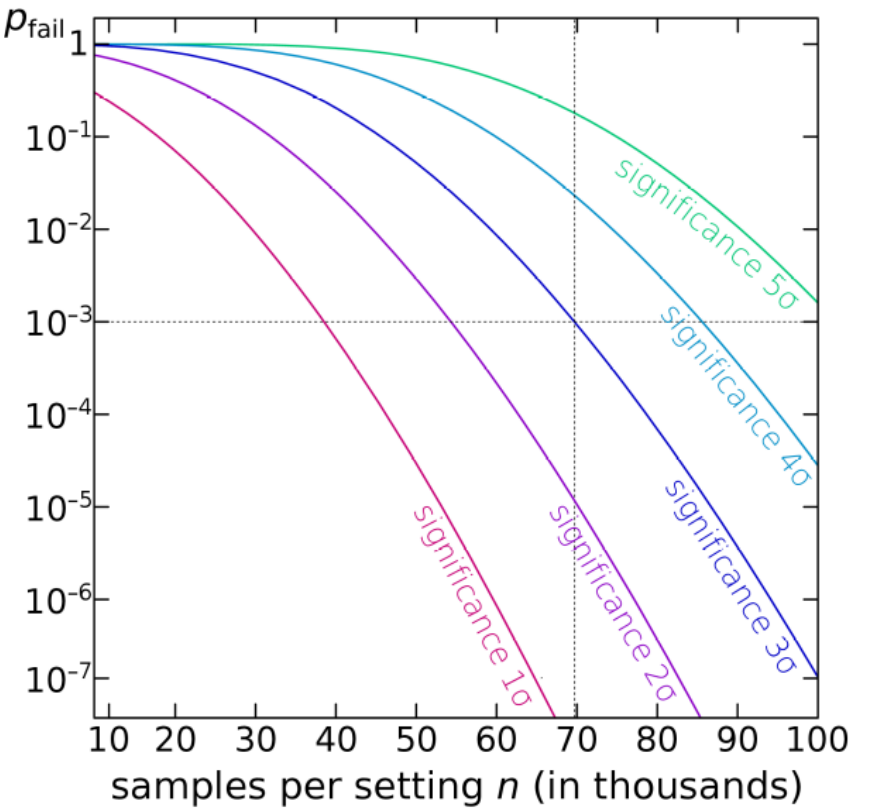
* **Even if $\|\rho_0 - \rho_{\text{exp}}\|_2 \leq r_0$, it could happen that $\hat{t}(\boldsymbol{x}) \geq t^*$.**
The probability that this happens decreases with the number of samples

Data eval. | Precision requirements

two qutrits



two ququarts



Probability p_{fail} that data

- does **NOT** confirm bound entanglement,
- at a level of significance of $k\sigma$ standard deviations,
- assuming 5% (2,5%) white noise for the qutrit (ququart) case.

Summary

- For suitable parametrized families of states, it is feasible to compute a target state ρ_0 with maximal r_0 such that

$$\|\rho_0 - \tau\|_2 \leq r_0 \Longrightarrow \tau \text{ is bound entangled}$$

- For families of qutrits and ququarts, $r_0 \approx 0.02$
- We show how to obtain a p-value for the null hypothesis “the state is NOT bound entangled” using tomographic data.
- With realistic assumptions, we obtain that ~10^5 samples are required to certify bound entanglement with 3σ significance.

GS, J.N. Greiner, J. Shang, J. Siewert, M. Kleinmann, arXiv:1804.07562