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Challenges in Non-Commutative Information Geometry

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Introduction

My goal: Find alternative description of quantum information theory

- Standard: a density matrix ρ describes a statistical mixture of quantum states
- Alternative: use the GNS-representation of mixed quantum states

Why? Find a more general theory

- not relying on the properties of the trace
- using elements of information geometry
- using elements of non-commutative geometry

J. Naudts, *Quantum Statistical Manifolds,* Entropy 20(6), 472 (2018); Correction submitted

Contents

Standard theory

See for instance Dénes Petz, *Quantum Information Theory and Quantum Statistics* (Springer, 2008)

The *density matrix* ρ is a complex *n*-by-*n* matrix satisfying
$\rho \geq 0$ and $\operatorname{Tr} \rho = 1$

A special role is played by the density matrix

$$\rho_0 = \frac{1}{n}\mathbb{I} = \frac{1}{n}\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The space of *n*-by-*n* matrices forms a Hilbert space $\mathcal{H}^{\mathrm{HS}}$
for the Hilbert-Schmidt scalar product

$$\langle A, B \rangle_{\mathrm{HS}} = \operatorname{Tr} A^\dagger B = n \operatorname{Tr} \rho_0 A^\dagger B = n \langle A^\dagger B \rangle_{\rho_0}$$

The relative modular operator

Fix two strictly positive density matrices ρ and σ

Note that in general ρ and σ do not commute

(Petz 86) introduced the *relative modular operator* $\Delta_{\rho,\sigma}$ on $\mathcal{H}^{\mathrm{HS}}$
defined by $\Delta_{\rho,\sigma} A = \rho A \sigma^{-1}$ for all A

The relative entropy (Umegaki 1962, Araki 1976), defined by

$$D(\sigma||\rho) = \mathrm{Tr}\, \sigma(\log \sigma - \log \rho)$$

can be written as $D(\sigma||\rho) = \langle \sigma^{1/2} | [\log \Delta_{\sigma,\rho,}] \sigma^{1/2} \rangle_{\mathrm{HS}}$

Proof Write $\Delta_{\sigma,\rho} = L_\rho R_{\sigma^{-1}}$ where $L_\rho R_{\sigma^{-1}} = R_{\sigma^{-1}} L_\rho$

The metric

From the relative entropy one derives Bogoliubov's scalar product, which can be written as a metric tensor

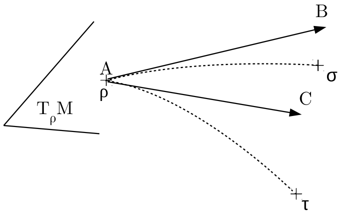
$$
\begin{aligned}
g_{\sigma,\tau}(\rho) &= \int_0^1 \mathrm{d}u \,\mathrm{Tr}\, \rho^u (\log\sigma - \log\rho)\rho^{1-u}(\log\tau - \log\rho) \\
&\quad - D(\rho||\sigma) D(\rho||\tau) \\
&= \int_0^1 \mathrm{d}u \,\langle \rho^{1/2} | [\log \Delta_{\tau,\rho}] \rho^u [\log \Delta_{\sigma,\rho}] \rho^{-u} | \rho^{1/2} \rangle_{\mathrm{HS}} \\
&\quad - \langle \rho^{1/2} | [\log \Delta_{\tau,\rho}] \rho^{1/2} \rangle_{\mathrm{HS}} \, \langle [\log \Delta_{\sigma,\rho}] \rho^{1/2} \rangle_{\mathrm{HS}}.
\end{aligned}
$$

ρ, σ, τ are strictly positive density matrices

The metric cont'd

ρ is a point A in the manifold $\mathbb{M}$ of strictly positive density matrices

With σ, τ correspond two points B, C in the tangent plane $T_\rho \mathbb{M}$



$g_{\sigma,\tau}(\rho)$ is the scalar product between the vectors $\overrightarrow{AB}, \overrightarrow{AC}$

The GNS representation (Gelfand, Naimark, Segal)

$n \times n$ matrices represented as $n^2 \times n^2$ block matrices

In this representation the density matrix ρ is replaced by a 'wave function' Ω_ρ.

- Diagonalize ρ: $\rho = \sum p_n |\psi_n\rangle\langle\psi_n|$ with $p_n > 0$, $\sum_n p_n = 1$.

- Let $\Omega_\rho = \sum_n \sqrt{p_n}\, \psi_n \otimes \psi_n$

- Then
$$\begin{aligned}\langle\Omega_\rho|A\otimes\mathbb{I}|\Omega_\rho\rangle &= \sum_{m,n}\sqrt{p_m p_n}\langle\psi_m\otimes\psi_m|(A\psi_n)\otimes\psi_n\rangle \\ &= \sum_{m,n}\sqrt{p_m p_n}\langle\psi_m|A\psi_n\rangle\langle\psi_m|\psi_n\rangle \\ &= \sum_n p_n\langle\psi_n|A\psi_n\rangle \\ &= \mathrm{Tr}\,\rho A = \langle A\rangle_\rho .\end{aligned}$$

The GNS representation Cont'd

In particular, with the tracial density matrix $\rho_0 = \frac{1}{n}\mathbb{I}$ corresponds

$$\Omega_0 = \frac{1}{n}\sum_{i=1}^{n} \psi_i \otimes \psi_i,$$

where $\psi_1, \cdots, \psi_n$ is any orthonormal bazis.

Let $\mathcal{A}$ denote the space of all 'operators' of the form $A \otimes \mathbb{I}$

The *commutant* $\mathcal{A}'$ consists of all operators of the form $\mathbb{I} \otimes A$

The chart centered at ρ

The metric tensor $g_{\sigma,\tau}(\rho)$ involves 3 density matrices
ρ fixes a point A in the manifold $\mathbb{M}$,
σ, τ fix the points B and C in the tangent plane $T_\rho\mathbb{M}$

ρ is described by the 'wave function' Ω_ρ
How to describe σ and τ?

Theorem Given ρ, σ there exists a unique $K = K^\dagger$ in $\mathcal{A}'$

such that $$K\Omega_\rho = \int_0^1 \mathrm{d}u\, \rho^u[\log \Delta_{\sigma,\rho}]\rho^{-u}\Omega_\rho + D(\rho||\sigma)\Omega_\rho$$

and $\langle\Omega_\rho|K\Omega_\rho\rangle = 0$

The map $\sigma \mapsto K$ is a chart for $\mathbb{M}$
It is centered at ρ: Indeed, $\sigma = \rho$ implies $K = 0$

The metric

Proposition Consider the chart $\chi_\rho : \sigma \mapsto K$ centered at ρ. Then

1) There exists a strictly positive operator G_ρ in $\mathcal{A}$ such that

$$G_\rho K \Omega_\rho = \left[(\Delta_{\rho,\sigma} + D(\rho||\sigma)\right] \Omega_\rho$$

2) For each pair $K = \chi_\rho(\sigma)$ and $L = \chi_\rho(\tau)$ is

$$g_{\sigma,\tau}(\rho) = (K\Omega_\rho, G_\rho L \Omega_\rho)$$

Note: G_ρ is in $\mathcal{A}$ while K and L belong to the commutant $\mathcal{A}'$

Positivity of the metric follows immediately from $G_\rho > 0$

The exponential connection

The geodesics $t \mapsto \rho_t$ are such that

$$\begin{aligned} \log \rho_t &= (1-t)\log \rho_0 + t \log \rho_1 - \zeta(t) \\ &= \log \rho_0 + tH - \zeta(t) \end{aligned}$$

$\zeta(t)$ is a normalizing function, H is defined by $H = \log \rho_1 - \log \rho_0$

Proposition

1) $\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \langle H \rangle_t = D(\rho_t || \rho_0) - D(\rho_t || \rho_1)$
2) $t \mapsto \zeta(t)$ is convex
3) $\chi_\rho(\rho_t) = (1-t)\chi_\rho(\rho_0) + t\chi_\rho(\rho_1)$

The latter shows that the chart χ_ρ is an affine coordinate

Summary

- It is possible to eliminate references to the trace operation using the GNS representation
- Label σ relative to ρ with an operator $K = K^\dagger$ in $\mathcal{A}'$
  This gives a chart $K = \chi_\rho(\sigma)$ of $\mathbb{M}$, centered at ρ
- Scalar product of Bogoliubov $\Rightarrow$ $G_\rho > 0$ in $\mathcal{A}$
- $K = \chi_\rho(\sigma)$ is an affine coordinate for the exponential connection

Challenges How much of this can be generalized?