Quantum entanglement from single particle information

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Problem

- System of L qubits, $\mathcal{H} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$.
- $\mathbb{P}(\mathcal{H})$, $\phi \in \mathcal{H}$, $[\phi] \in \mathbb{P}(\mathcal{H})$.
- One qubit base: $\{|0\rangle, |1\rangle\}$.
- **Problem**: Classification of pure states by their entanglement type
- What can we say about entanglement type by looking at 1-particle reduced density matrices?

Setting

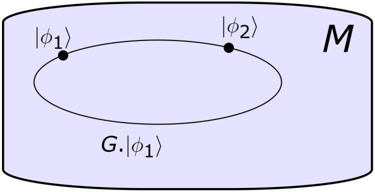
- SLOCC operations – $G = SL(2,\mathbb{C})^{\times L}$

$$[g.\phi] = [g_1 \otimes \cdots \otimes g_L \phi],\; g_k \in SL(2,\mathbb{C}),\; \phi \in \mathcal{H}.$$

- Two states $[\phi_1]$ and $[\phi_2]$ are equally entangled iff

$$[g.\phi_1] = [\phi_2],\, g \in G$$

- Let $\mathcal{C}_\phi := G.\phi = \{g.\phi : g \in G\}$



Two qubits

$G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$

Schmidt decomposition:

$$|\Psi\rangle = p_1|00\rangle + p_2|11\rangle, \;\; |p_1|^2 + |p_2|^2 = 1$$

There are two entanglement classes (two G-orbits)

1. $p_1 \neq 0$ and $p_2 \neq 0$ – entangled states
2. $p_1 = 0$ and $p_2 \neq 0$ or *vice versa* – separable states

$$\rho_1(\Psi) = \begin{pmatrix} |p_1|^2 & 0 \\ 0 & |p_2|^2 \end{pmatrix}, \;\; \rho_2(\Psi) = \begin{pmatrix} |p_1|^2 & 0 \\ 0 & |p_2|^2 \end{pmatrix}$$

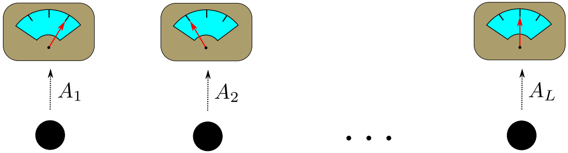
Conclusion: One can distinguish between entanglement classes by measuring spectra of 1-qubit RDMs

Three qubits

- $G = SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$
- There is no Schmidt decomposition
- There are six entanglement classes (G-orbits in $\mathbb{P}(\mathcal{H})$):
  1. Separable states
  2. 3 orbits of BiSeparable states
  3. $[G.\phi_W]$, where $\phi_W = |001\rangle + |010\rangle + |100\rangle$
  4. $[G.\phi_{GHZ}]$, where $\phi_{GHZ} = |000\rangle + |111\rangle$
- Can we distinguish between these classes by measuring spectra of 1-qubit RDMs?
- How about more qubits?

Problems of the classification

- Number of classes $\mathcal{C}_\phi = [G.\phi]$ is infinite starting from the system of four qubits.
- Number of parameters required to distinguish between classes C_ϕ grows exponentially with the number of qubits.
- These parameters, e.g. invariant polynomials typically lack physical meaning and are not measureable.
- We want to introduce a classification, which is much more robust by organising classes $\mathcal{C}$ into a finite number of families using spectra of 1-qubit RDMs.



1-qubit RDMs

- $\rho_i([\phi])$ - the i-th one-qubit Reduced Density Matrix (RDM)

$$\mu([\phi]) = \left(\rho_1([\phi]) - \frac{1}{2}I, \ldots, \rho_L([\phi]) - \frac{1}{2}I\right)$$

- The ordered spectrum of $\rho_i([\phi]) - \frac{1}{2}I$ is given by

$$\sigma\left(\rho_i([\phi]) - \frac{1}{2}I\right) = (-\lambda_i, \lambda_i)\,, \; \lambda_i \in [0, \frac{1}{2}].$$

- The collection of spectra for $[\phi] \in \mathbb{P}(\mathcal{H})$:

$$\Psi : \mathbb{P}\mathcal{H} \to \left[0, \frac{1}{2}\right]^{\times L}, \;\; \Psi([\phi]) = \{\lambda_1, \lambda_2, \ldots, \lambda_L\}.$$

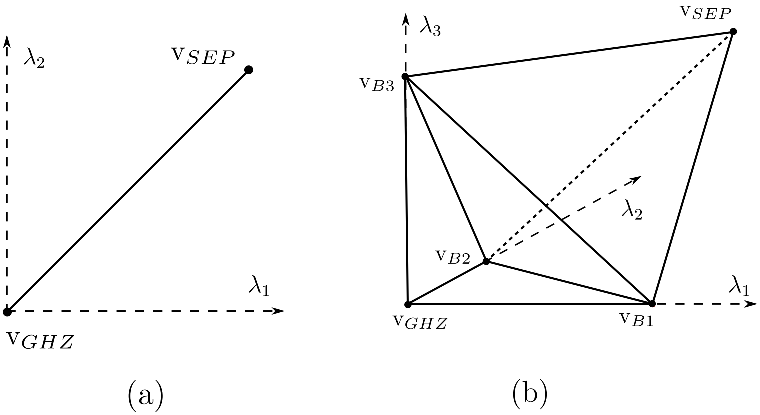
First Convexity Theorem

- $\Delta_{\mathcal{H}} := \Psi(\mathbb{P}\mathcal{H})$ is a convex polytope.
- Follows from the momentum map convexity theorem (Kirwan '84)
- Higuchi, Sudbery, Szulc '03 - This polytope is given by the intersection of

$$\forall_i \ \left(\frac{1}{2} - \lambda_i\right) \leq \sum_{j \neq i} \left(\frac{1}{2} - \lambda_j\right),$$

with the cube $\left[0, \frac{1}{2}\right]^{\times L}$.

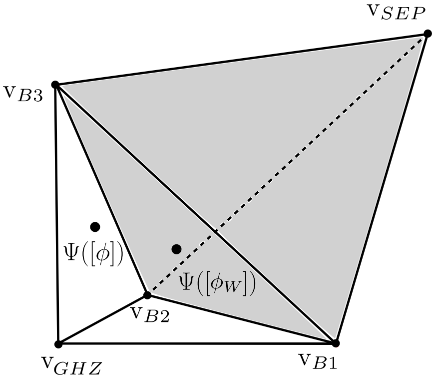
Polytopes for 2 and 3 qubits



Second Convexity Theorem

- $\mathcal{C}_\phi = [G.\phi]$
- $\Delta_{\mathcal{C}_\phi} = \Psi(\overline{\mathcal{C}_\phi})$ is a convex polytope.
- Follows from the convexity theorem of Brion '87
- $\Delta_{\mathcal{C}_\phi}$ is called an Entanglement Polytope (EP)
- Introduced to QI in '12 (AS, Oszmaniec, Kuś) and (Walter, Doran, Gross, Christandl)
- Although for $L \geq 4$ the number of classes $\mathcal{C}$ is infinite, the number of polytopes $\Delta_{\mathcal{C}}$ is always finite!
- Brion's theorem: Finding EPs requires knowing the generating set of covariants.
- This was solved only up to 4 qubits (Briand, Luque, J.-Y. Thibon 2003).

Entanglement Polytopes for three qubits

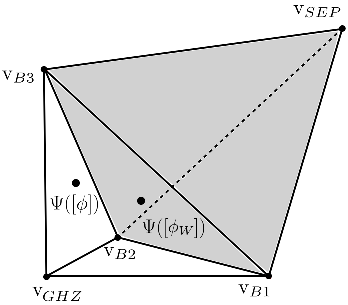


$$|\phi_W\rangle = \frac{1}{\sqrt{3}}\left(|011\rangle + |101\rangle + |110\rangle\right)$$

$$|\phi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

Properties of Entanglement Polytopes

- Entanglement polytopes are typically not disjoint, $\Delta_{\mathcal{C}} \cap \Delta_{\mathcal{C}'} \neq \emptyset$.
- Example: $\Delta_{\mathcal{C}_{GHZ}} = \Delta_{\mathcal{H}}$ thus $\Delta_{\mathcal{C}_\phi} \subset \Delta_{\mathcal{C}_{GHZ}}$ for every C_ϕ
- Entanglement polytopes can be regarded as entanglement witnesses.



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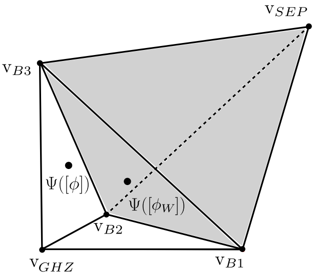

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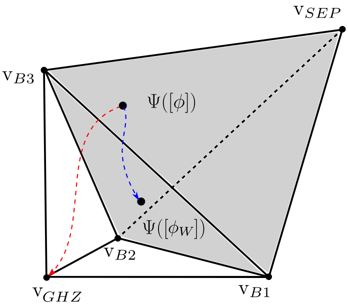
EPs as entanglement witnesses:

- For $[\phi] \in \mathbb{P}(\mathcal{H})$ we give a list of polytopes that do not contain $\Psi([\phi])$.
- The decision-making power of EPs is determined by the volume of the region in $\Delta_{\mathcal{H}}$ where many EPs overlap.
- Problem: Finding entanglement polytopes, even for five qubits, is in fact intractable!

Resolving overlaps

- For a polytope Δ_C let $\overline{\lambda}_C$ be the point that is closest to the origin $\overline{0}$.
- Using momentum map techniques we can construct a protocol that transforms a given state ϕ using G operations to a state with $\overline{\lambda} = \overline{\lambda}_{\mathcal{C}_\phi}$.
- This way we can (at least partially) resolve overlaps between polytopes)



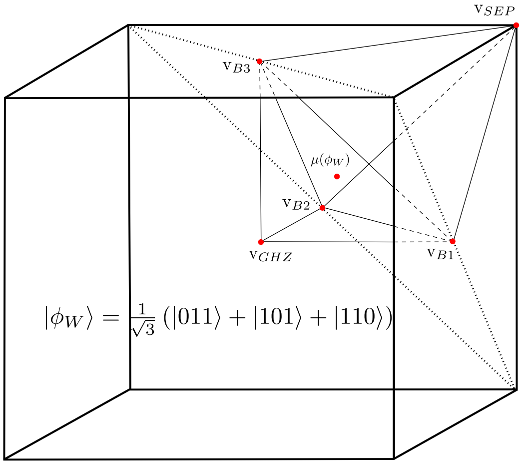
Understand the distribution of
$\|\overline{\lambda}_{\mathcal{C}}\|^2$ in $\Delta_{\mathcal{H}}$
for large number of qubits.

Procedure for finding $\overline{\lambda}_C$ for L qubits

- '15 TM and A. Sawicki - the procedure for finding $\overline{\lambda}_C$ using momentum map results of Kirwan and Ness

1. Construct L-dimensional hypercube whose vertices have coordinates $\pm\frac{1}{2}$.
2. Chose L out of 2^L vertices and consider the plane P containing the chosen points .
3. Find the closest point p to the origin $\overline{0}$ in P.
4. Point $p = \overline{\lambda}_{\mathcal{C}}$ for some $\Delta_{\mathcal{C}}$ iff p does not lie on an edge of the hypercube.

3 qubits



$$|\phi_W\rangle = \frac{1}{\sqrt{3}}\left(|011\rangle + |101\rangle + |110\rangle\right), \; |\phi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|\phi_{B1}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle \otimes (|11\rangle + |00\rangle)\right)$$

Finding $\|\overline{\lambda}_{\mathcal{C}}\|^2$

- Linear Entropy (mean purity)

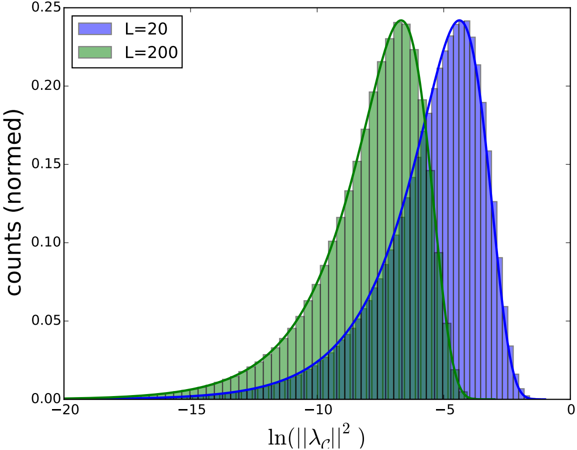
$$E([\phi]) = 1 - \frac{1}{L}\sum_{i=1}^{L} \mathrm{tr}\rho_i^2([\phi])$$

- $[\phi]$ is critical iff

$$dE([\phi]) = 0$$

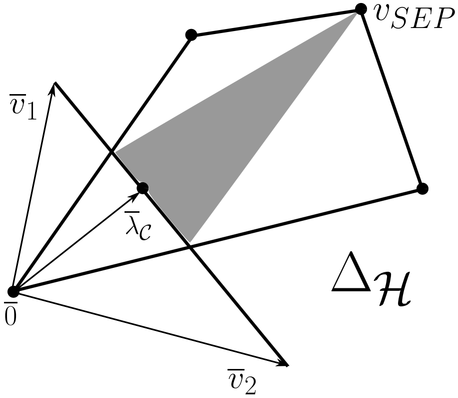
- **Fact** $\|\overline{\lambda}_{\mathcal{C}}\|^2 = -LE_c + \frac{L}{2}$, where E_c is a critical value of $E(\cdot)$

Histograms for 20 and 200 qubits, sample of 10^6 points



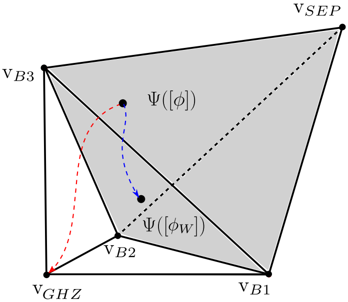
$$\|\overline{\lambda}_C\|^2 \sim Gamma\left(\frac{1}{2}, 2L\right),\ \mathbb{E}\left[\|\overline{\lambda}_C\|^2\right] = \frac{1}{4L}.$$

Implications - region with a weak entanglement witnessing power



Implications - the feasibility of entanglement distillation protocols

Such a protocol (gradient flow of E) transforms a given state $\phi \in \mathcal{C}$ using SLOCC operations to a state with critical local spectra $\overline{\lambda}_{\mathcal{C}}$.



Implications - required purity of states

For a mixed state ρ with $\mathrm{Tr}\rho^2 = p$ there exists a pure state ψ such that $\langle\psi|\rho|\psi\rangle \geq p$ and $\|\Psi(\rho) - \Psi(\phi)\| \leq \delta_L(p) = \frac{L}{2}(1 - \sqrt{2p-1})$.

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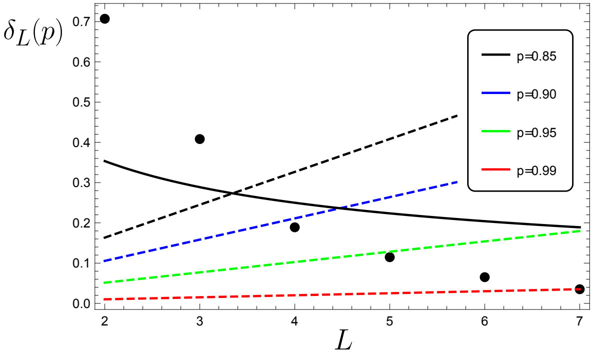


Dashed lines – $\delta_L(p)$, black line – $\mathbb{E}\left[\|\overline{\lambda}_C\|^2\right] = \frac{1}{4L}$, black dots – polytopes closest to zero.

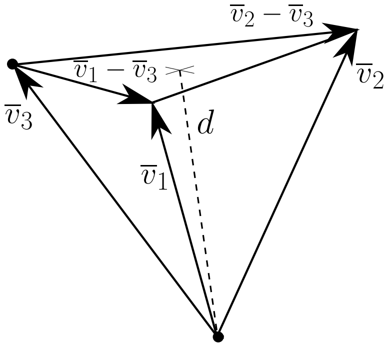
Procedure for finding $\|\overline{\lambda}_{\mathcal{C}}\|^2$ for L qubits

- For vectors $\overline{v}_1, \dots, \overline{v}_k \in \mathbb{R}^n$ let

$$G(\overline{v}_1, \dots, \overline{v}_k) = \begin{pmatrix} (\overline{v}_1|\overline{v}_1) & \dots & (\overline{v}_1|\overline{v}_k) \\ \vdots & \ddots & \dots \\ (\overline{v}_k|\overline{v}_1) & \dots & (\overline{v}_k|\overline{v}_k) \end{pmatrix}$$

- $|G(\overline{v}_1, \dots, \overline{v}_k)| := \det G(\overline{v}_1, \dots, \overline{v}_k)$

Example



$$V = \frac{1}{3} S d, \quad d^2 = \frac{3^2 V^2}{S^2}$$

$$V^2 = \frac{1}{3!} |G(\overline{v}_1, \overline{v}_2, \overline{v}_3)| \qquad S^2 = \frac{1}{2!} |G(\overline{v}_1 - \overline{v}_3, \overline{v}_2 - \overline{v}_3)|$$

$$d^2 = \frac{|G(\overline{v}_1, \overline{v}_2, \overline{v}_3)|}{|G(\overline{v}_1 - \overline{v}_3, \overline{v}_2 - \overline{v}_3)|},$$

Formula for $\|\overline{\lambda}_{\mathcal{C}}\|^2$

$$\|\overline{\lambda}_{\mathcal{C}}\|^2 = \frac{1}{4}\frac{|G(\overline{v}_1, \ldots, \overline{v}_L)|}{|G(\overline{v}_1 - \overline{v}_L, \ldots, \overline{v}_{L-1} - \overline{v}_L)|},$$

where $\overline{v}_i \in \mathbb{R}^L$ are vectors with ± 1 entries – Bernoulli vectors

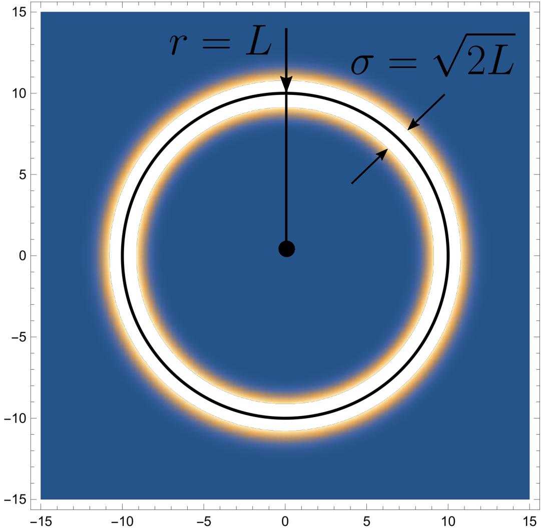
The model

- Vertices of the L-dimensional cube with Bernoulli vertices are uniformly distributed on S^{L-1} with $r^2 = L$.
- Let $\overline{v} = (v_1, \ldots, v_L)^t \in \mathbb{R}^L$ be a Gaussian vector, i.e. $v_i \sim N(0,1)$.

$$\overline{v} \sim \frac{\exp\left(-\frac{1}{2}\|v\|^2\right)}{\sqrt{(2\pi)^L}}$$

- The distribution of $\overline{v}$ is isotropic. $\|\overline{v}\|^2$ is χ^2_L with the mean L and $\sigma = \sqrt{2L}$
- When $L \to \infty$ the ratio $\frac{\sqrt{2L}}{L} \to 0$
- Problem: Calculate distribution of $\frac{|G(\overline{v}_1,\ldots,\overline{v}_L)|}{|G(\overline{v}_1-\overline{v}_L,\ldots,\overline{v}_{L-1}-\overline{v}_L)|}$ for $\overline{v}_i \sim N(\overline{0}, I)$

The model



Conclucisons

- The closet points to the origin of the EPs accumulate close to the origin
- The mean of $|\overline{\lambda}_{\mathcal{C}}|^2$ is $\frac{1}{4L}$
- The usefulness of EPs depends on purity and experimental precision for large L

References

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