Seminar in Wigner Research Centre for Physics

ALGEBRAIC CURVES AND INTEGRABILITY IN STRING THEORY

Minkyoo Kim
(Sogang & Ewha University)
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Contents

Old aspects of string theory (1)

- Quantum & Relativistic theory for 1-dim. Object
- Worldsheet conformal invariance
- Spacetime supersymmetry

  - lead to 5 consistent superstring theories in 10-dim. target spacetime with YM gauge symmetry based on two kinds of Lie algebras $SO(32)$ & $E_8 \times E_8$

- Compactification of extra dimension

  - CY_3 for low energy N=1 SUSY

Old aspects of string theory (2)

- Finding a specific & unique compactification which allows already known particle physics results and predicts new phenomena like supersymmetry etc.

  - till now, unsuccessful
  - huge numbers of false vacua $\sim 10^{500}$
  - Landscape, anthropic principle?

- In old approach, quantum gauge field theories for nature are included in String theory.

AdS/CFT (1)

- Surprising duality between string and gauge theory
  - String theory on some supergravity background which obtained from near horizon limit of multiple D-branes can be mapped to the conformal gauge field theory defined in lower dimensional spacetime.
  - Holographic principle

- In AdS/CFT, specific string configurations on AdS backgrounds have corresponding dual descriptions as composite operators in CFT.

AdS/CFT (2)

- Maldacena's conjecture

- IIB string on $AdS_5 \times S^5$ — N=4 SYM in 4D
- effective tension $\frac{R^2}{\alpha'}$ — coupling constant $\lambda = {g_{YM}}^2 N$
- string coupling g_s — number of colors N

$$\{E_n(\lambda, N)\} = \{D_n(\lambda, N)\}$$

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}, \quad \frac{\lambda}{N} = g_s$$

- Planar limit

$$N^{\#\text{Faces}-\#\text{Edges}+\#\text{vertices}}(g^2N)^{\text{Power}} = N^{2-2h}(g^2N)^{\text{Power}}$$

- Large N with fixed λ → "$g_s \to 0$" "planar diagram"

AdS/CFT (3)

- In this duality, QFT is not included in but equivalent to String theory!!

- Consider string theory as a framework for nature
  - Use it to study non-perturbative aspects of QFT
  - Pursue to check more and prove
  - Non-perturbative regime of string (gauge) theory has dual, perturbative regime of gauge (string) theory.

- What idea was helpful for us?

AdS/CFT and Integrability

- In both sides of AdS/CFT, integrable structures appeared.
- Drastic interpolation method for integrability ->

  - all-loop Bethe ansatz equations

  - exact S-matrices

  - TBA and Y-system
- Very successful for spectral problem

- "Integrable aspects of String theory"

String non-linear sigma model

- Type IIB superstring on $AdS_5 \times S^5$

$$S = \frac{1}{\alpha'} \int d^2\xi \, [\sqrt{-g} g^{ab} G_{\mu\nu}(x) \partial_a x^\mu \partial_b x^\nu + (\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \rho_a D_b \theta^J + O(\theta^4)]$$

- Super-isometry group : PSU(2,2|4) ⊃ SO(4,2) x SO(6)
- Difficult to quantize strings in curved background
- One way to circumvent

 - Penrose limit and its curvature correction

$$ds^2 = R^2 \left[- \left(\frac{1 + \frac{1}{4} z^2}{1 - \frac{1}{4} z^2} \right)^2 dt^2 + \left(\frac{1 - \frac{1}{4} y^2}{1 + \frac{1}{4} y^2} \right)^2 d\phi^2 + \frac{dz_k dz_k}{(1 - \frac{1}{4} z^2)^2} + \frac{dy_{k'} dy_{k'}}{(1 + \frac{1}{4} y^2)^2} \right]$$

$$\begin{aligned} ds^2 \approx \; & 2\, dx^+ dx^- + dz^2 + dy^2 - (z^2 + y^2)\,(dx^+)^2 + \\ & [2\,(z^2 - y^2)\, dx^- dx^+ + z^2 dz^2 - y^2 dy^2 - (z^4 - y^4)\,(dx^+)^2]\, \frac{1}{2R^2} \\ & + \mathcal{O}(1/R^4)\,. \end{aligned}$$

$$\frac{H_{\rm LC}}{\mu} = 2\sqrt{1 + \frac{n^2 \lambda}{J^2}}$$

Reduced model (1)

- 3+3 Cartan generators of $SO(4,2) \times SO(6)$

$$ds^2_{(AdS_5)} = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\left(d\gamma^2 + \cos^2\gamma\, d\phi_1^2 + \sin^2\gamma\, d\phi_2^2\right),$$
$$ds^2_{(S^5)} = d\psi^2 + \cos^2\psi\, d\varphi_3^2 + \sin^2\psi\left(d\theta^2 + \cos^2\theta\; d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2\right).$$

- Rotating string ansatz

$$\xi_i(\tau,\sigma) = r_i(\sigma)\exp\{i\varphi_i(\tau,\sigma)\} = r_i(\sigma)\exp\{i\left[w_i\tau + \alpha_i(\sigma)\right]\}, \quad (i = 1,\, 2,\, 3),$$
$$\eta_r(\tau,\sigma) = s_r(\sigma)\exp\{i\phi_r(\tau,\sigma)\} = s_r(\sigma)\exp\{i\left[\omega_r\tau + \beta_r(\sigma)\right]\}, \quad (r = 0,\, 1,\, 2)$$
$$r_i(\sigma + 2\pi) = r_i(\sigma), \qquad \alpha_i(\sigma + 2\pi) = \alpha_i(\sigma) + 2\pi m_i \quad (m_i \in \mathbb{Z}),$$
$$s_r(\sigma + 2\pi) = s_r(\sigma), \qquad \beta_r(\sigma + 2\pi) = \beta_r(\sigma) + 2\pi k_r \quad (k_0 = 0;\ k_{1,2} \in \mathbb{Z}).$$

- Neumann-Rosochatius model

$$\mathcal{L}_{(AdS_5\times S^5)} = \frac{1}{2}\left[\delta^{ij}\left(r_i' r_j' - w_i^2 r_i r_j - \frac{v_i v_j}{r_i r_j}\right) - \Lambda(\delta^{ij} r_i r_j - 1)\right] + $$
$$+\frac{1}{2}\left[\eta^{rs}\left(s_r' s_s' - \omega_r^2 s_r s_s - \frac{u_r u_s}{s_r s_s}\right) - \widetilde{\Lambda}(\eta^{rs} s_r s_s + 1)\right]$$

$$v_i \equiv r_i^2 \alpha_i'$$
$$u_r \equiv s_r^2 \beta_r'$$

Reduced model (2)

- NR integrable system – particle on sphere with the potential $\sum_i (w_i^2 r_i^2 + v_i^2 r_i^{-2})$
- Infinite conserved charges construced from integrals of motion of NR integrable system

- Coupled NR systems with Virasoro constraints

$$0 = \eta^{rs}\left(s'_r s'_s + \omega_r^2 s_r s_s + \frac{u_r u_s}{s_r s_s}\right) + \delta^{ij}\left(r'_i r'_j + w_i^2 r_i r_j + \frac{v_i v_j}{r_i r_j}\right) ,$$

$$0 = \eta^{rs}\omega_r u_r + \delta^{ij} w_i v_i .$$

- if we confine to sphere part, then the 1st equation becomes a SG equation.

Reduced model (3)

- String theory on $R \times S^2$ in static gauge could be reduced to SG model from so called Polmeyer reduction.

$$\partial_a \partial^a \phi - \sin\phi = 0 \qquad \partial_a \vec{\xi} \cdot \partial^a \vec{\xi}^* \equiv \cos\phi$$

$$\partial_a \partial^a \vec{\xi} + (\partial_a \vec{\xi} \cdot \partial^a \vec{\xi}^*)\vec{\xi} = \vec{0} \qquad \partial_a \partial^a \vec{\xi} + (\cos\phi)\vec{\xi} = \vec{0}$$

- For example, SG kink → Giant magnon
- O(4) model → complex SG model

$$\partial_+ \partial_- \psi + \psi^* \frac{\partial_+ \psi \, \partial_- \psi}{1 - |\psi|^2} + \psi(1 - |\psi|^2) = 0$$

Classical Integrability (1)

- Coset construction of string action : SU(2) sector

$$S_{\sigma m} = -\frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \left[\frac{1}{2}\mathrm{Tr} j_a^2 + (\partial_a X_0)^2\right]$$

$$j_a = g^{-1}\partial_a g \qquad \partial_+\partial_- X_0 = 0,$$

$$\partial_+ j_- + \partial_- j_+ = 0,$$

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0,$$

- Rescaled currents are flat too.

$$J_\pm(x) = \frac{j_\pm}{1 \mp x}$$

- Monodromy & Transfer matrix

$$\Omega(x) = P\exp\left(-\int_0^{2\pi} d\sigma\, J_\sigma\right) = P\exp\int_0^{2\pi} d\sigma\, \frac{1}{2}\left(\frac{j_+}{x-1} + \frac{j_-}{x+1}\right),$$

$$\mathrm{tr}\,\Omega(x) = 2\cos p(x),$$

- Local charges can be obtained from quasi-momentum.
- Resolvent and integral equation

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1},$$

$$G(x+\mathrm{i}0) + G(x-\mathrm{i}0) = 2 \fint \mathrm{d}y \frac{\rho(y)}{x-y} = \frac{4\pi\kappa x}{x^2-1} + 2\pi n_C.$$

Classical Integrability (2)

- Lax representation of full coset construction
- Generally, strings on semisymmetric superspace are integrable from Z4 symmetry.

  - All known examples
- Classical algebraic curve

  - Fully use the integrability
  - Simple poles in Monodromy
  - 8 Riemann surfaces with poles and cuts
  - Can read analytic properties
  - Efficient way to obtain charges and leading quantum effects

Classical Integrability (3)

- MT superstring action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}\, \left(J^{(2)} \wedge * J^{(2)} - J^{(1)} \wedge J^{(3)} \right) + \Lambda \wedge \mathrm{str}\, J^{(2)}$$

$$J = -g^{-1} dg$$

- Bosonic case

$$g = \left(\begin{array}{c|c} \mathcal{Q} & 0 \\ \hline 0 & \mathcal{R} \end{array} \right)$$

$$u^j \Sigma_j^S = \mathcal{R} E \mathcal{R}^T \ , \ v^j \Sigma_j^A = \mathcal{Q} E \mathcal{Q}^T$$

$$\begin{aligned} 1 &= u_6^2 + u_5^2 + u_4^2 + u_3^3 + u_2^2 + u_1^2 \, , \\ 1 &= v_6^2 + v_5^2 - v_4^2 - v_3^2 - v_2^2 - v_1^2 \, . \end{aligned}$$

$$S_b = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \sqrt{h} \left(h^{\mu\nu}\, \partial_\mu u \cdot \partial_\nu u + \lambda_u \left(u \cdot u - 1 \right) - \left(u \to v \right) \right)$$

- Lax connection

$$A(x) = J^{(0)} + \frac{x^2+1}{x^2-1} J^{(2)} - \frac{2x}{x^2-1} \left(* J^{(2)} - \Lambda \right) + \sqrt{\frac{x+1}{x-1}}\, J^{(1)} + \sqrt{\frac{x-1}{x+1}}\, J^{(3)}$$

Classical Integrability (4)

- Flat connection -> path independ. -> Monodromy

$$\Omega(x) = \mathrm{Pexp} \oint_{\gamma} A(x)$$

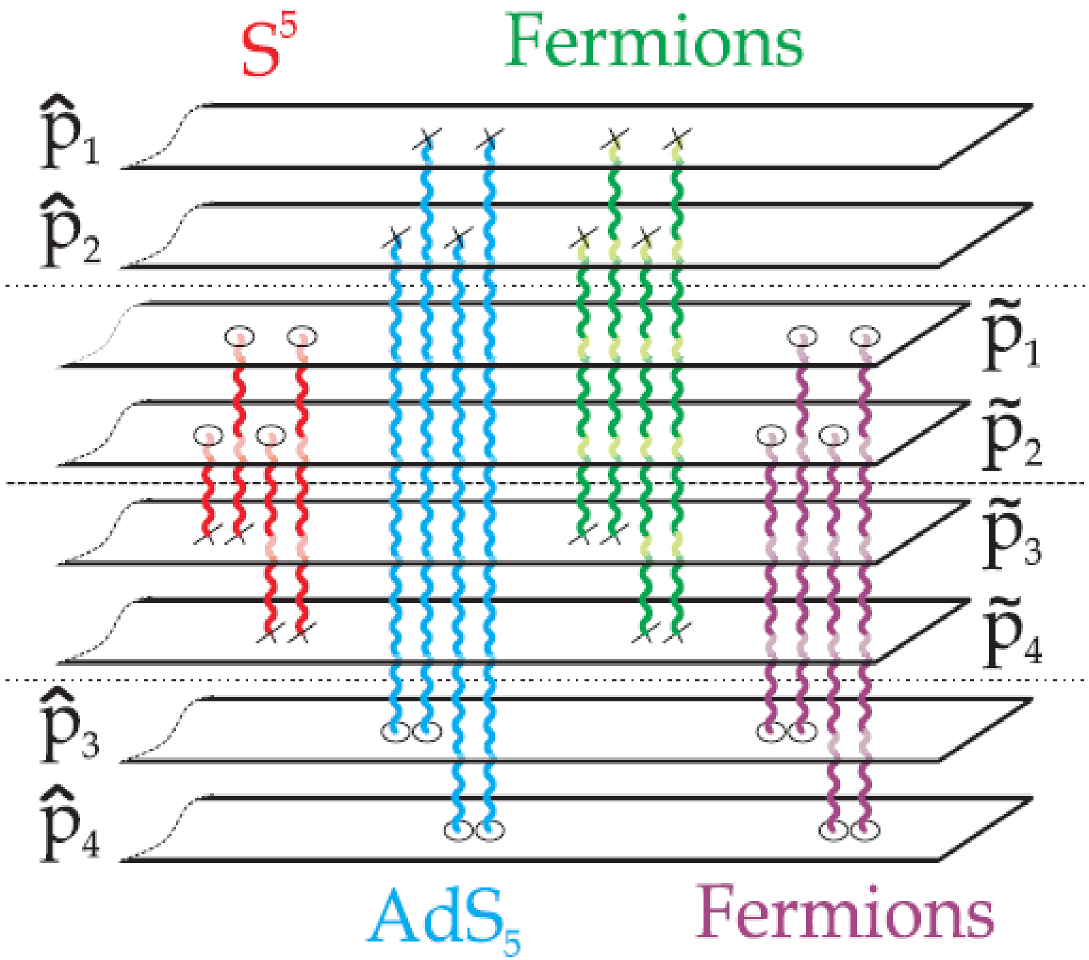
- Collection of quasi-momenta as eigenvalues of Monodromy

$$p_i^+ - p_j^- = 2\pi n_{ij} \ , \ x \in \mathcal{C}_n^{ij}$$

$$S_{ij} = \pm \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{C}_{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x) dx.$$

- Analogy of Mode numbers and Fourier mode amplitudes in flat space

Classical Integrability (5)



$$S^5 \;:\; (\tilde{1},\tilde{3}),(\tilde{1},\tilde{4}),(\tilde{2},\tilde{3}),(\tilde{2},\tilde{4})$$

$$AdS_5 \;:\; (\hat{1},\hat{3}),(\hat{1},\hat{4}),(\hat{2},\hat{3}),(\hat{2},\hat{3})$$

$$\text{Fermions} \;:\; (\tilde{1},\hat{3}),(\tilde{1},\hat{4}),(\tilde{2},\hat{3}),(\tilde{2},\hat{4})$$

$$(\hat{1},\tilde{3}),(\hat{1},\tilde{4}),(\hat{2},\tilde{3}),(\hat{2},\tilde{3})$$

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} \simeq \frac{2\pi}{x\sqrt{\lambda}} \begin{pmatrix} +E - S_1 + S_2 \\ +E + S_1 - S_2 \\ -E - S_1 - S_2 \\ -E + S_1 + S_2 \\ +J_1 + J_2 - J_3 \\ +J_1 - J_2 + J_3 \\ -J_1 + J_2 + J_3 \\ -J_1 - J_2 - J_3 \end{pmatrix}$$

$$\tilde{p}_{1,2}(x) = -\tilde{p}_{2,1}(1/x) - 2\pi m$$

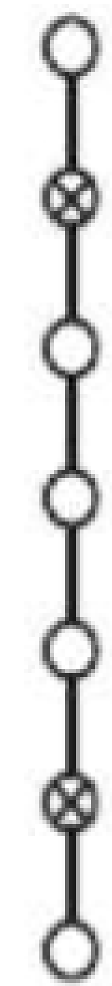
$$\tilde{p}_{3,4}(x) = -\tilde{p}_{4,3}(1/x) + 2\pi m$$

$$\hat{p}_{1,2,3,4}(x) = -\hat{p}_{2,1,4,3}(1/x)$$

Classical Integrability (6)

- Scaling limit of all-loop BAES

$$
\begin{aligned}
1 &= \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{1,k}-u_{2,j}-\frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{1-g^2/2x_{1,k}x^{+\eta_1}_{4,j}}{1-g^2/2x_{1,k}x^{-\eta_1}_{4,j}} \\
1 &= \prod_{j=1,j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i\eta_1}{u_{2,k}-u_{2,j}+i\eta_1} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{3,j}-\frac{i}{2}\eta_1} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{1,j}-\frac{i}{2}\eta_1} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{3,k}-u_{2,j}-\frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^{+\eta_1}_{4,j}}{x_{3,k}-x^{-\eta_1}_{4,j}} \\
1 &= \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{j=1}^{K_4} \left(\frac{x^{+\eta_1}_{4,k}-x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k}-x^{+\eta_2}_{4,j}} \frac{1-g^2/2x^+_{4,k}x^-_{4,j}}{1-g^2/2x^-_{4,k}x^+_{4,j}} \sigma^2_{BES}(x_{4,k},x_{4,j}) \right) \\
&\quad \times \prod_{j=1}^{K_1} \frac{1-g^2/2x^{-\eta_1}_{4,k}x_{1,j}}{1-g^2/2x^{+\eta_1}_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{3,j}-x^{-\eta_1}_{4,k}}{x_{3,j}-x^{+\eta_1}_{4,k}} \prod_{j=1}^{K_5} \frac{x_{5,j}-x^{-\eta_2}_{4,k}}{x_{5,j}-x^{+\eta_2}_{4,k}} \prod_{j=1}^{K_7} \frac{1-g^2/2x^{-\eta_2}_{4,k}x_{7,j}}{1-g^2/2x^{+\eta_2}_{4,k}x_{7,j}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{5,k}-u_{6,j}-\frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^{+\eta_2}_{4,j}}{x_{5,k}-x^{-\eta_2}_{4,j}} \\
1 &= \prod_{j=1,j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i\eta_2}{u_{6,k}-u_{6,j}+i\eta_2} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{5,j}-\frac{i}{2}\eta_2} \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{7,j}-\frac{i}{2}\eta_2} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{7,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{7,k}-u_{6,j}-\frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{1-g^2/2x_{7,k}x^{+\eta_2}_{4,j}}{1-g^2/2x_{7,k}x^{-\eta_2}_{4,j}},
\end{aligned}
$$



Examples : Circular strings (1)

- Circular string solution

$$u_2 + i u_1 = \sqrt{\frac{\mathcal{J}_3}{w_3}}\, e^{i(w_3\tau + m_3\sigma)} \quad , \quad v_2 + i v_1 = \sqrt{\frac{\mathcal{S}_2}{\mathrm{w}_2}}\, e^{i(\mathrm{w}_2\tau + k_2\sigma)} ,$$

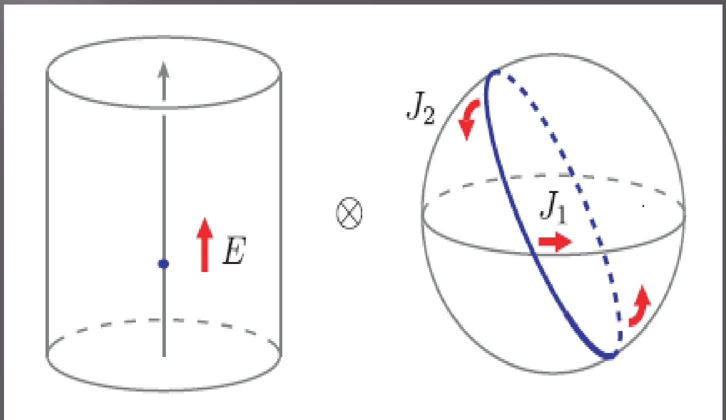
$$u_4 + i u_3 = \sqrt{\frac{\mathcal{J}_2}{w_2}}\, e^{i(w_2\tau + m_2\sigma)} \quad , \quad v_4 + i v_3 = \sqrt{\frac{\mathcal{S}_1}{\mathrm{w}_1}}\, e^{i(\mathrm{w}_1\tau + k_1\sigma)} ,$$

$$u_6 + i u_5 = \sqrt{\frac{\mathcal{J}_1}{w_1}}\, e^{i(w_1\tau + m_1\sigma)} \quad , \quad v_6 + i v_5 = \sqrt{\frac{\mathcal{E}}{\kappa}}\, e^{i\kappa\tau} ,$$

$$1 = \sum_{i=1}^{3} \frac{\mathcal{J}_i}{w_i} \ , \ 1 = \frac{\mathcal{E}}{\kappa} - \sum_{j=1}^{2} \frac{\mathcal{S}_j}{\mathrm{w}_j} \ , \ 0 = \sum_{j=1}^{2} k_j \mathcal{S}_j + \sum_{i=1}^{3} m_i \mathcal{J}_i ,$$

$$\mathrm{w}_j^2 = \kappa^2 + k_j^2 \ , \ \kappa^2 = \sum_{j=1}^{2} \mathcal{S}_j \frac{2k_j^2}{\mathrm{w}_j} + \sum_{i=1}^{3} \mathcal{J}_i \frac{w_i^2 + m_i^2}{w_i} ,$$

$$w_i^2 = \nu^2 + m_i^2 \ , \ \nu^2 \equiv \sum_{i=1}^{3} \mathcal{J}_i \frac{w_i^2 - m_i^2}{w_i} .$$



Examples : Circular strings (1)

- Quasi-momenta

$$\tilde{A}(x) = \pi \begin{pmatrix} -\tilde{a}_+(1/x) & \tilde{b}_+ & -\tilde{c}(1/x) & \tilde{d}(x) \\ \tilde{b}_+ & \tilde{a}_+(x) & \tilde{d}(1/x) & \tilde{c}(x) \\ -\tilde{c}(1/x) & \tilde{d}(1/x) & \tilde{a}_-(x) & \tilde{b}_- \\ \tilde{d}(x) & \tilde{c}(x) & \tilde{b}_- & -\tilde{a}_-(1/x) \end{pmatrix}$$

$$\begin{aligned} \tilde{a}_\pm(x) &= \pm\tilde{a}(x) - m_3\cos\theta \\ \tilde{a}(x) &= -\frac{m_1 - w_1x + (m_2 - w_2x)\cos\theta + x\cos 2\gamma(-w_1 + m_1x + (w_2 - m_2x)\cos\theta)}{x^2-1} \\ \tilde{b}_\pm &= (m_2 \mp m_3)\cos\gamma\sin\theta \\ \tilde{c}(x) &= \frac{(m_2+m_3)x^2 - (m_2 - m_3) - 2w_3x}{x^2-1}\sin\gamma\sin\theta \\ \tilde{d}(x) &= \frac{-m_1 + w_1x + (m_2 - w_2\,x)\cos\theta}{x^2-1}\sin 2\gamma \end{aligned}$$

$$\hat{A}(x) = \pi \begin{pmatrix} -\hat{a}_+(1/x) & \hat{b}_+ & -\hat{c}(x) & \hat{d}(x) \\ \hat{b}_+ & \hat{a}_+(x) & \hat{d}(x) & \hat{c}(x) \\ \hat{c}(x) & -\hat{d}(x) & \hat{a}_-(x) & \hat{b}_- \\ -\hat{d}(x) & -\hat{c}(x) & \hat{b}_- & -\hat{a}_-(1/x) \end{pmatrix}$$

$$\begin{aligned} \hat{a}_\pm(x) &= \pm\frac{2\pi\kappa - k_1\,(x^2-1)\cos\theta}{x^2-1}\cosh\rho + k_2\cos\psi \\ \hat{b}_\pm &= (k_2\cosh\rho \mp k_1)\sin\psi \\ \hat{c}(x) &= \frac{k_2\,(x^2+1) - 2\mathrm{w}_2x}{x^2-1}\sin\psi\sinh\rho \\ \hat{d}(x) &= \frac{k_1\,(x^2+1) - 2\mathrm{w}_1x}{x^2-1}\cos\psi\sinh\rho \end{aligned}$$

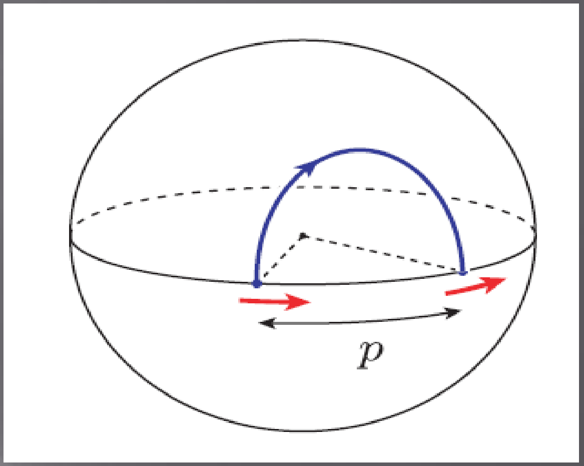
$$\hat{p}_{1,2} = -\hat{p}_{3,4} = \frac{2\pi\kappa\,x}{x^2-1}$$

$$\begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} = 2\pi \begin{pmatrix} +\frac{x}{x^2-1}K(1/x) \\ +\frac{x}{x^2-1}K(x) - m \\ -\frac{x}{x^2-1}K(x) + m \\ -\frac{x}{x^2-1}K(1/x) \end{pmatrix}, \quad K(x) \equiv \sqrt{m^2x^2 + \mathcal{J}^2}$$

Giant magnon solutions

- Giant magnon solutions
  - dual to fundamental excitation of spin-chain
  - Dispersion relation

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\left|\sin\left(\frac{p}{2}\right)\right|$$



$$G_{\mathrm{magnon}} = -i\log\left(\frac{x-X^+}{x-X^-}\right)$$

  - Log cut solution

$$\begin{aligned} p_{\hat{1},\hat{2}}(x) = -p_{\hat{3},\hat{4}}(x) &= \frac{2\pi\Delta}{\sqrt{\lambda}}\frac{x}{x^2-1} \\ p_{\tilde{2}}(x) \;=\; -p_{\tilde{3}}(x) &= \frac{2\pi\Delta}{\sqrt{\lambda}}\frac{x}{x^2-1} + \frac{1}{i}\log\frac{x-X^+}{x-X^-} + \tilde{\phi}_2 \\ p_{\tilde{1}}(x) \;=\; -p_{\tilde{4}}(x) &= \frac{2\pi\Delta}{\sqrt{\lambda}}\frac{x}{x^2-1} + \frac{1}{i}\log\frac{x-1/X^-}{x-1/X^+} + \tilde{\phi}_1\,. \end{aligned}$$

Quantum effects from Al. curve (1)

- Quasi-momenta make multi-sheet algebraic curve.
- Giant-magnon -> logarithmic cut solution in complex plane

$$G_{\text{magnon}} = -i \log \left(\frac{x - X^+}{x - X^-} \right)$$

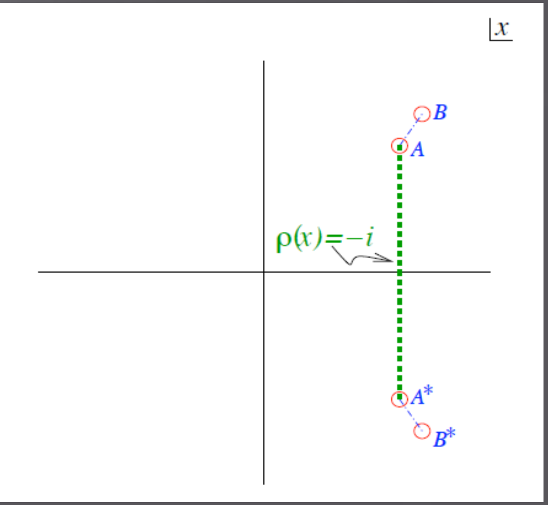
- Deforming quasi-momenta - Finite-size effects

  - Classical effects : Resolvent deformation

$$G_{\text{finite}} = -2i \log \left(\frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}} \right)$$

  - Quantum effects :

  Adding poles to original curve



Quantum effects from Al. curve (2)

- Fluctuations of quasi-momenta

$$\delta p_i(x) \sim \eta_i \, N_n^{ij} \frac{\alpha(x_n^{ij})}{x - x_n^{ij}}$$

$$\eta_{\hat{1}} = \eta_{\hat{2}} = \eta_{\tilde{3}} = \eta_{\tilde{4}} = -\eta_{\hat{3}} = -\eta_{\hat{4}} = -\eta_{\tilde{1}} = -\eta_{\tilde{1}} = 1$$

$$\alpha(x) \equiv \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}$$

$$\delta\epsilon_{1-loop} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{(ij)} (-1)^{F_{ij}} \Omega_n^{ij}$$

- GM

$$\Omega(x) = \frac{2}{x^2 - 1} \left(1 - \frac{X^+ + X^-}{X^+ X^- + 1} \, x \right)$$

- Circular string

	eigenmodes	notation
$\mathbf{S^5}$	$\sqrt{2\mathcal{J}^2 + n^2 \pm 2\sqrt{\mathcal{J}^4 + n^2\mathcal{J}^2 + m^2 n^2}}$ $\sqrt{\mathcal{J}^2 + n^2 - m^2}$	$\omega_n^{S\pm}$ ω_n^{S}
Fermions	$\sqrt{\mathcal{J}^2 + n^2}$	ω_n^{F}
$\mathbf{AdS_5}$	$\sqrt{\mathcal{J}^2 + n^2 + m^2}$	ω_n^{A}

Quantum effects from Al. curve (3)

- One-loop energy shifts

$$\delta\epsilon_{1-loop} = \oint_{\mathrm{U}+} \frac{dx}{2\pi i}\partial_x\Omega(x)\sum_{(ij)}(-1)^{F_{ij}}e^{-i(p_i-p_j)}$$

$$\delta\epsilon_{1-loop} = \frac{8\sin^2\frac{p}{4}\,e^{-\frac{2\pi\Delta}{\sqrt{\lambda}}}}{\pi\left(\sin\frac{p}{2}-1\right)\left(\frac{\Delta}{\sqrt{\lambda}}\right)^{1/2}}\left[1-\frac{7+4\sin p-4\cos p+\sin\frac{p}{2}}{16\pi\left(\sin\frac{p}{2}-1\right)\frac{\Delta}{\sqrt{\lambda}}}+\mathcal{O}\left(\frac{1}{\left(\frac{\Delta}{\sqrt{\lambda}}\right)^2}\right)\right]+\dots$$

- From exact dispersion, we know one-loop effects are finite-size piece.

$$\epsilon_\infty(p) = \sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}$$

$$\epsilon_\infty(p) = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{p}{2}\right)+0+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

TsT-transformed $AdS_5 \times S^5$

- Twisted Algebraic curve

$$
\begin{aligned}
p_{\hat{1}}(x) &= \frac{\alpha x}{x^2-1} + \phi_{\hat{1}};\ p_{\hat{2}}(x) = \frac{\alpha x}{x^2-1} + \phi_{\hat{2}};\ p_{\hat{3}}(x) = \frac{-\alpha x}{x^2-1} + \phi_{\hat{3}};\ p_{\hat{4}}(x) = \frac{-\alpha x}{x^2-1} + \phi_{\hat{4}};\\
p_{\tilde{1}}(x) &= \frac{\alpha x}{x^2-1} + i\log\left(\frac{1/x - X^+}{1/x - X^-}\right) + \phi_{\tilde{1}};\ p_{\tilde{2}}(x) = \frac{\alpha x}{x^2-1} - i\log\left(\frac{x - X^+}{x - X^-}\right) + \phi_{\tilde{2}}\\
p_{\tilde{3}}(x) &= \frac{-\alpha x}{x^2-1} + i\log\left(\frac{x - X^+}{x - X^-}\right) + \phi_{\tilde{3}};\ p_{\tilde{4}}(x) = \frac{-\alpha x}{x^2-1} - i\log\left(\frac{1/x - X^+}{1/x - X^-}\right) + \phi_{\tilde{4}}\,, \quad (2.17)
\end{aligned}
$$

$$\phi_{\tilde{1}} = p/2 + \pi\beta Q\,; \quad \phi_{\tilde{2}} = -p/2 - \pi\beta Q\,;$$

$$\phi_{\tilde{3}} = p/2 + \pi\beta(2L - 3Q)\,; \quad \phi_{\tilde{4}} = -p/2 - \pi\beta(2L - 3Q)$$

- Curves with extra phases
- Quantum finite size effects from al.curve matched well with those using Luscher’s method.

IIA superstring on $AdS_4 \times CP_3$

- Algebraic curve of AdS4 x CP3 Superstring

$$
\begin{aligned}
q_1 &= -q_{10} = \frac{\alpha x}{x^2-1} \\
q_2 &= -q_9 = \frac{\alpha x}{x^2-1} \\
q_3 &= -q_8 = \frac{\alpha x}{x^2-1} + G_u(0) - G_u\left(\frac{1}{x}\right) + G_v(0) - G_v\left(\frac{1}{x}\right) + G_r(x) - G_r(0) + G_r\left(\frac{1}{x}\right) \\
q_4 &= -q_7 = \frac{\alpha x}{x^2-1} + G_u(x) + G_v(x) - G_r(x) + G_r(0) - G_r\left(\frac{1}{x}\right) \\
q_5 &= -q_6 = G_u(x) - G_u(0) + G_u\left(\frac{1}{x}\right) - G_v(x) + G_v(0) - G_v\left(\frac{1}{x}\right).
\end{aligned}
$$

- 10 Riemann sheets
- CP3 structure -> multi-resolvents -> Various GMs

Open strings - Y=0 brane (1)

- Open string boundary conditions

$$ds^2_{S^5} = dXd\bar{X} + dYd\bar{Y} + dZd\bar{Z}, \quad \text{with } |X|^2 + |Y|^2 + |Z|^2 = 1$$

$$\begin{aligned} \text{Dirichlet} &: \quad Y|_{\sigma=0,\pi} = 0 \\ \text{Neumann} &: \quad \partial_\sigma X|_{\sigma=0,\pi} = 0, \quad \partial_\sigma Z|_{\sigma=0,\pi} = 0 \end{aligned}$$

- Difficult to directly construct Monodromy
- Scaling limit of all-loop Bethe equations

$$e^{-2ip_j(L+1)} \prod_{k=j-1}^{1} S_0(p_j,p_k) R_0^-(p_j) \prod_{k=1:k\neq j}^{N} S_0(p_k,-p_j) R_0^+(-p_j) \prod_{k=N}^{j+1} S_0(p_j,p_k) = 1$$

$$\sqrt{\lambda} \sim u_a \sim K_a \sim L \gg 1, \quad a = 1,2,...,7,$$

$$x^\pm = x \pm \frac{i}{2}\alpha(x) + O(\frac{1}{\lambda}), \quad \alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2-1}.$$

$$G_a(x) = \sum_{j=1}^{K_a} \frac{\alpha(y_{a,j})}{x - y_{a,j}}, \quad H_a(x) = \sum_{j=1}^{K_a} \frac{\alpha(x)}{x - y_{a,j}}$$

Open strings - Y=0 brane (2)

- SU(2) sector

$$\tilde{p}_2(x+i0) - \tilde{p}_3(x-i0) = 2\pi n_{\tilde{2}\tilde{3}}$$

- (a) Bulk and boundary S-matrices part
- (b), (c) Dressing factors

$$(a) \quad \to \quad \sum_{j=1}^{N} \frac{2xy_j}{g(x-y_j)(xy_j-1)}$$

$$(b) \quad \to \quad -N\frac{2x}{g(-1+x^2)^2(1+x^2)}$$

$$(c) \quad \to \quad -\sum_{j=1}^{N} \frac{2x}{g\left(-1+x^2\right)\left(-1+x^2y_j^2\right)}$$

$$(a) \quad \to \quad H(x) + \bar{H}(x) - H(-x) - \bar{H}(-x)$$

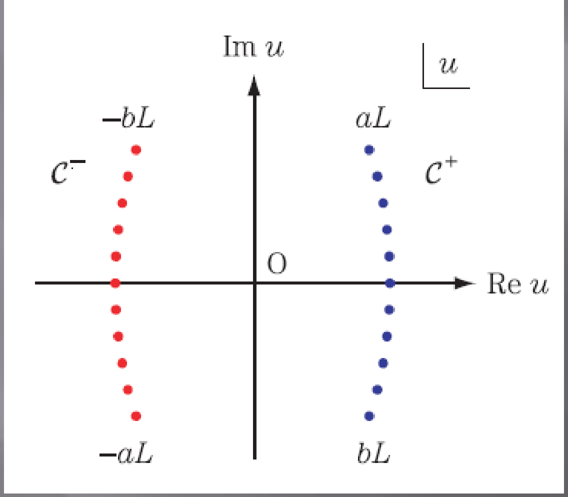
$$(c) \quad \to \quad -\bar{H}(x) + \bar{H}(-x)$$

- Quasi-momentum can be constructed and compared with Y-system result by L. Palla.

Open strings - Y=0 brane (3)

- Double contour technique

$$\pm\frac{2}{x_j} - 2\pi n = \frac{2}{L}\sum_{k=1,\,k\neq j}^{M}\left(\frac{1}{x_j - x_k} + \frac{1}{x_j + x_k}\right)$$



- Strong coupling effects

$$\frac{1}{2g}\frac{x}{x^2+1}$$

Discussions

- Integrable structures in string theory
- Beyond some reduced models, we can fully use integrability – Algebraic curve.
- Direct way and scaling limit of BAEs and Y-system
- Efficient computation of quantum effects
- Spectral curves in other models
  - $AdS_5, AdS_4, AdS_3, AdS_2$
  - Beta-deformed theory, Open string theory
- Final destination : Quantum algebraic curve (?)
  - Today, in 1305:1939, P-system (?)