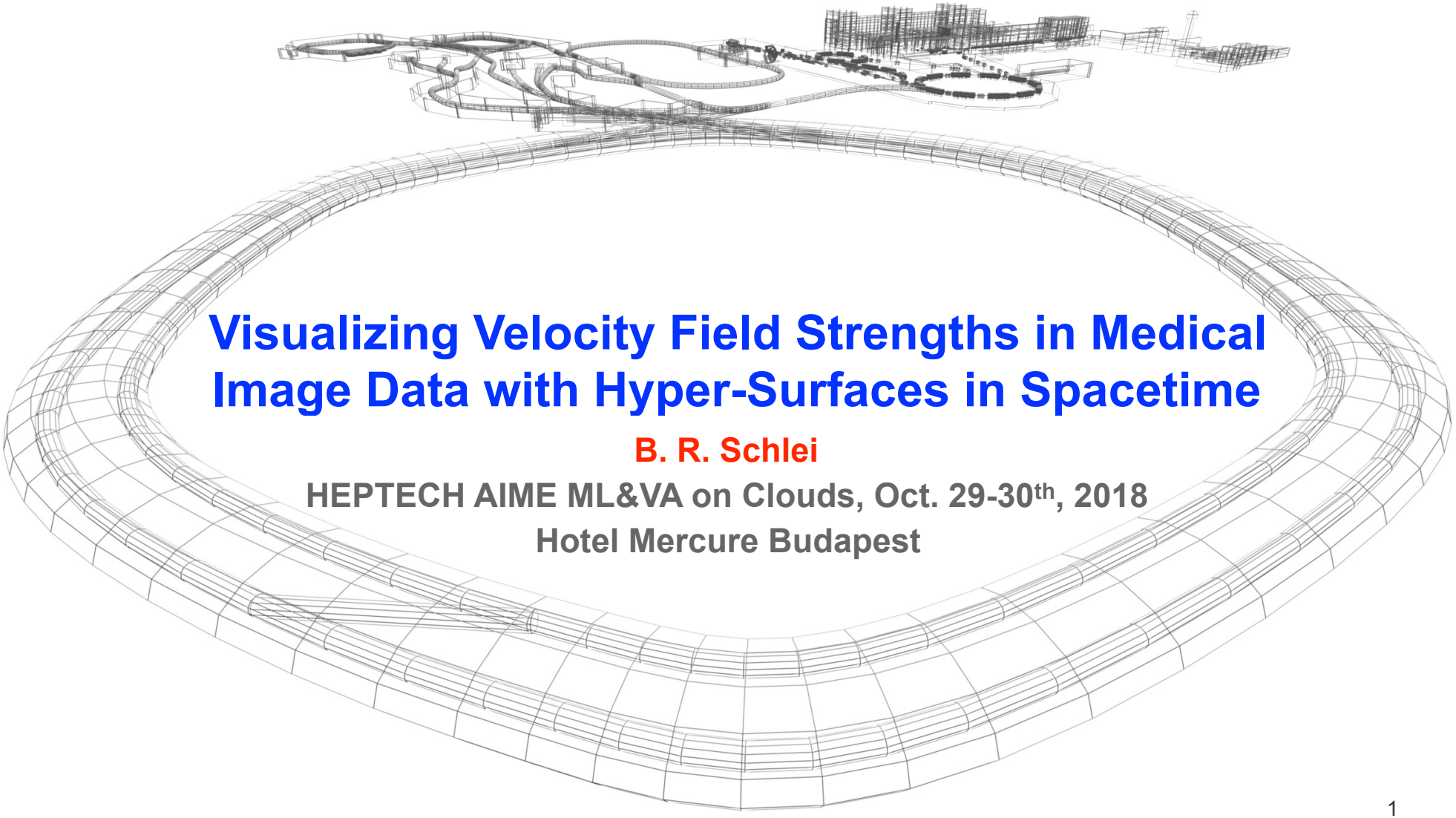


Visualizing Velocity Field Strengths in Medical Image Data with Hyper-Surfaces in Spacetime

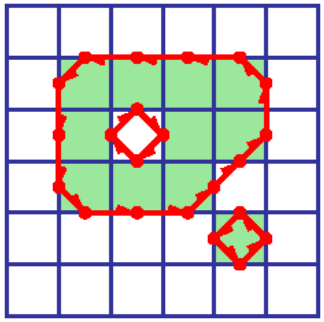
B. R. Schlei

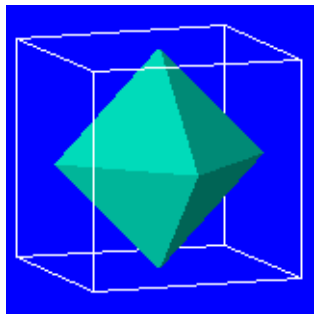
HEPTECH AIME ML&VA on Clouds, Oct. 29-30th, 2018

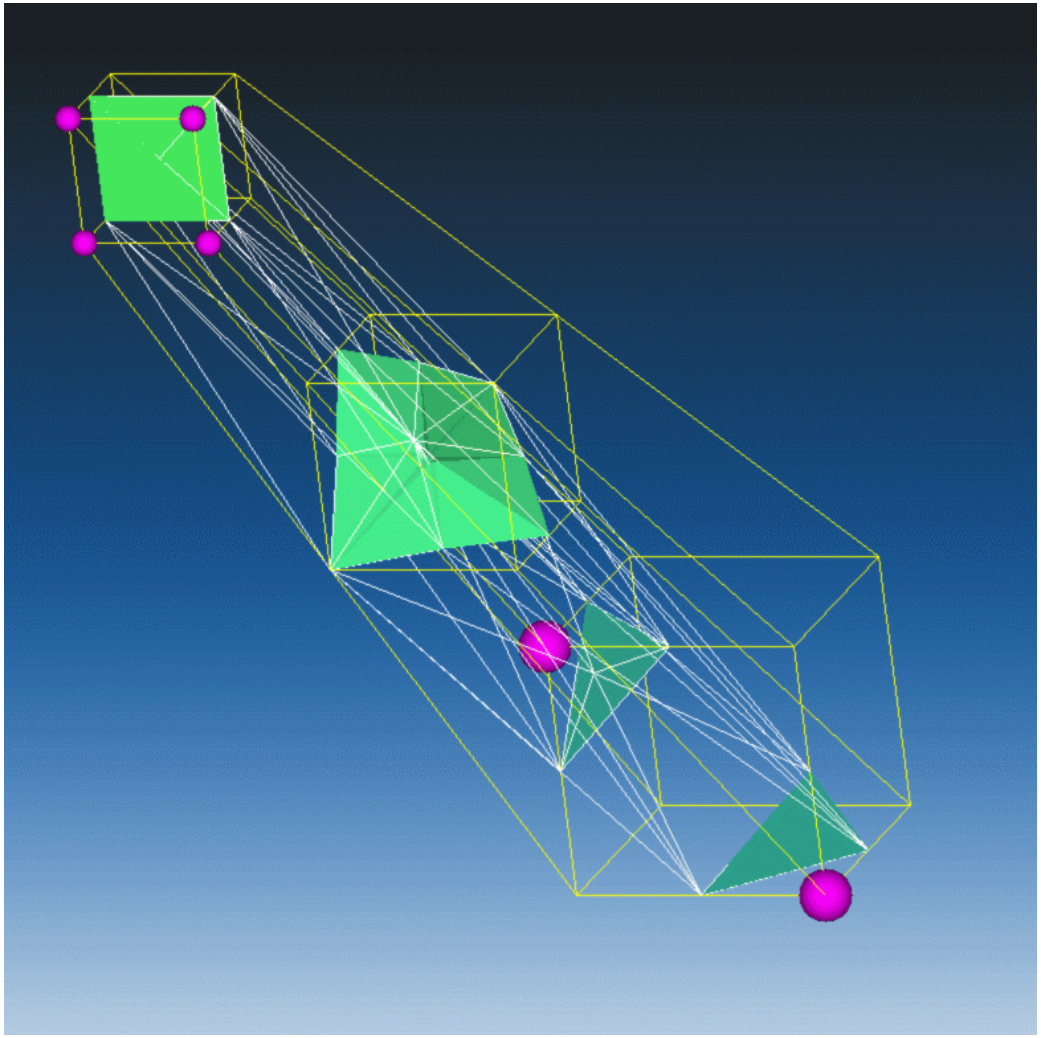
Hotel Mercure Budapest

Contents

1. Introduction
2. Grids
3. Contours
4. Surfaces
5. Hyper-Surfaces
6. Medical Image Data (4D-CT)
7. Summary and Outlook



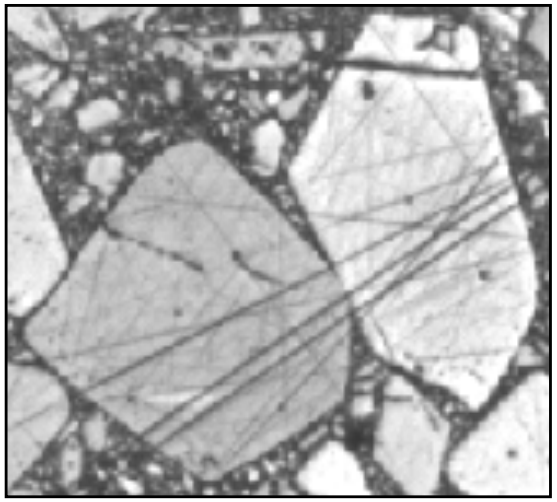


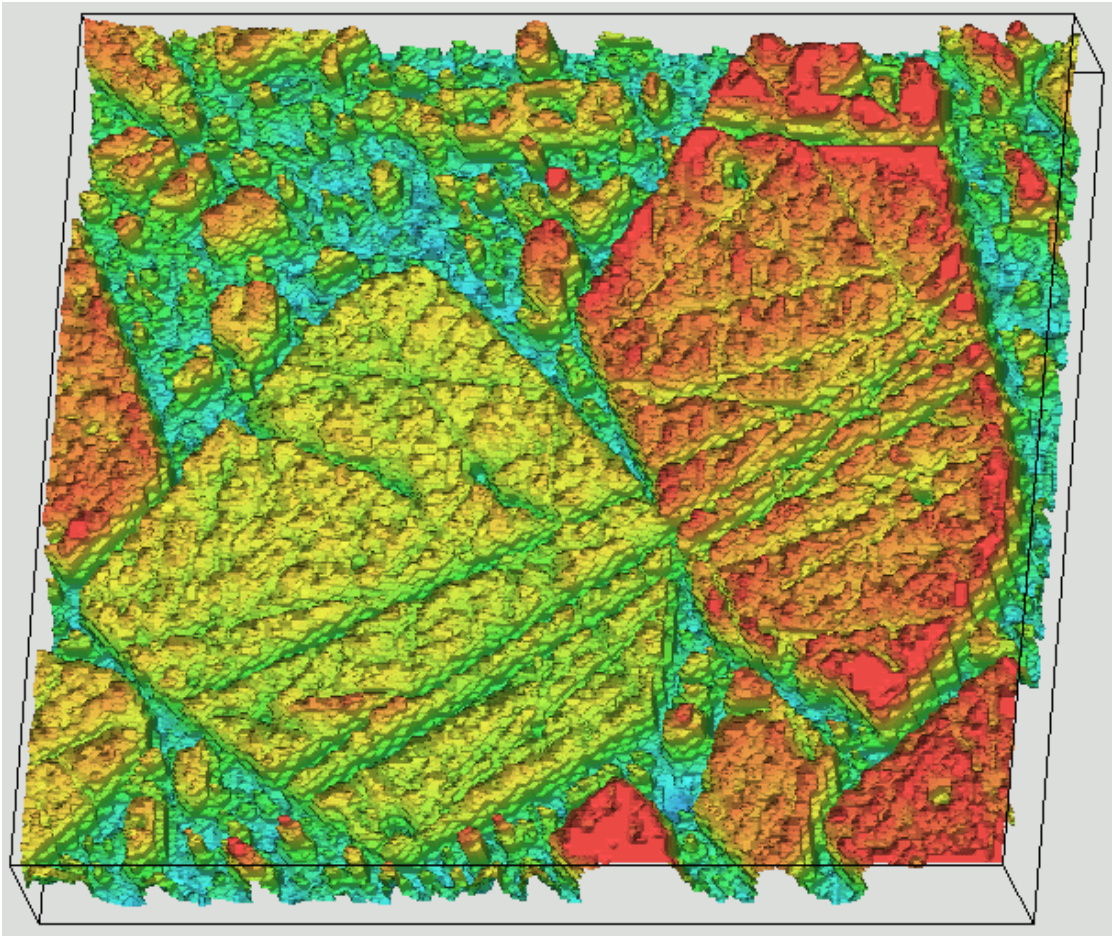


Introduction

Alternate Approach

This is NOT what we are going to do!



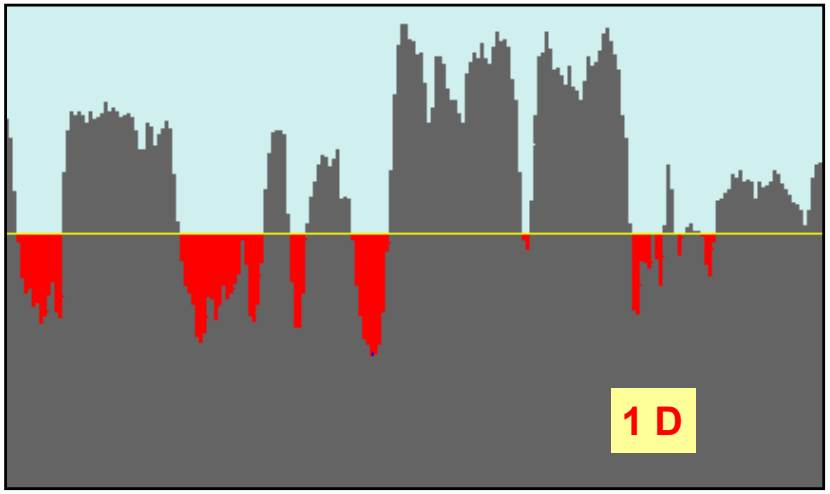


2D: Gray-Level (GL) Image

mapped into 3D while using GL as the third Dimension

→ **Relief**

More on Reliefs

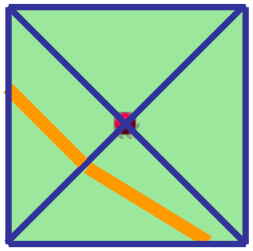


1D: Gray-Level Strip mapped into 2D while using GL as the second Dimension.

Intersections at constant GL, may yield both: points (0D), AND line segments (1D)!

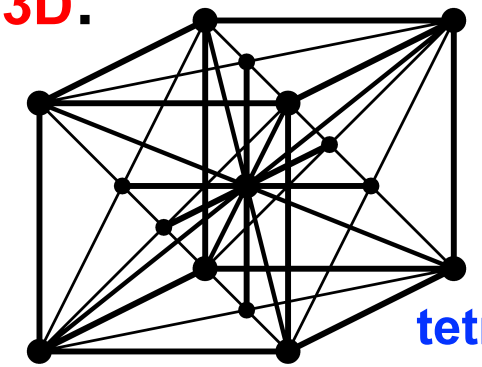
Similar in higher Dimensions:

2D:



embedded in 3D:
points (0D)
lines (1D)
triangles (2D)

3D:



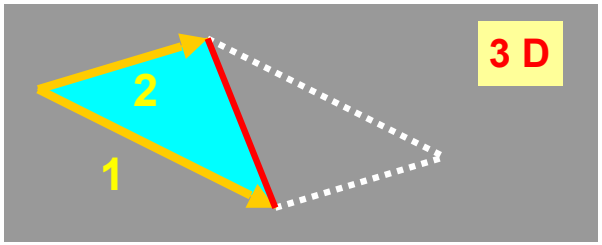
embedded in 4D:
points (0D)
lines (1D)
triangles (2D)
tetrahedrons (3D)

Furthermore: Too many Elements, Orientations not always clear …

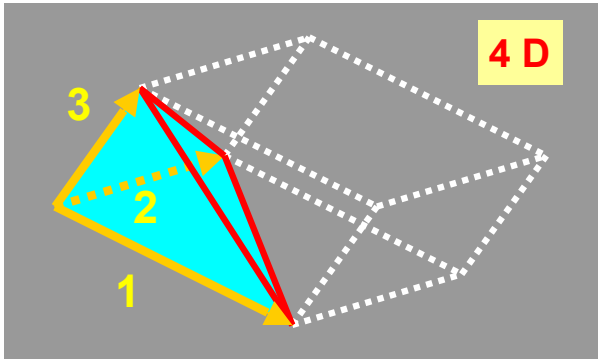
Instead: Manifolds of Co-Dimension 1

are numerically approximated best by:

Line-Elements in 2 D → Contours
"DICONEX"



Triangles in 3 D → Surfaces
"VESTA"



Tetrahedrons in 4 D → Hyper-Surfaces
"STEVE"

Such Choices may lead to Size Correction Factors for the particular Tiling.

Higher Dimensions

This presentation refers to the invention disclosure GSI / P309, “Verfahren zur Hyper-flächenkonstruktion in n Dimensionen.”

German Patent Application

No.s 10 2011 050 721.3

& 10 2011 051 203.9

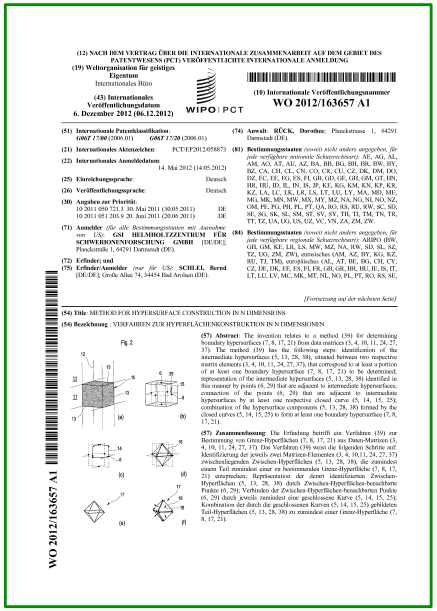
PCT Application

No. PCT/EP2012/058873

(EP2715673 A1,

US20140104270 A1,

WO2012163657 A1)



(12) NACH DEM VERTRAG ÜBER DIE INTERNATIONALE ZUSAMMENARBEIT AUF DEM GEBIET DES PATENTWESENS (PCT) VERÖFFENTLICHTE INTERNATIONALE ANMELDUNG

(19) Weltorganisation für geistiges Eigentum
Internationales Büro

(43) Internationales Veröffentlichungsdatum
6. Dezember 2012 (06.12.2012)

WIPO | PCT

(10) Internationale Veröffentlichungsnummer
WO 2012/163657 A1

(51) Internationale Patentklassifikation: *G06F 17/00* (2006.01) *G06T 17/20* (2006.01)

(21) Internationales Aktenzeichen: PCT/EP2012/058873

(22) Internationales Anmeldedatum: 14. Mai 2012 (14.05.2012)

(25) Einreichungssprache: Deutsch

(26) Veröffentlichungssprache: Deutsch

(30) Angaben zur Priorität:
10 2011 050 721.3 30. Mai 2011 (30.05.2011) DE
10 2011 051 203.9 20. Juni 2011 (20.06.2011) DE

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(72) Erfinder; und
(75) Erfinder/Anmelder *(nur für US)*: SCHLEI, Bernd [DE/DE]; Große Allee 74, 34454 Bad Arolsen (DE).

(74) Anwalt: RÜCK, Dorothee; Planckstrasse 1, 64291 Darmstadt (DE).

(81) Bestimmungsstaaten *(soweit nicht anders angegeben, für jede verfügbare nationale Schutzrechtsart)*: AE, AG, AL, AM, AO, AT, AU, AZ, BA, BB, BG, BH, BR, BW, BY, BZ, CA, CH, CL, CN, CO, CR, CU, CZ, DK, DM, DO, DZ, EC, EE, EG, ES, FI, GB, GD, GE, GH, GM, GT, HN, HR, HU, ID, IL, IN, IS, JP, KE, KG, KM, KN, KP, KR, KZ, LA, LC, LK, LR, LS, LT, LU, LY, MA, MD, ME, MG, MK, MN, MW, MX, MY, MZ, NA, NG, NI, NO, NZ, OM, PE, PG, PH, PL, PT, QA, RO, RS, RU, RW, SC, SD, SE, SG, SK, SL, SM, ST, SV, SY, TH, TJ, TM, TN, TR, TT, TZ, UA, UG, US, UZ, VC, VN, ZA, ZM, ZW.

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[Fortsetzung auf der nächsten Seite]

(54) Title: METHOD FOR HYPERSURFACE CONSTRUCTION IN N DIMENSIONS

(54) Bezeichnung: VERFAHREN ZUR HYPERFLÄCHENKONSTRUKTION IN N DIMENSIONEN

Fig. 2

(57) Abstract: The invention relates to a method (39) for determining boundary hypersurfaces (7, 8, 17, 21) from data matrices (5, 4, 10, 11, 24, 27, 37). The method (39) has the following steps: identification of the intermediate hypersurfaces (5, 13, 28, 38), situated between two respective matrix elements (3, 4, 10, 11, 24, 27, 37), that correspond to at least a portion of at least one boundary hypersurface (7, 8, 17, 21) to be determined; representation of the intermediate hypersurfaces (5, 13, 28, 38) identified in this manner by points (6, 29) that are adjacent to intermediate hypersurfaces; connection of the points (6, 29) that are adjacent to intermediate hypersurfaces by at least one respective closed curve (5, 14, 15, 25); combination of the hypersurface components (5, 13, 28, 38) formed by the closed curves (5, 14, 15, 25) to form at least one boundary hypersurface (7, 8, 17, 21).

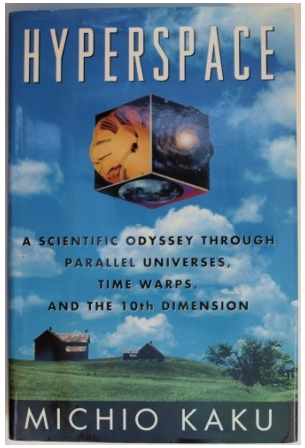
(57) Zusammenfassung: Die Erfindung betrifft ein Verfahren (39) zur Bestimmung von Grenz-Hyperflächen (7, 8, 17, 21) aus Datenmatrizen (5, 4, 10, 11, 24, 27, 37). Das Verfahren (39) weist die folgenden Schritte auf: Identifizierung der jeweils zwei Matrizen-Elementen (3, 4, 10,11, 24, 27, 37) zwischenliegenden Zwischen-Hyperflächen (5, 13, 28, 38), die zumindest einem Teil zumindest einer zu bestimmenden Grenz-Hyperfläche (7, 8, 17, 21) entsprechen; Repräsentation der dieser identifizierten Zwischen-Hyperflächen (5, 13, 28, 38) durch Zwischen-Hyperflächen-benachbarte Punkte (6, 29); Verbinden der Zwischen-Hyperflächen-benachbarten Punkte (6, 29) durch jeweils zumindest eine geschlossene Kurve (5, 14, 15, 25); Kombination der durch die geschlossenen Kurven (5, 14, 15, 25) gebildeten Teil-Hyperflächen (5, 13, 28, 38) zu zumindest einer Grenz-Hyperfläche (7, 8, 17, 21).

WO 2012/163657 A1

Cf., also

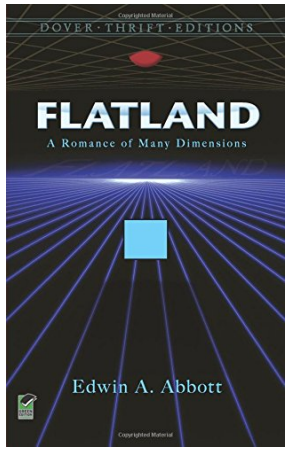
B. R. Schlei, “STEVE - Space-Time-Enclosing Volume Extraction,” arXiv:1302.5683 [cs.CG].

Proposed Literature:



“HYPERSPACE A scientific odyssey through parallel universes, time warps, and the 10th dimension,” by Michio Kaku, Oxford University Press, 1st edition (1994).

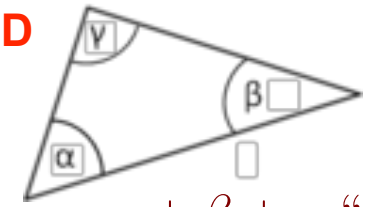
“FLATLAND A Romance of Many Dimensions,” by Edwin A. Abbott, 1st published in 1884 by Seeley & Co. of London.



Further Introductory Remarks

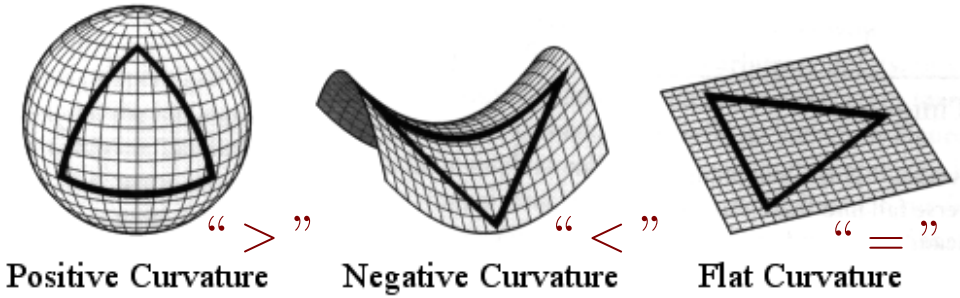
1854 - **Georg Friedrich Bernhard Riemann**, geometry on non-flat, i.e., curved, surfaces. In contrast to **Euclidean geometry**, e.g., the **Triangle Angle Sum Theorem**:

2D



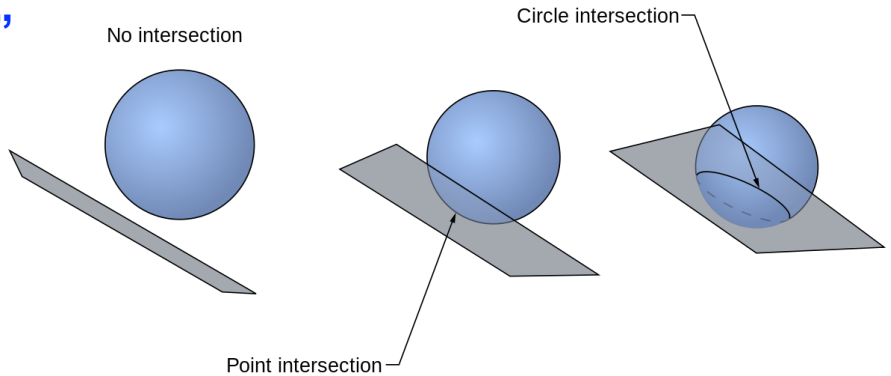
$$\alpha + \beta + \gamma \text{ “}=\text{” } 180^{\circ}$$

3D



Riemannian geometry works in n dimensions (Differential geometry).
It enabled Einstein's **General Relativity Theory** (published in **1915**; generalization of the **Special Theory of Relativity**, originally proposed in **1905** by **Albert Einstein**).

1884 - A **2D "flatlander"** perceives a **3D object**, i.e., a sphere.



Analogously:
A person in 3D could actually observe an intersection of a 4D object with his world.

Grids

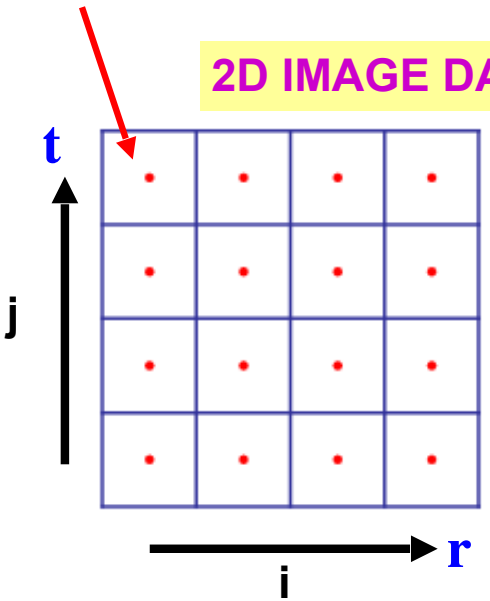
Example: Pixels, Voxels and Structured Grids in Numerical Hydrodynamics

Grid Points of Hydrodynamic Lattices may be viewed as **Centers of Pixels in 2D**, or **Centers of Voxels in 3D.**

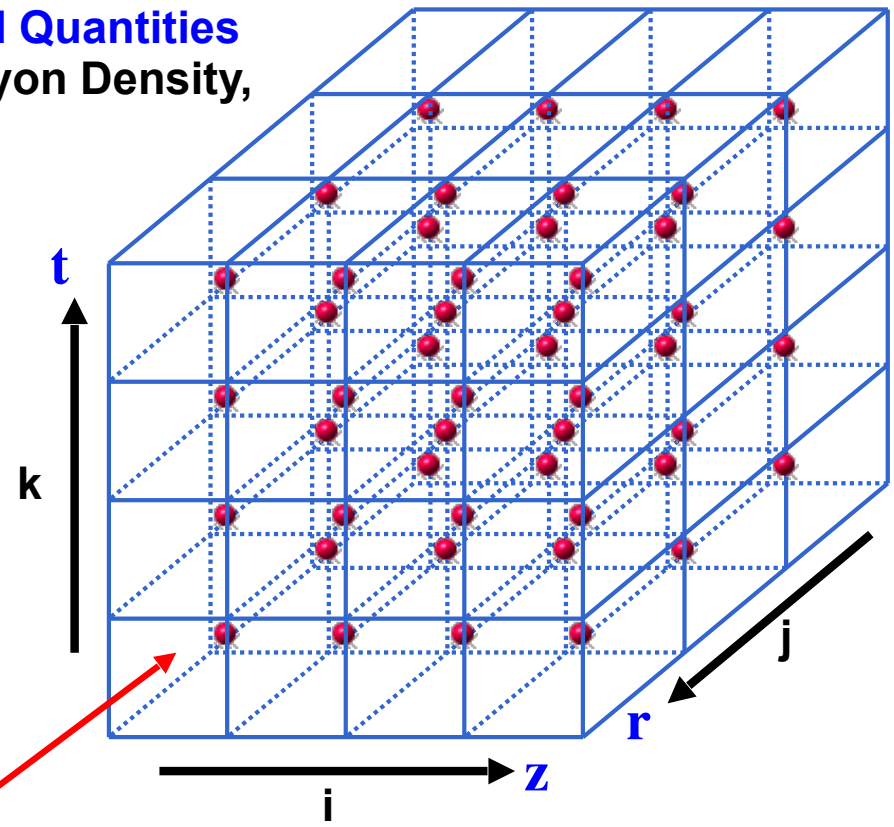
Each Center may be characterised by Field Quantities such as Temperature, Energy Density, Baryon Density, Fluid Velocity, … etc.

Pixel = Picture Element (2D)

2D IMAGE DATA

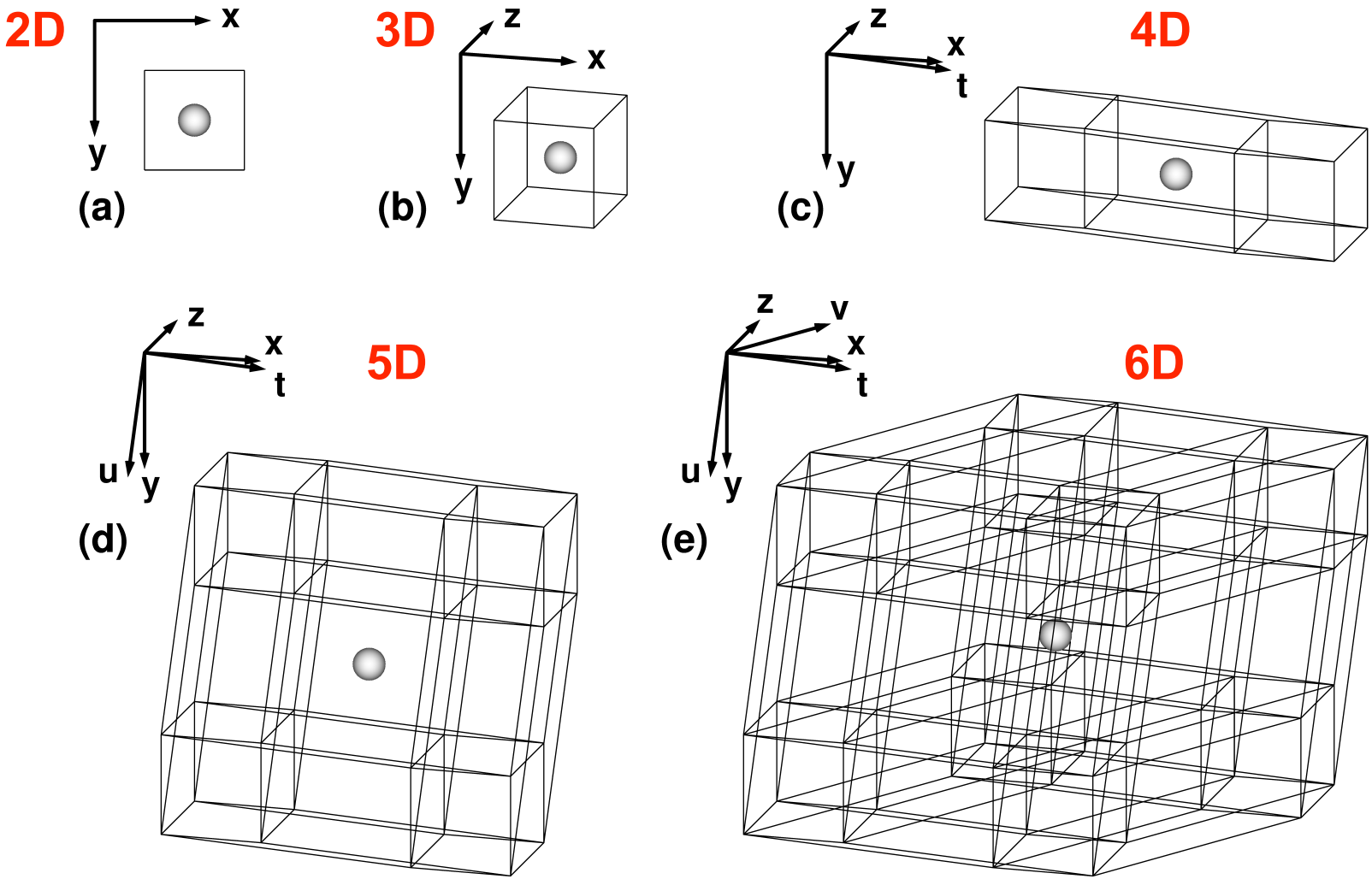


3D IMAGE DATA



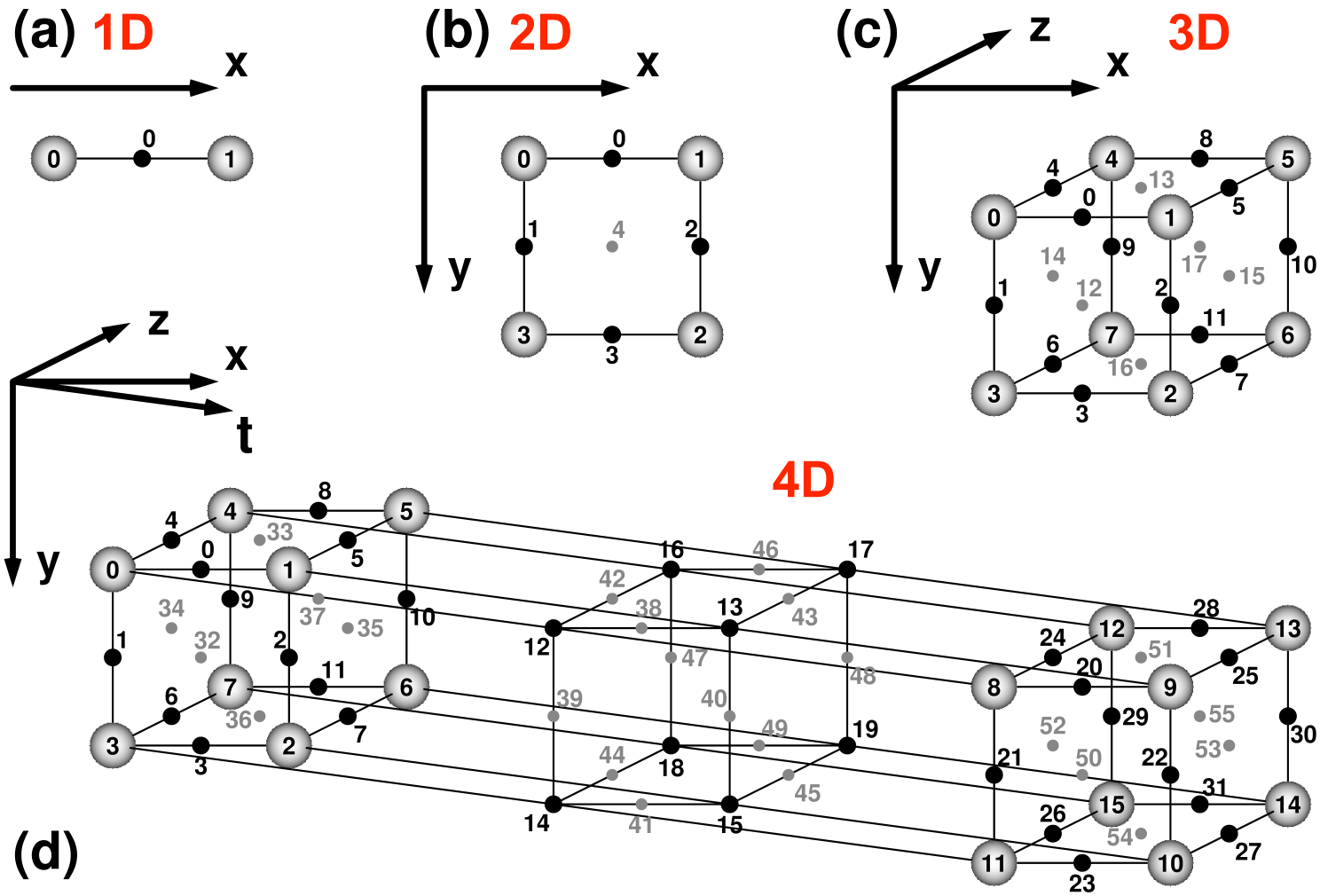
Voxel = VOlume piXEL (3D)

Single ND Picture Elements: Pixel, Voxel, Toxel, 5-Cube, and 6-Cube



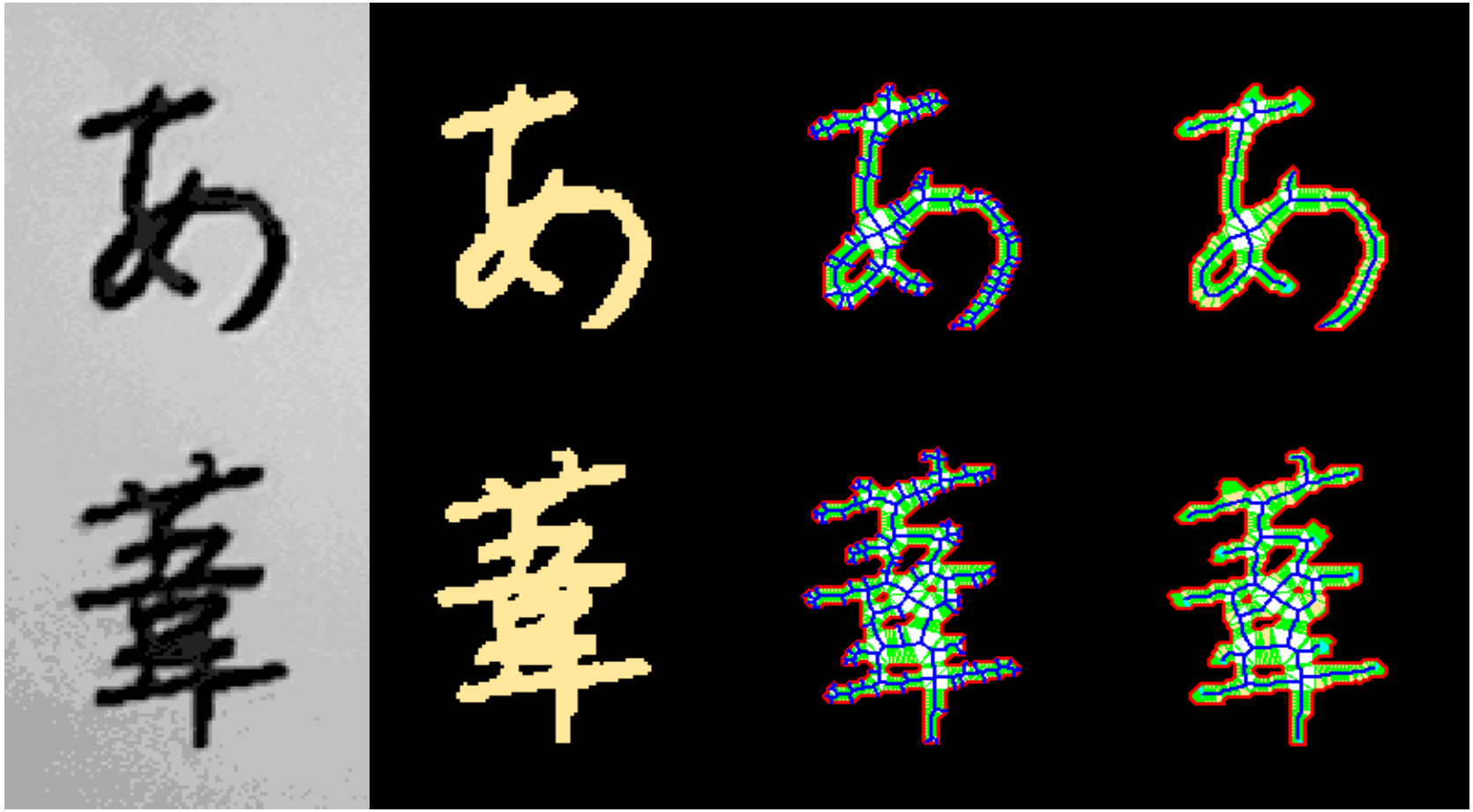
The Spheres indicate the center of each Picture Element.

N-Dimensional Neighborhoods (N = 1,2,3,4)



2 Dimensions

Example: Skeletonization of Handwritten Characters

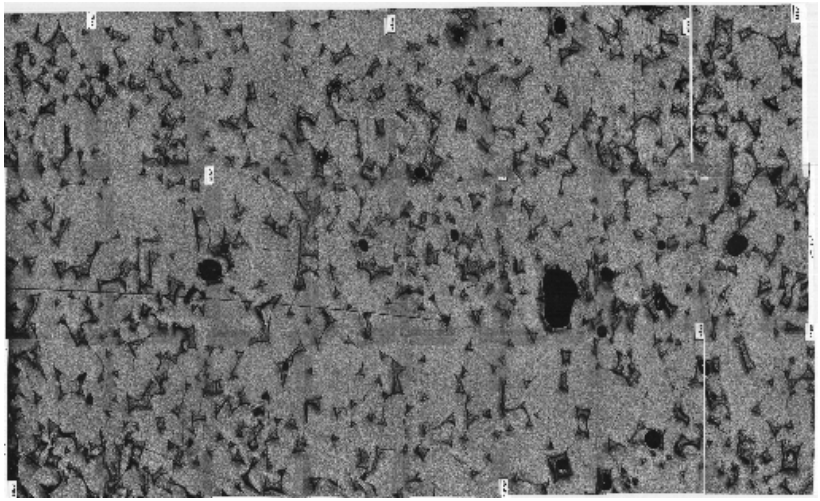


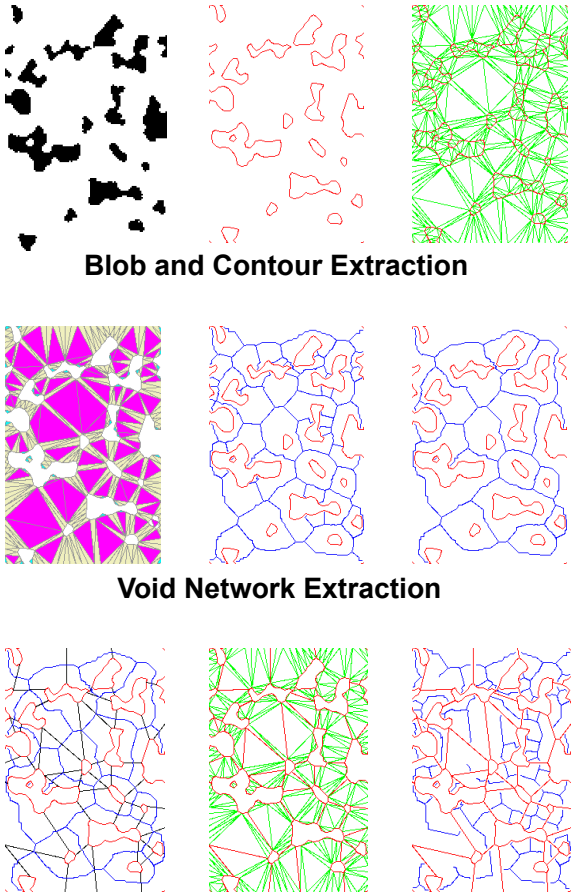
Original **Segmented** **Skeleton (before and after Pruning)**

Example: Polymeric Foams

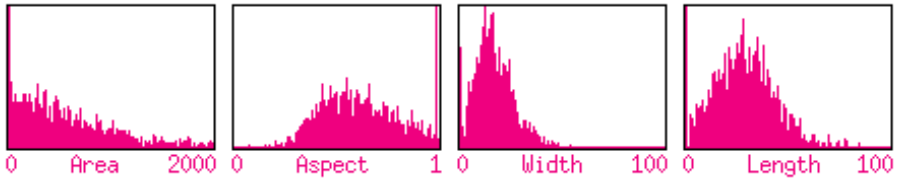
B. R. Schlei, L. Prasad, A. N. Skourikhine, Proc. SPIE 4476 (2001) 73.

Micrograph of a Polymeric Foam:



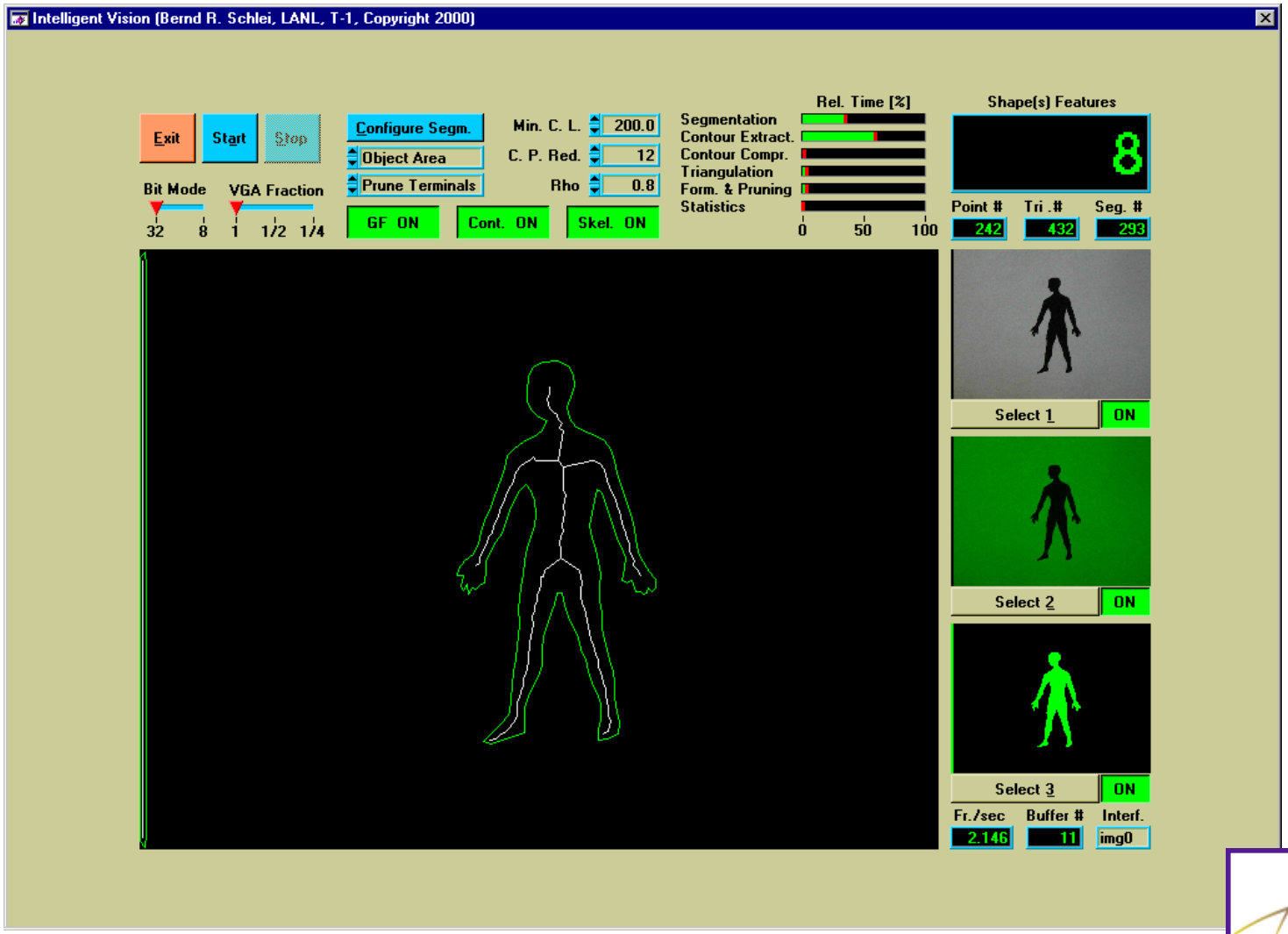


Histograms for single void cells [1 unit = 1 pixel]



Result: Automated Generation of Statistics.

Example: Intelligent Video Camera, Version 1.0



SONY - EVI Video Camera
National Instruments Frame Grabber
LabWindows/ CVI based GUI

Shape Feature Extraction is invariant under

- Translations
- Rotations
- Scaling

A few References on Contour Extraction

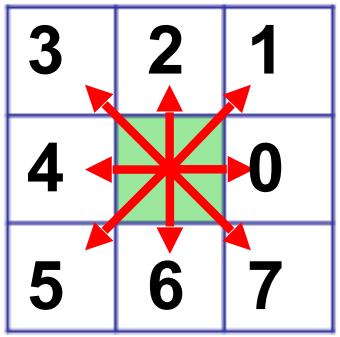
Many Papers have been written on Contour Extraction in Binary Images.
Here is a selection:

[1] H. Freeman, IRE Trans. Electronic Computers, EC-10, 2, June 1961, 260 – 268.
[2] H. Freeman, in *Picture Processing and Psychopictorics*, B. S. Lipkin, and A. Rosenfeld, Eds. Academic Press, New York, 1970, 241 – 266.
[3] H. Freeman, Computational Surveys, 6, 1974, 57 – 97.
[4] Z. Kulpa, Computer Graphics and Image Processing, Vol. 6, 4, December 1977, 434 – 451.
[5] T. Pavlidis, *Algorithms for Graphics and Image Processing*, Computer Science Press, Rockville, MD, 1982.
[6] Y.-T. Liow, Computer Vision Graphics and Image Processing, 53(3), 1991, 313 – 321.
[7] S. Di Zentro, L. Cinque, S. Levialdi, IEEETrans. Pattern Analysis and Machine Intelligence, 18(1), 1996, 83 – 89.
[8] E. Bribiesca, Pattern Recognition 32 (1999) 235 – 251.
[9] N. L. Jones, M. J. Kennard, A. K. Zundel, Computers & Geosciences 26 (2000) 831 – 837.
[10] S. Kaygin, M. M. Bulut, Pattern Recognition Letters 22 (2001) 1169 – 1178.
[11] Y. B. Bai, X. W. Xu, Computers in Industry 46 (2001) 65 – 74.
[12] M. Ren, J. Yang, H. Sun, Image and Vision Computing 20 (2002) 125 – 131.
[13] B. R. Schlei, in *Vision Geometry XI*, Eds. L. J. Latecki, D. M. Mount, A. Y. Wu, SPIE Vol. 4794 (2002) 63 – 70.

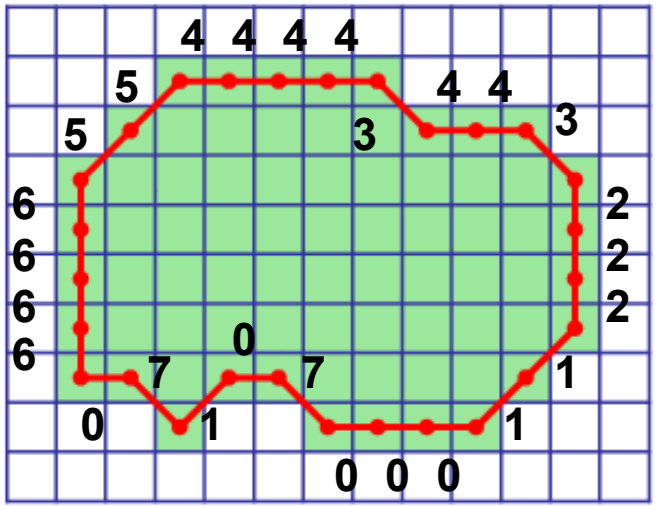
In the Following, we shall discuss the Approach of Freeman [1 – 3] in more Detail.

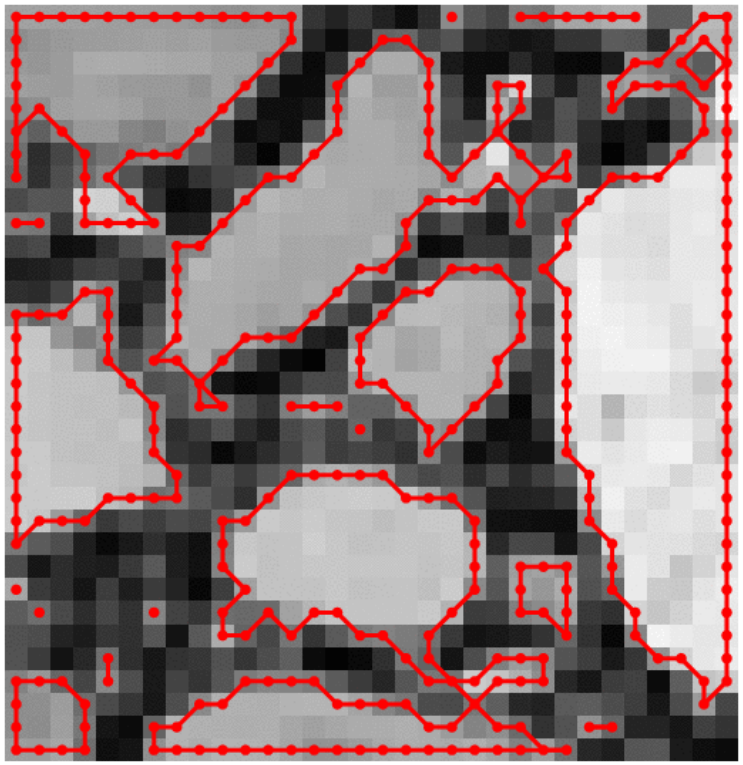
The Chain Code Algorithm

W. K. Pratt, Digital Image Processing, Wiley-Interscience, New York, 2001. (Papers by H. Freeman.)



8 Neighbors



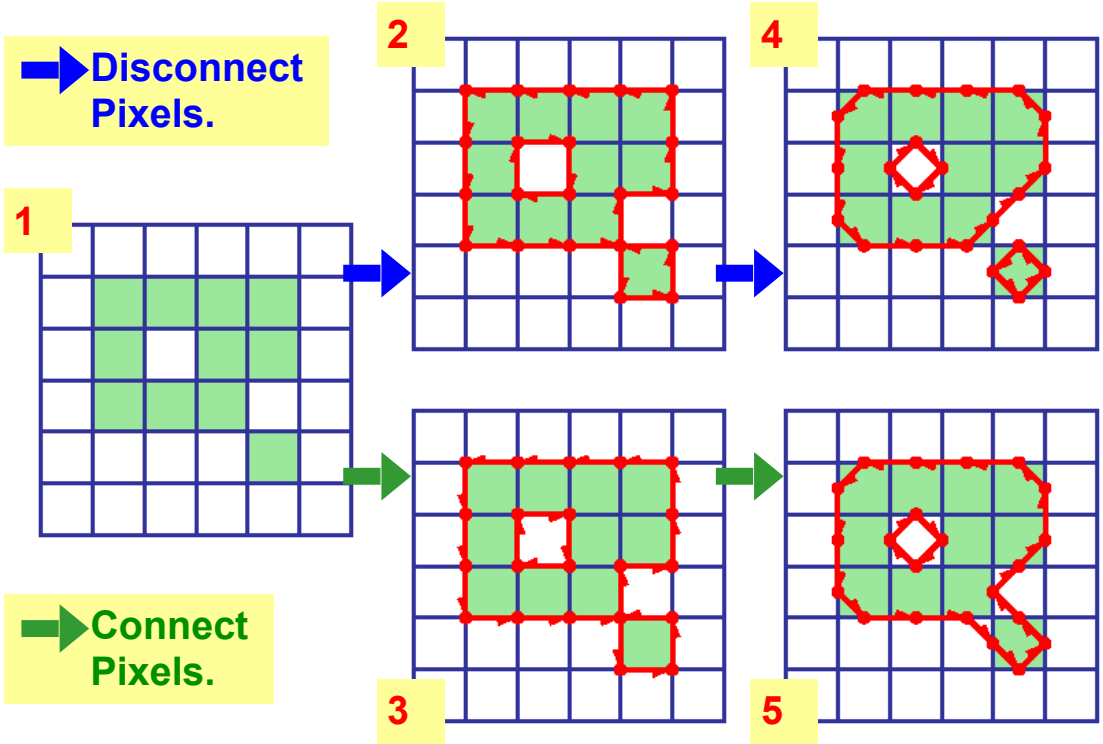


1. A Code Table is used for Boundary Coding.
2. "Active" Pixels are traced along the Shape Boundary and result in a Chain Code:

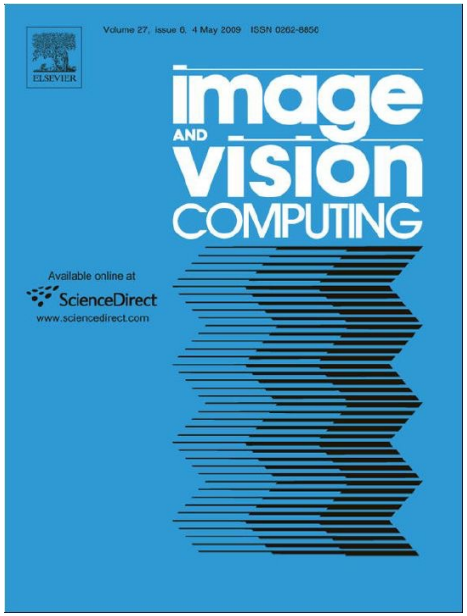
   07107000112223443444456666

Problem: The Contours of the Chain Code Algorithm may be self-intersecting and degenerated, i.e., NOT all enclosed Areas are > 0 !

DICONEX – DIlated CONtour EXtraction



B. R. Schlei,
SPIE Vol. 4794 (2002) 63;
Image and Vision Computing
27 (2009) 637 – 647.

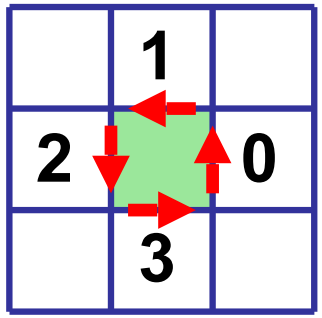


1: Initial Distribution of "Active" Pixels.
2 / 3: Left / Right-Turning Vectors.
4 / 5: Dilated Contours.

Fast 3-Step Algorithm:

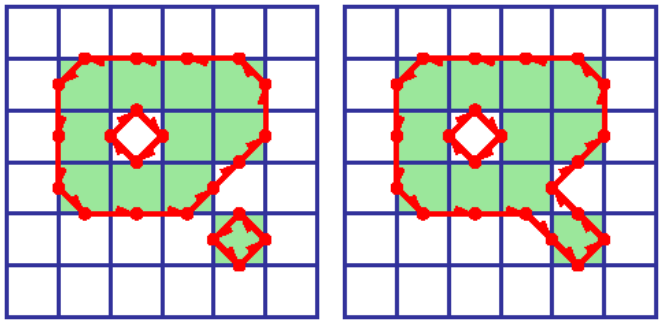
"Vectors – Connect – Shift"

Features of DICONEX Contours

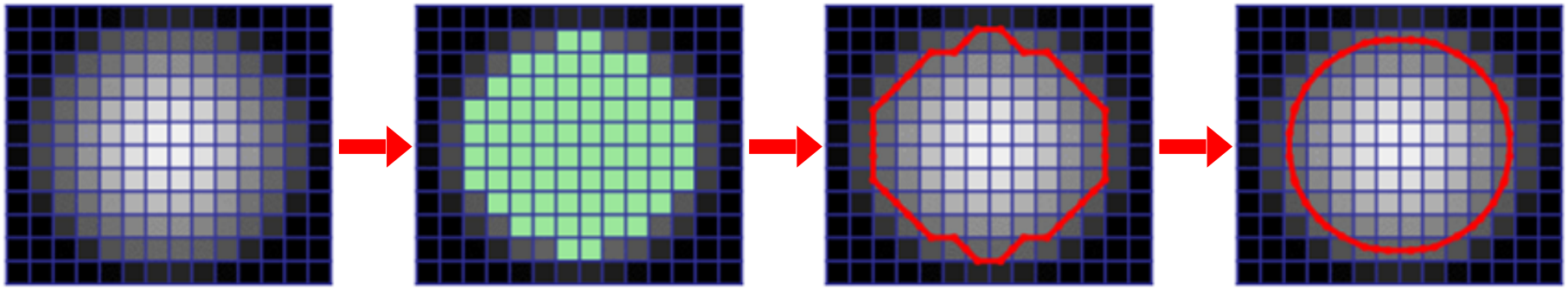


4 Neighbours

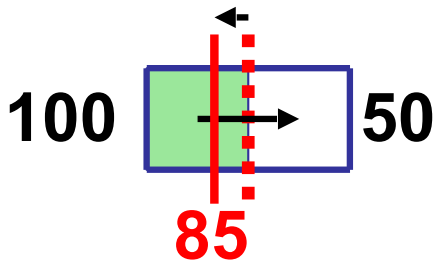
1. Only four Pixel Neighbours have to be considered during DICONEX Contour Construction.
2. DICONEX Contours are never self-intersecting or degenerated !



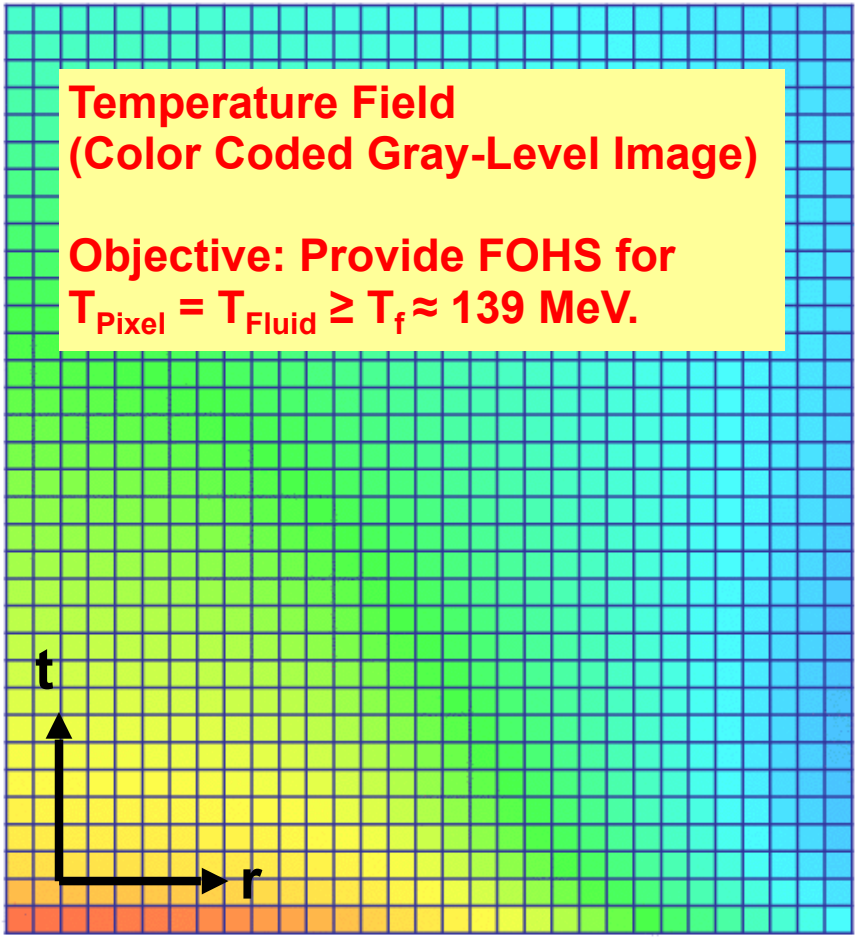
Displacement of DICONEX Contours:

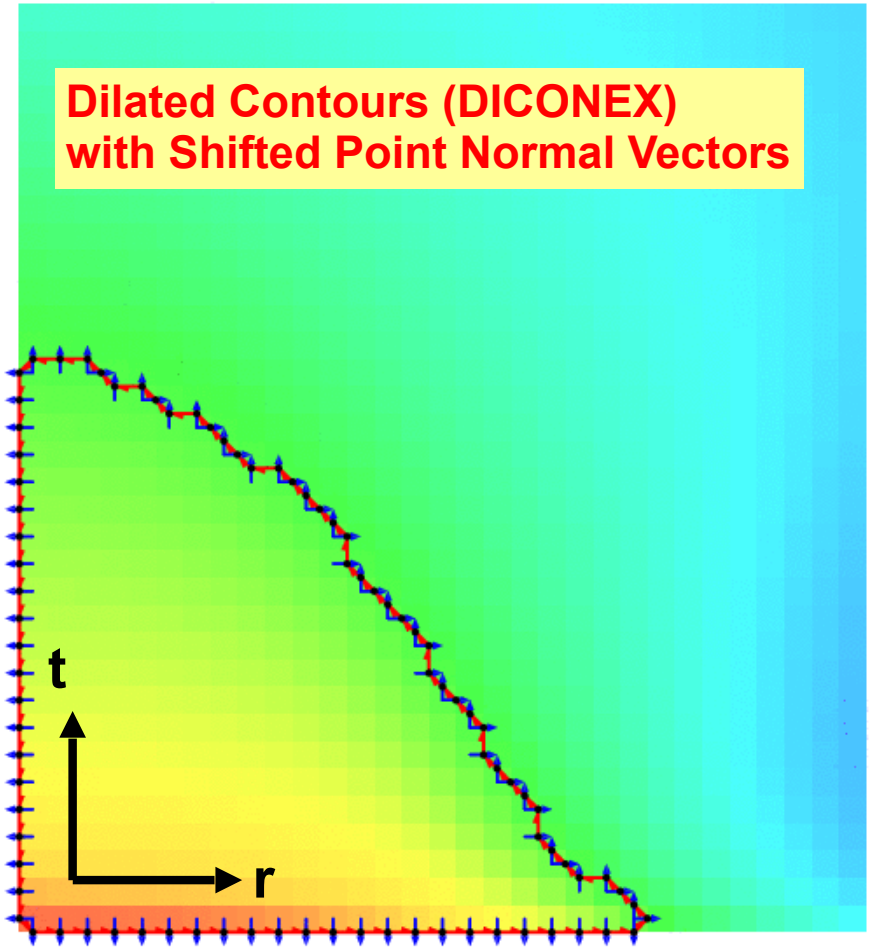


Initially given Gray-Level Images provide additional Spectral Information, which allow the Dislocation of the Points, which support the DICONEX Contours:



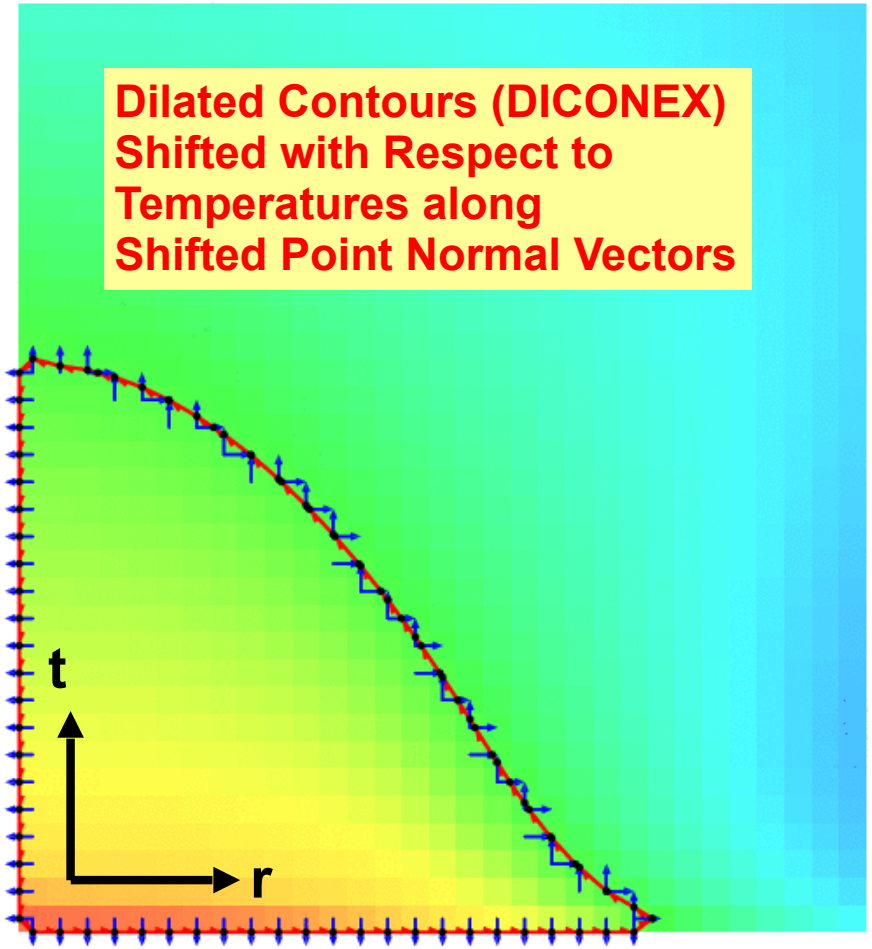
Example: Extraction of 1+1 D FOHS (Part 1)

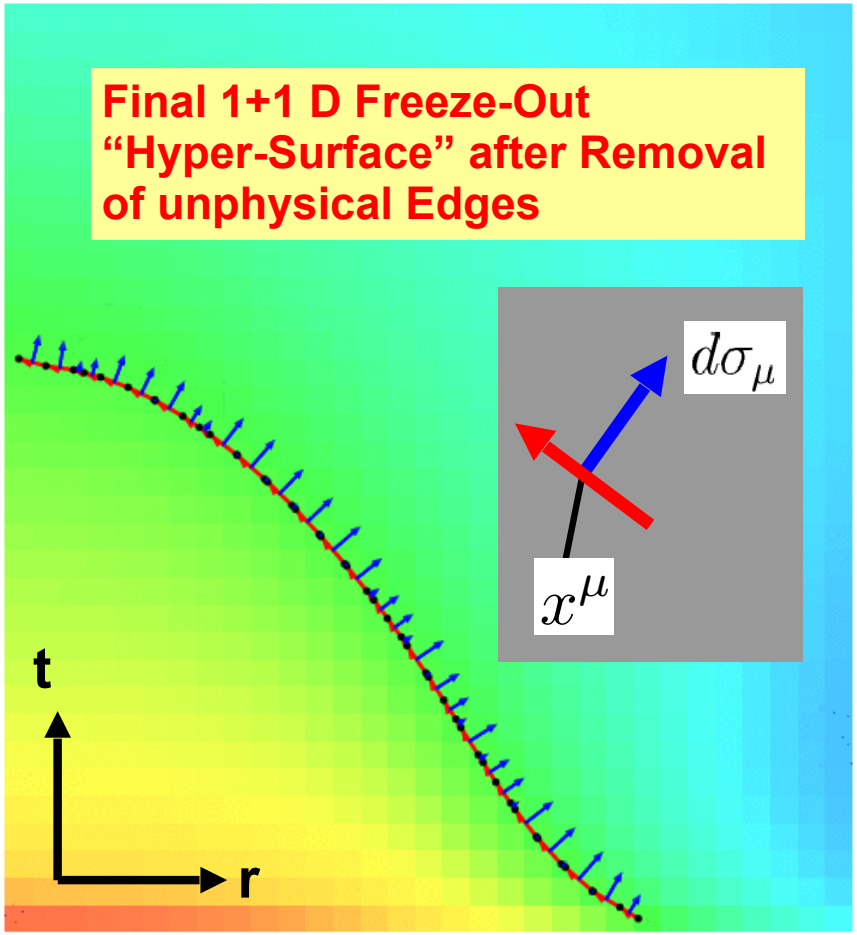




B. R. Schlei, Image and Vision Computing 27 (2009) 637 – 647.

Example: Extraction of 1+1 D FOHS (Part 2)





B. R. Schlei, Image and Vision Computing 27 (2009) 637 – 647.

Single- and Double-Inclusive Momentum Spectra

The Knowledge of a Freeze-Out Hyper-Surface (FOHS) and its corresponding Field Quantities (e.g., from Relativistic Hydrodynamics) allows to calculate both, Single-Inclusive Momentum Spectra and Bose-Einstein Correlations.

Single-Inclusive Momentum Spectra are given by (i.e., Particles of 1st Generation):

$$E\frac{dN_i}{d\vec{k}} = \frac{g_i}{(2\pi)^3}\int_\Sigma \frac{k^\mu\, d\sigma_\mu}{\exp\left(\frac{k^\mu u_\mu - \mu_i}{T_f}\right) \pm 1}$$

: FOHS Space-Time Coordinates

: FOHS 4-Normals

: Particle 4-Momenta

: 4-Velocity of Fluid

Bose-Einstein Correlations (e.g., for identical Pion Pairs) are given by:

$$C_2(\vec{k}_1,\vec{k}_2) = 1 + \frac{\langle a^\dagger(\vec{k}_1)a(\vec{k}_2)\rangle\langle a^\dagger(\vec{k}_2)a(\vec{k}_1)\rangle}{\langle a^\dagger(\vec{k}_1)a(\vec{k}_1)\rangle\langle a^\dagger(\vec{k}_2)a(\vec{k}_2)\rangle}$$

$$\sqrt{E_iE_j}\langle a^\dagger(\vec{k}_i)a(\vec{k}_j)\rangle = \frac{1}{(2\pi)^3}\int_\Sigma \frac{\frac{1}{2}(k_i^\mu + k_j^\mu)d\sigma_\mu(x_\mu)}{\exp\left[\frac{\frac{1}{2}(k_i^\mu + k_j^\mu)u_\mu(x_\mu)}{T_f(x_\mu)}\right] - 1}\cdot\exp[ix_\mu(k_i^\mu - k_j^\mu)]$$

μ_i **: Chemical Potentials (e.g., for B,S)**

T_f **: Freeze-Out Temperature**

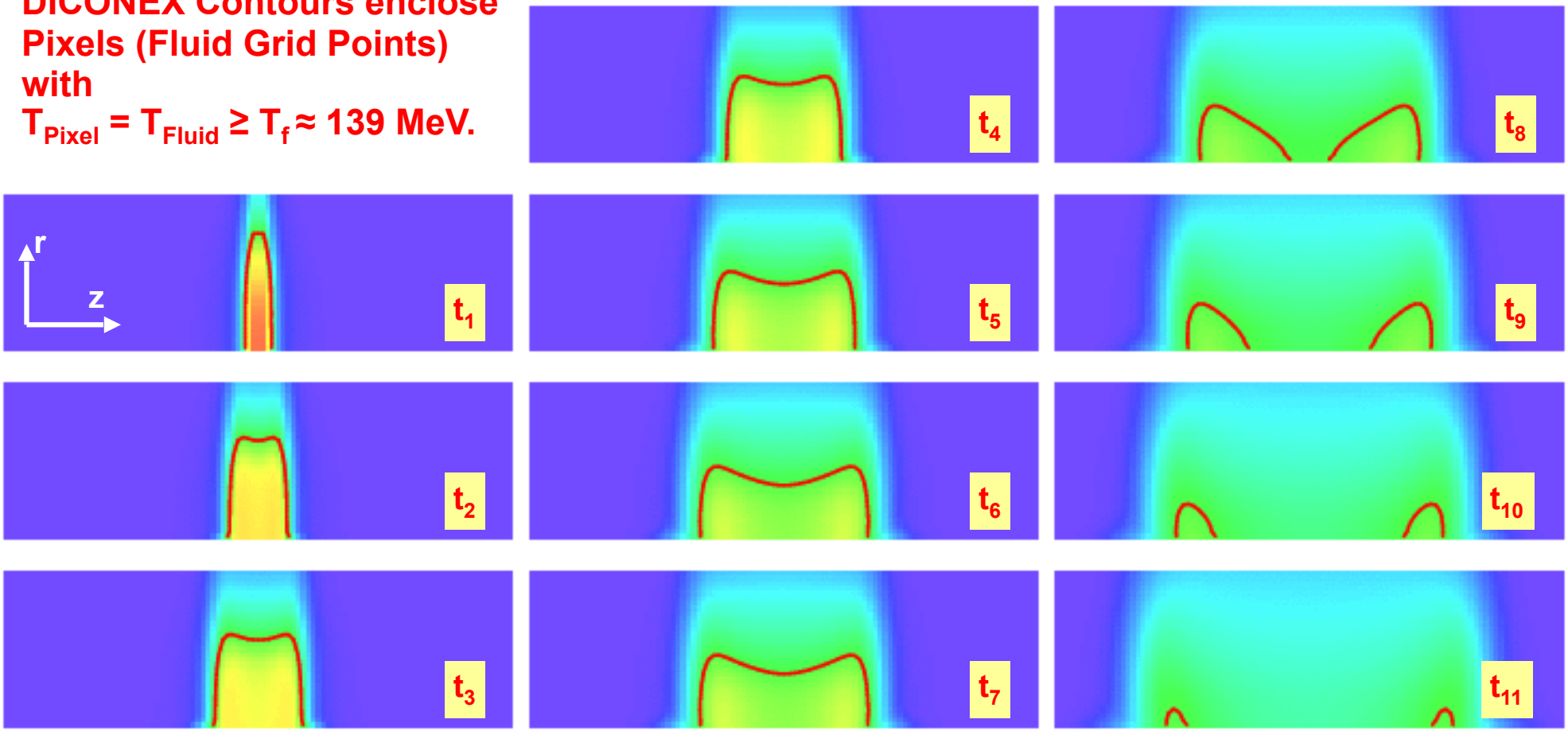
B. R. Schlei, U. Ornik, M. Plümer, R. M. Weiner, Phys. Lett. B293 (1992) 275.

3 Dimensions

Fluid Expansion and Cooling in HYLANDER-C (e.g., S+S @ 200A GeV)

U. Ornik, M. Plümer, B. R. Schlei, D. Strottman, R. M. Weiner, Phys. Rev. C54 (1996) 1381.

DICONEX Contours enclose Pixels (Fluid Grid Points) with $T_{Pixel} = T_{Fluid} \geq T_f \approx 139$ MeV.

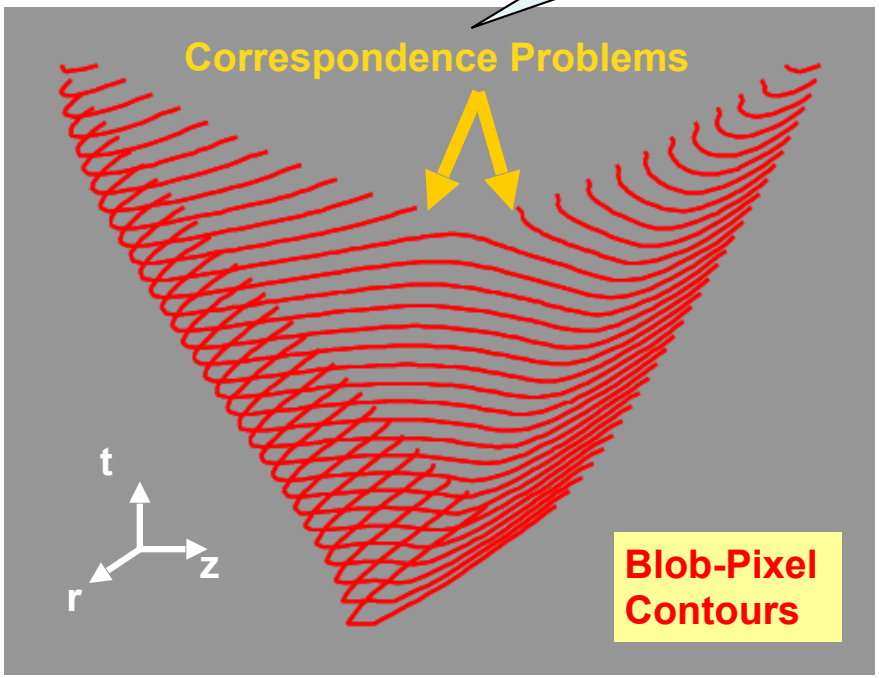


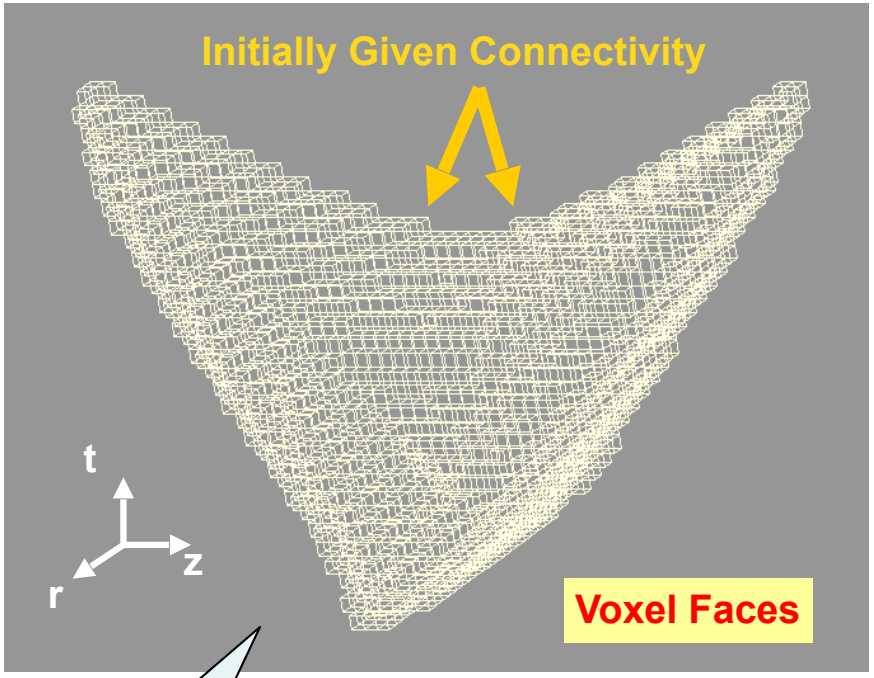
Note the Break-Up of *ONE* Contour into *TWO* Contours between Images t_7 and t_8.

Blob-Pixel Contours vs. Voxel Faces

While Building Temporal Correlations between the Contours, one may encounter Correspondence Problems.

Discontinue Approach.
STOP !





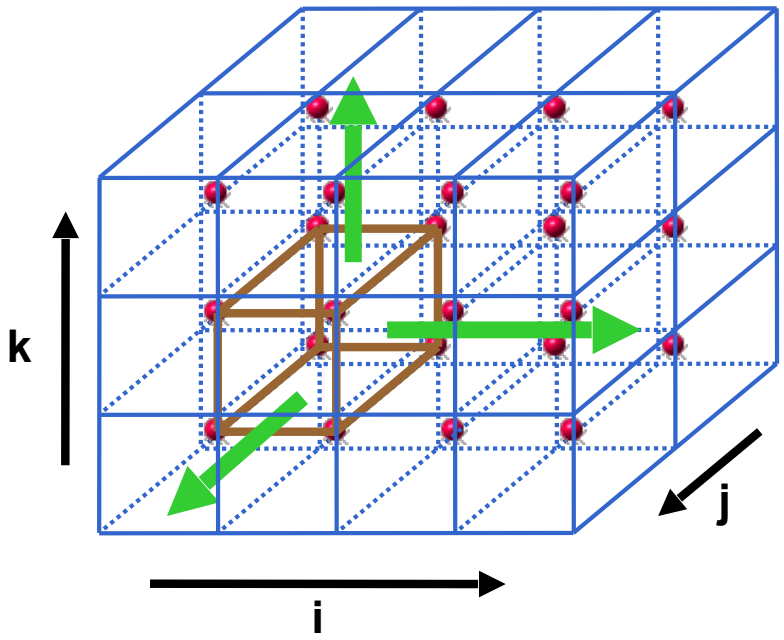
Rather Begin Here.
GO !

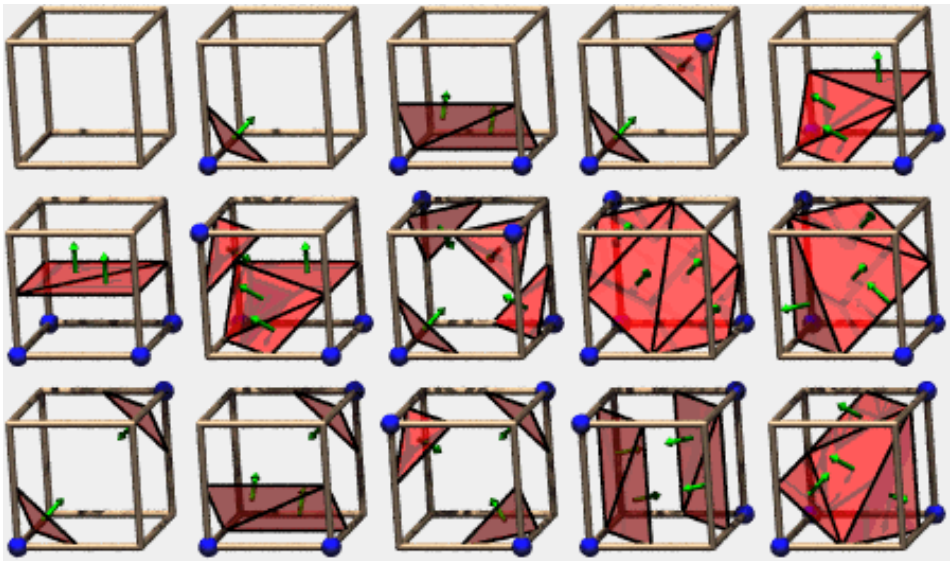
Consider a Temporal Stack of Blob-Pixels (➔ Voxels). Use only Voxel Faces, which separate the Shapes Interior from its exterior Regions.

The Marching Cubes Algorithm

W. E. Lorensen, H. E. Cline, Computer Graphics, 21 (1987) 163.

1. 2x2x2 Voxel Cubes are inspected for their Configurations of "active" Voxels (Volume Scan or Random Access).
   There are 256 = 2^8 different Configurations of "active" Voxels in a 2x2x2 Cube.





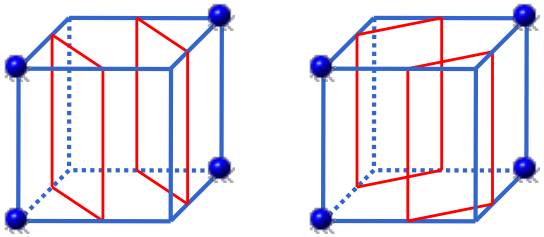
The 15 Cube Combinations

2. However, only 15 principal Configurations remain.
   Surface Templates are provided by a Look-Up Table.

http://exaflop.org/docs/marchcubes

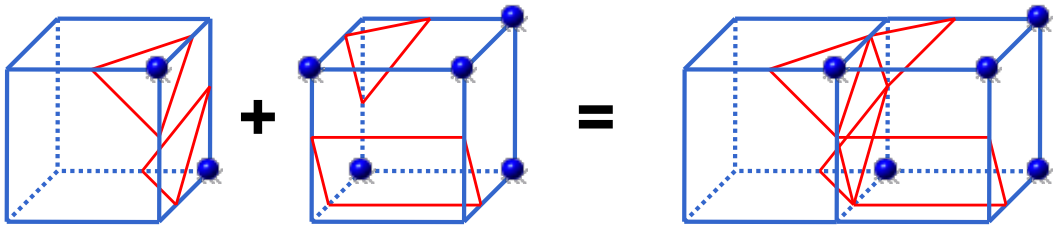
Problems with and Fixes for the Marching Cubes Algorithm

Problems with the Marching Cubes Algorithm:



1. The exact Location of the Surface is not always known from the Configuration of "active" Voxels.

2. As a Result, the Marching Cubes Algorithm is not guaranteed to produce a closed Surface :



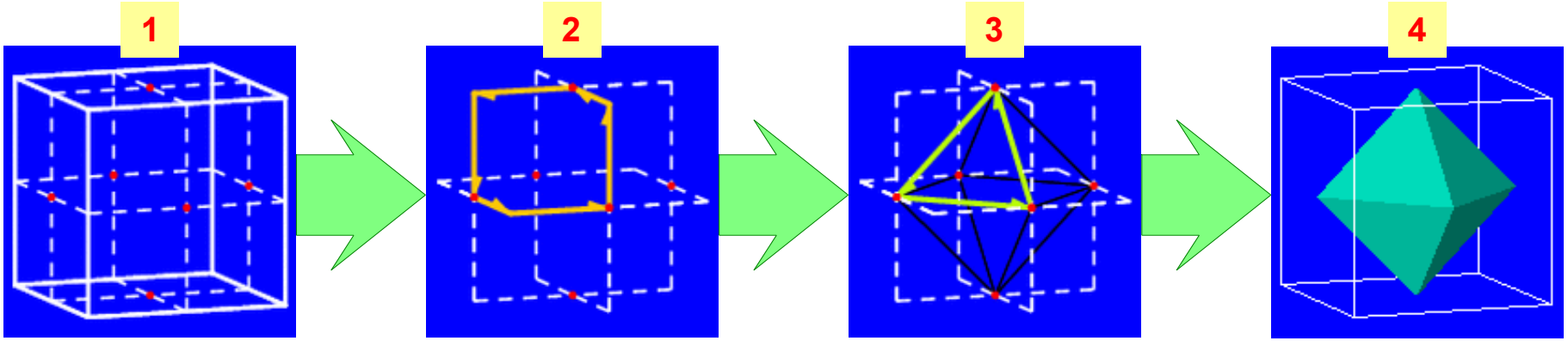
Fixes for Marching Cubes Algorithm through:

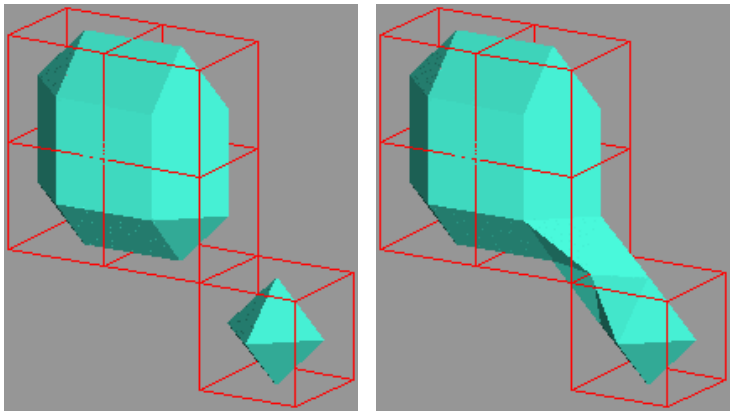
- Static Analysis (Uniform Orientation, Face Adjacency, Simplex Decomposition)
- Interpolation Analysis (Closest Orientation, Interpolation, Subdivision, … etc.)

S. Hill, J. C. Roberts, in "Graphics Gems V", eds. A. W. Peath, Academic Press, 1995, p. 98.; and Ref.s therein.

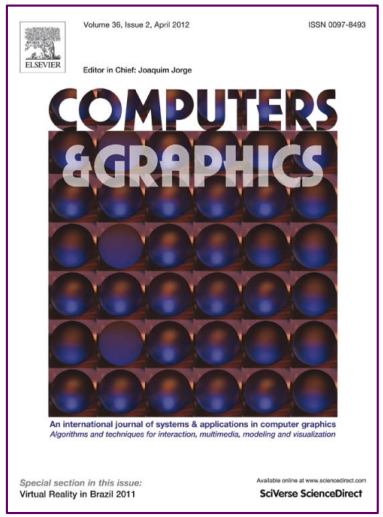
A Surface Construction Algorithm that requires Fixes is not really desirable !

3D: VESTA — Volume-Enclosing Surface exTraction Algorithm





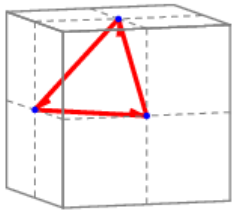
B. R. Schlei,
"Volume-Enclosing Surface Extraction,"
Computers & Graphics 36 (2012) 111,
doi: 10.1016/j/cag.2011.12.008.



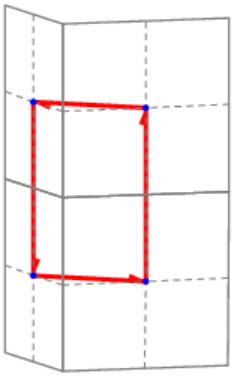
Voxels, which are in Contact with another Voxel only through one single Edge may be disconnected or connected.

VESTA - Surface Cycles

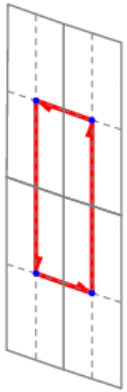
3-Cycle



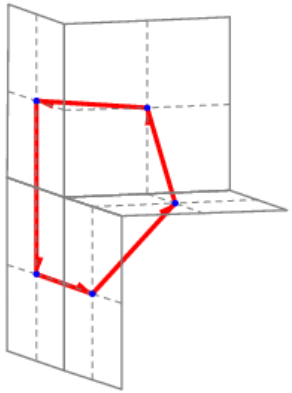
4-Cycle



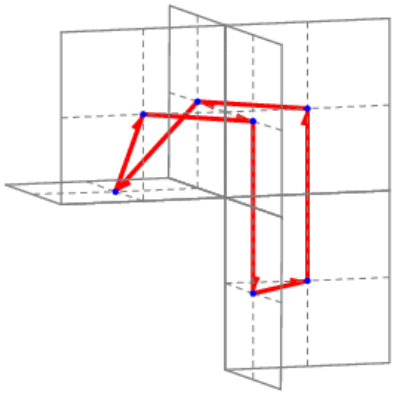
4-Cycle



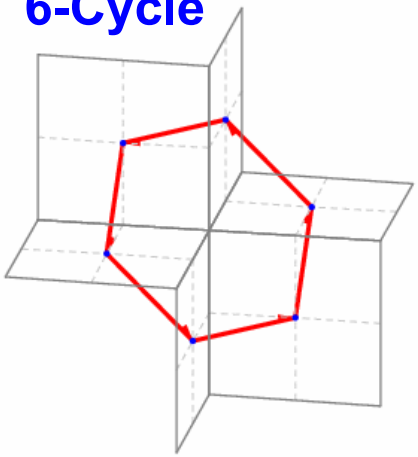
5-Cycle



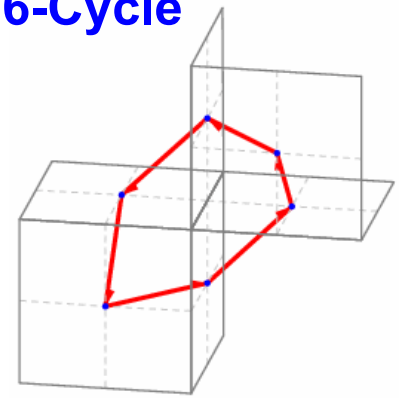
7-Cycle



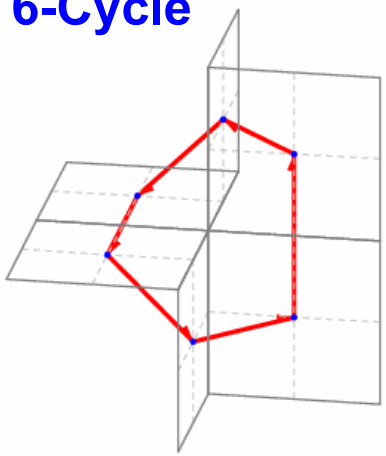
6-Cycle



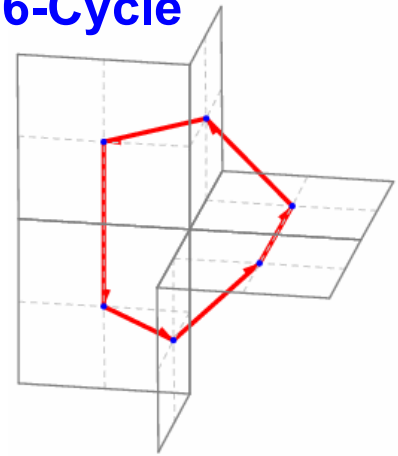
6-Cycle



6-Cycle

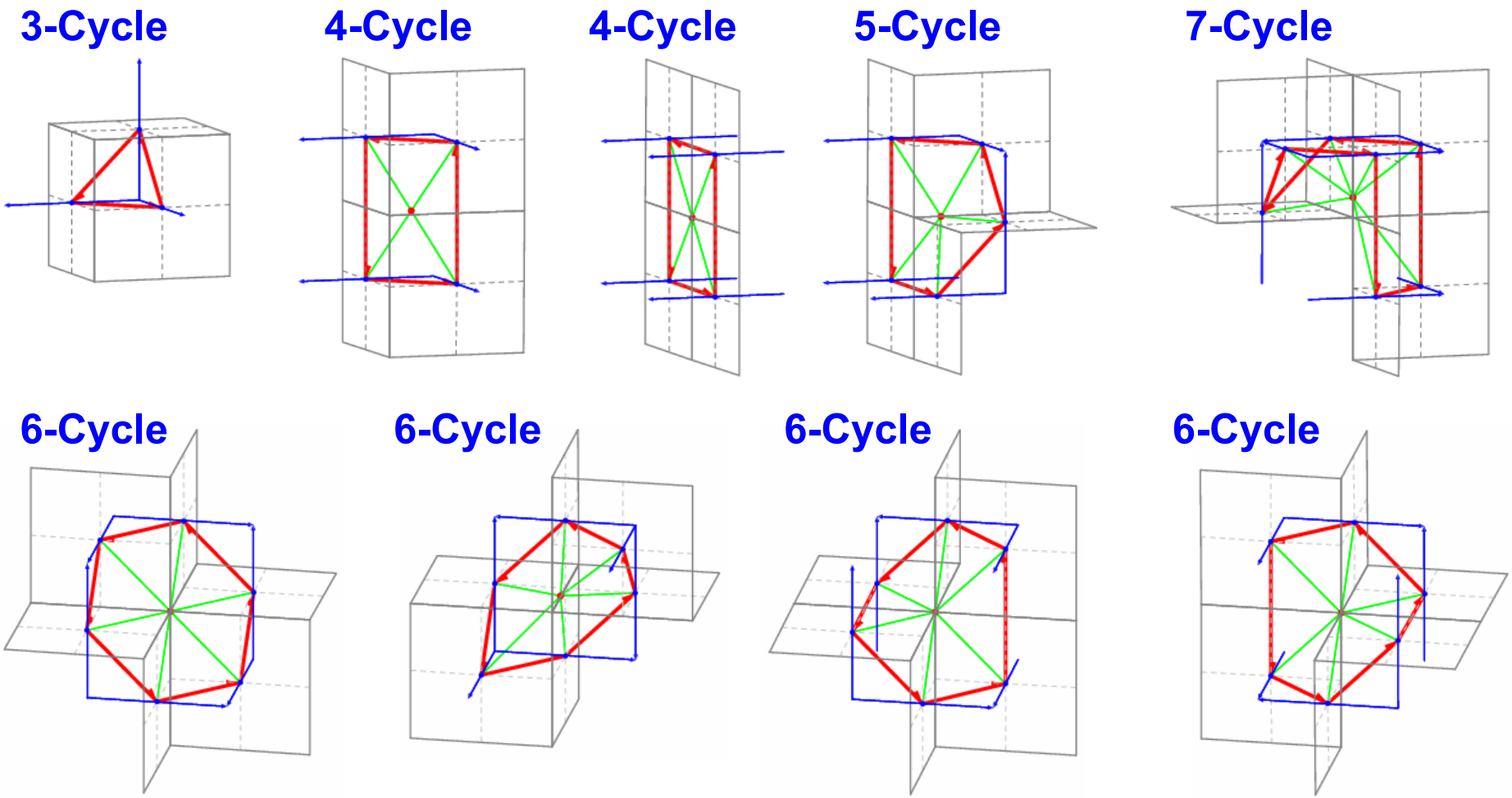


6-Cycle



Some N-Cycles are planar, some are non-planar !

VESTA - Surface Cycle Break-Up

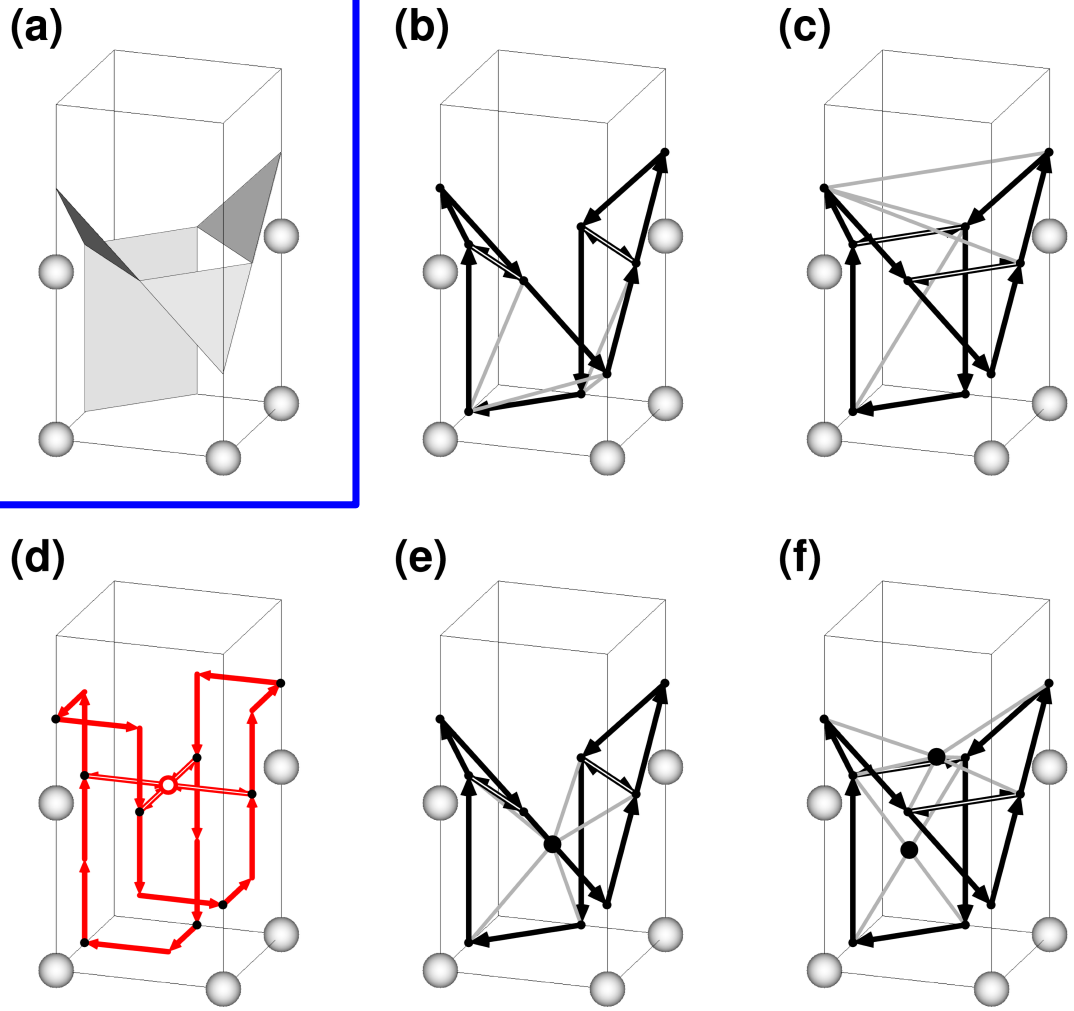


Break-Up into Triangles allows for Acceptable Shape Adaptation

VESTA vs. Marching Cubes Algorithm

Holes may appear while using the original MCA

i.e., W. E. Lorensen, H. E. Cline, Computer Graphics, 21 (1987) 163.



In general, there are more than one solutions possible!

(b) low resolution, disconnect mode

(c) low resolution, connect mode

(e) high resolution, disconnect mode

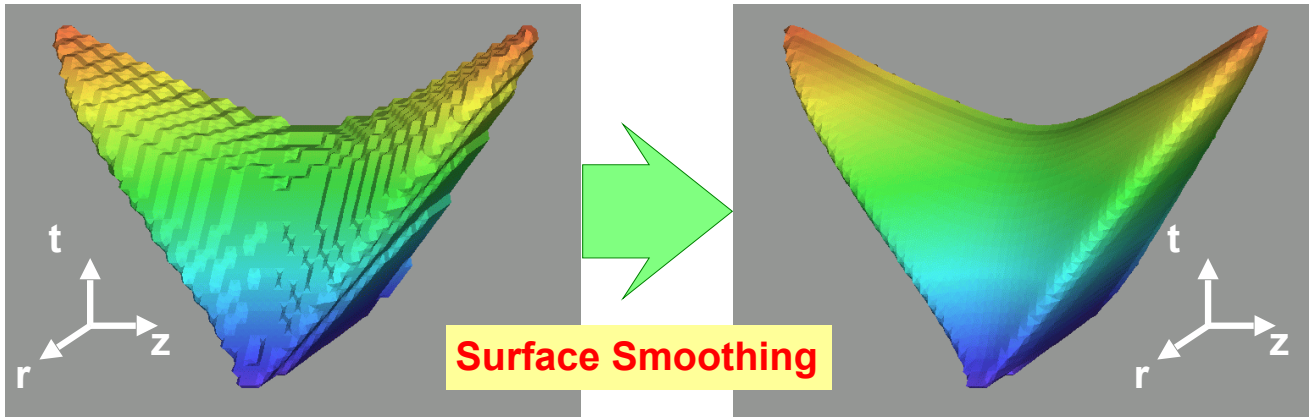
(f) high resolution, connect mode

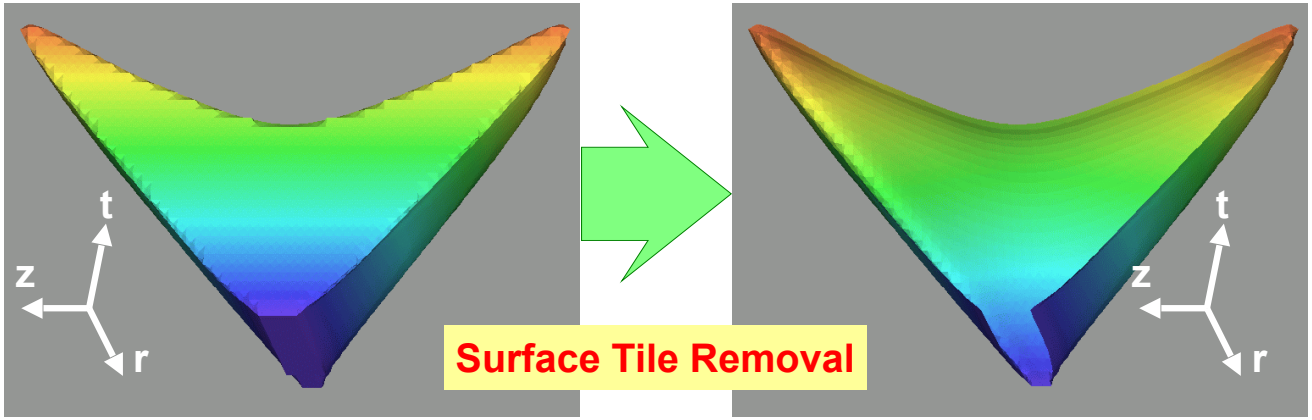
VESTA: Ambiguities will be explicitly resolved.

B. R. Schlei, Computers & Graphics 36 (2012) 111 – 130.

VESTA Surface Manipulations

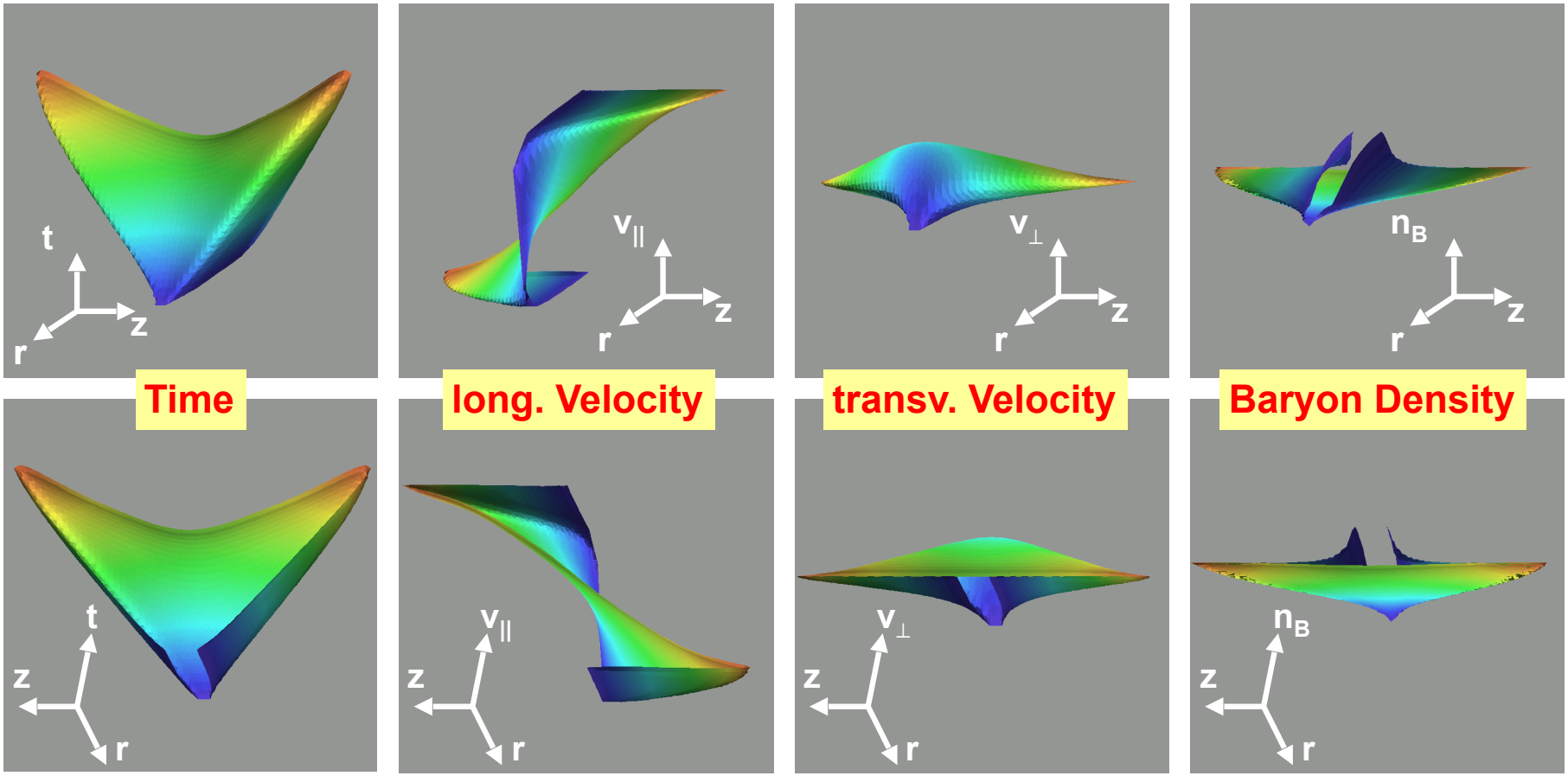
While using the actual Temperature Values within a Voxel and one of its Neighbours, one may dislocate the Face Centers of the VESTA Surface with respect to T_f (or ε_f, or n_f, … etc.). This usually leads to a Surface Smoothing.





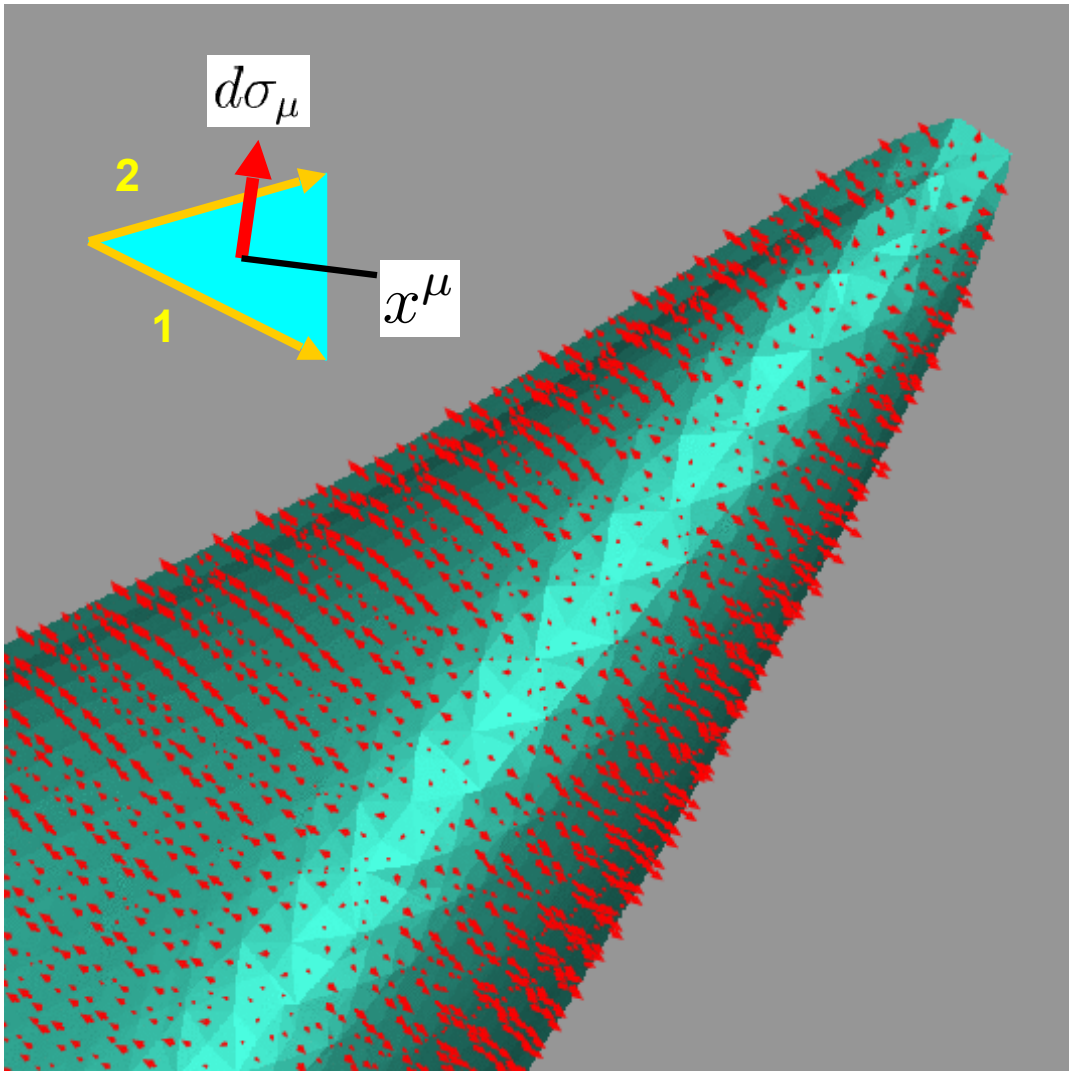
Surface Tiles with all of their Components $t \leq 0$, or $r \leq 0$, have no meaning for 2+1 D Hydrodynamics. They are therefore removed.

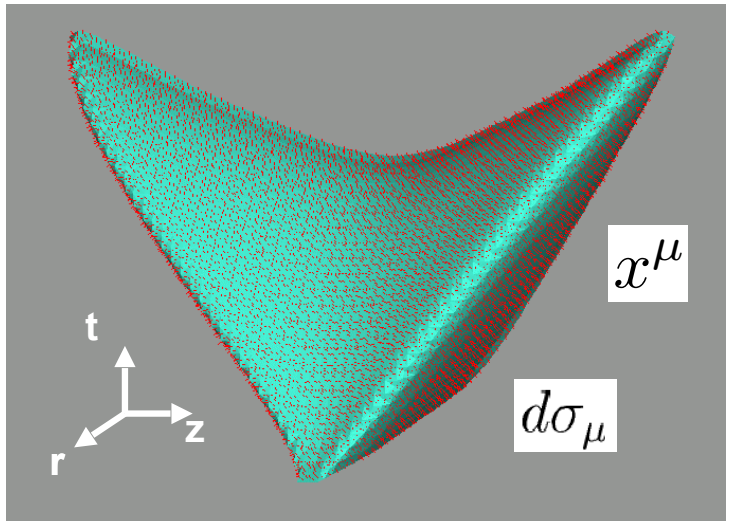
Rendering of Corresponding Field Quantities



Here, all other Field Quantities, such as Energy Density ε, Speed of Sound c_0^2, etc., are constant, and therefore not shown.

FOHS and Normal Vectors

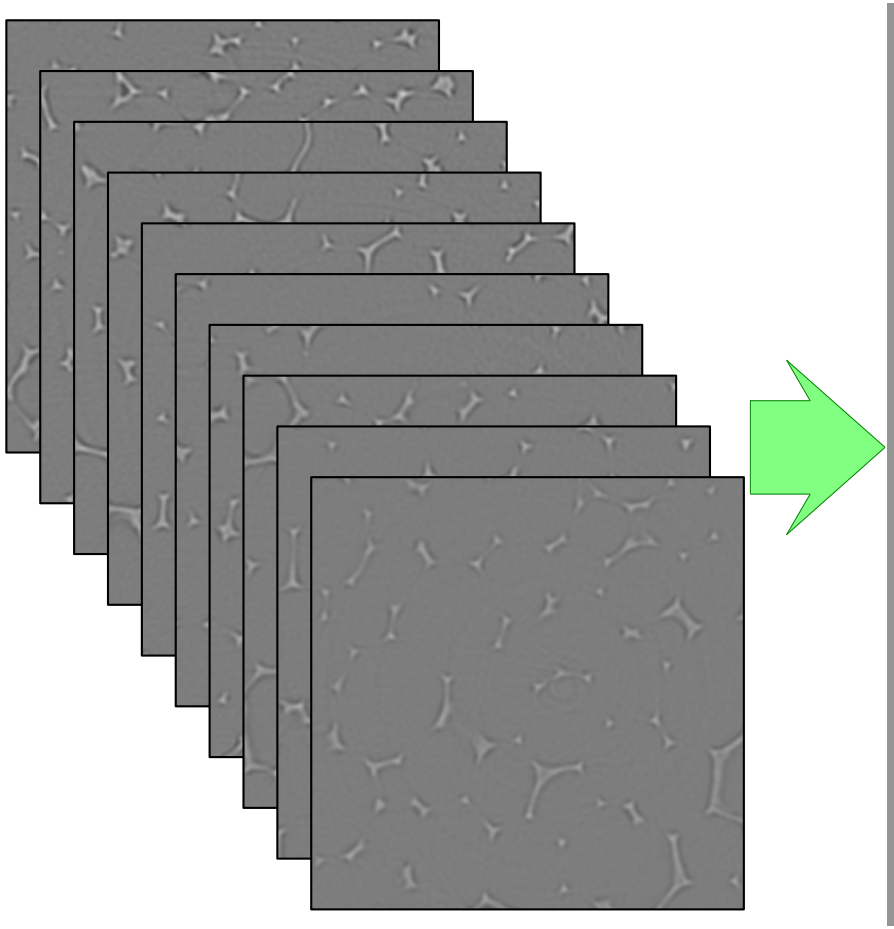


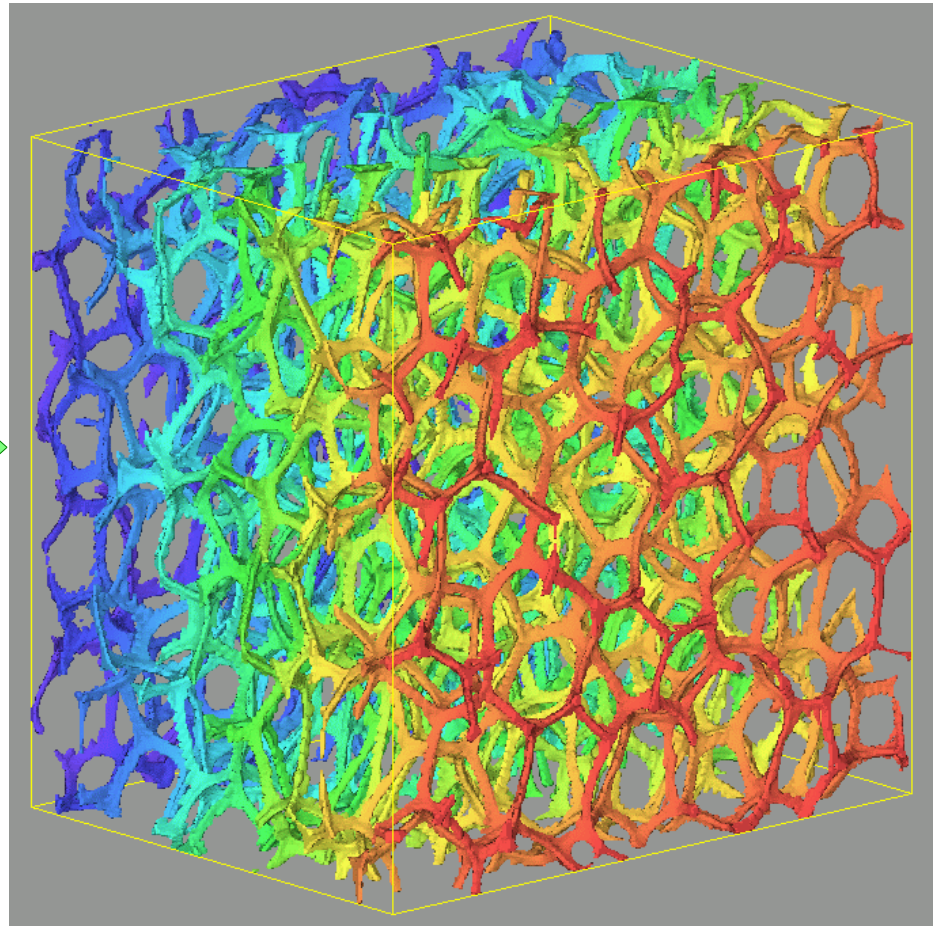


$d\sigma_\mu$ are the **Normal Vectors** for each FOHS Triangle; their Lengths are equal to their corresponding Triangle Area.

x^μ are the **Centers of Mass** for each FOHS Triangle; **all related Field Quantities are evaluated at these Points.**

3D: Surface Extraction for Data Sets from Computed Tomography (CT)





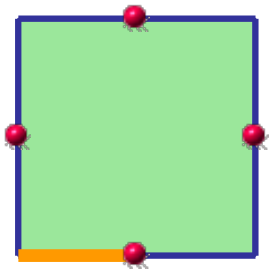
3D Image Data from X-Ray Micro-Tomography, courtesy, Jerry Seidler et al., Univ. WA, Seattle.

B. R. Schlei, "Volume-Enclosing Surface Extraction," Computers & Graphics 36 (2012) 111, doi: 10.1016/j/cag.2011.12.008.

4 Dimensions

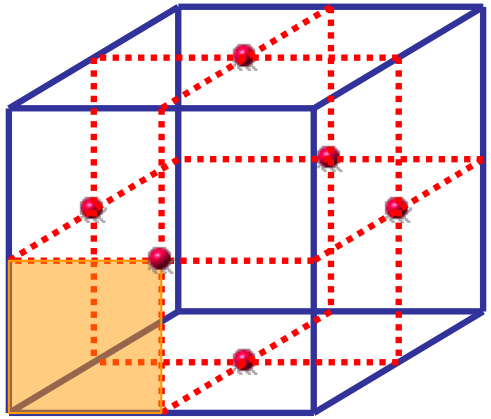
4D Analogies with 2D and 3D (Part 1)

2D



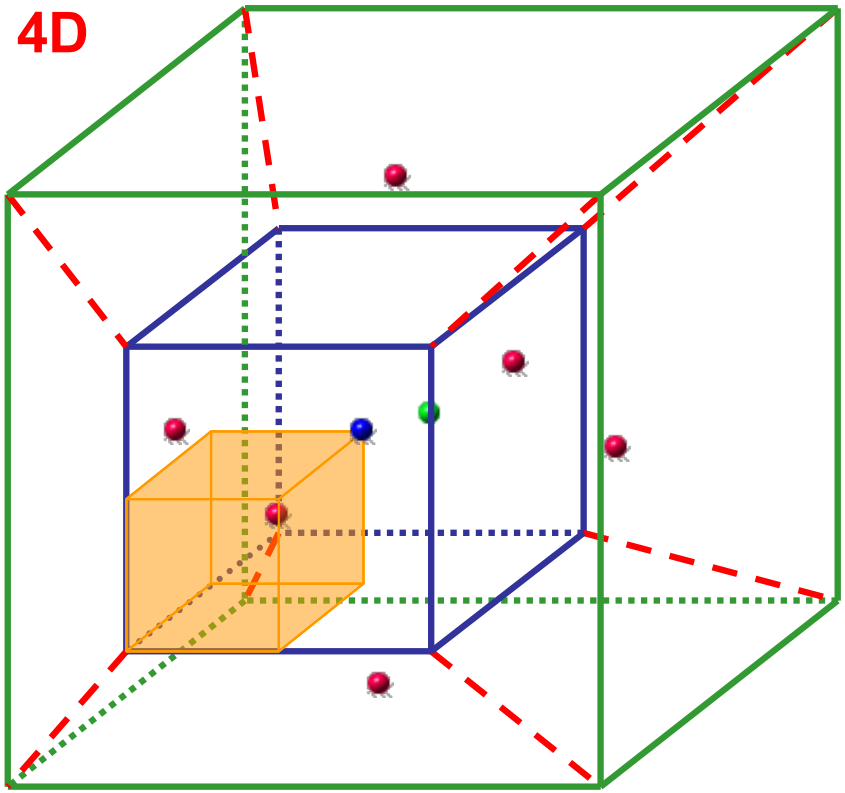
Pixel,
Edges,
Edge Centers,
… etc.

3D



Voxel,
Faces,
Face Centers,
… etc.

4D



Toxel,
Volumes,
Volume Centers,
… etc.

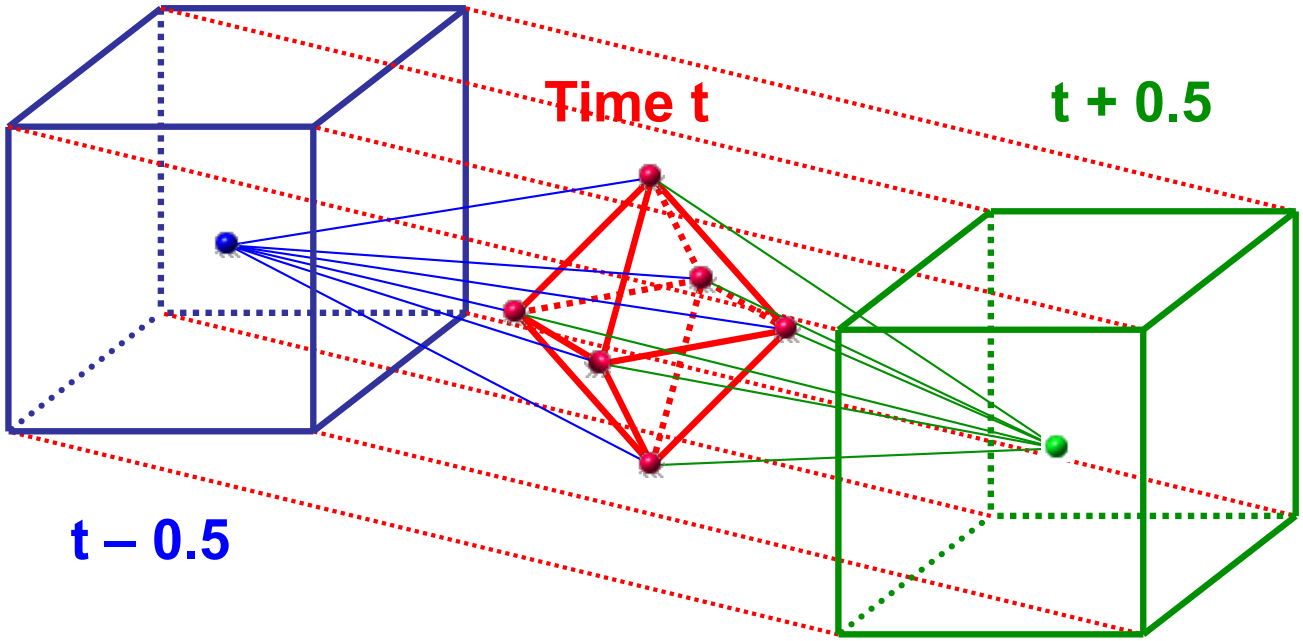
Generalisations via Analogies
DICONEX → VESTA → STEVE

(STEVE = Space-Time Enclosing Volume Extraction,
B. R. Schlei (Copyright © 2003 - 2018))

4D Analogies with 2D and 3D (Part 2)

2D: 4 Edges
3D: 6 Faces
4D: 8 Volumes
nD: 2n Hyper-Faces

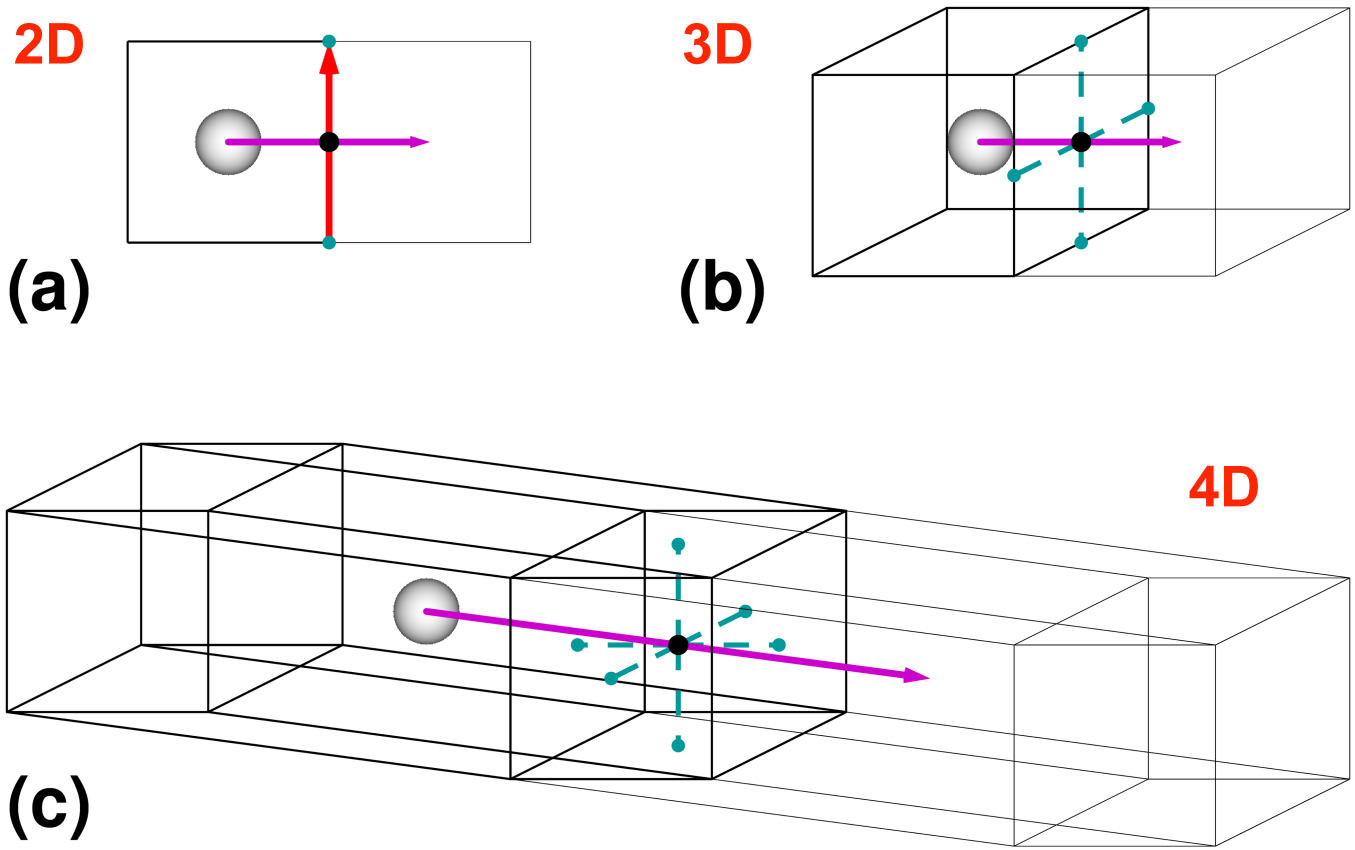
2D: 4 Line Segments
3D: 8 Triangles
4D: 16 Tetrahedrons
nD: 2^n Hyper-Triangles
= (n - 1)-Simplices



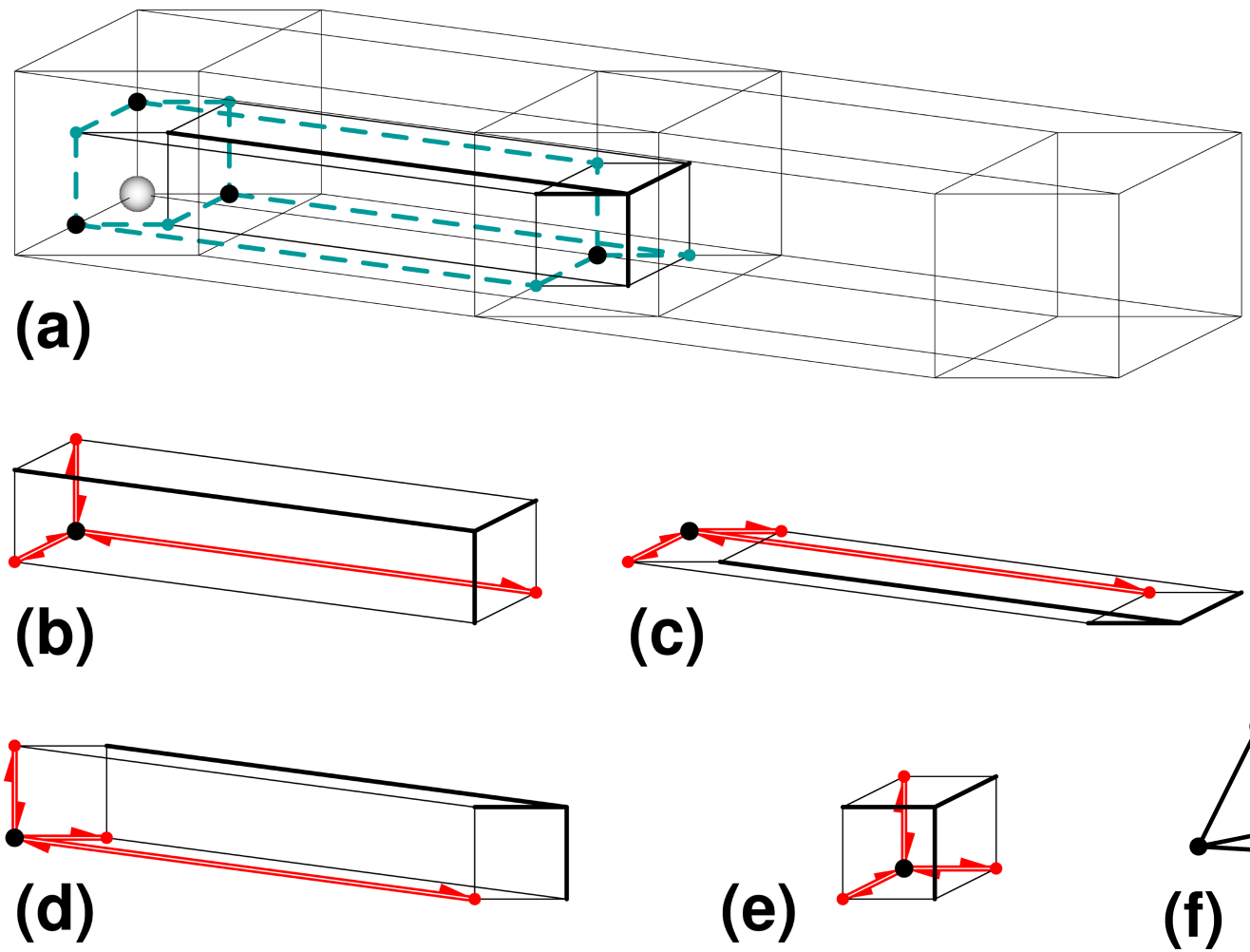
Furthermore:

2D: 2 Pixels, which touch in only one Point may be connected or disconnected.
3D: 2 Voxels, which touch in only one Edge may be connected or disconnected.
4D: 2 Toxels, which touch in only one Face may be connected or disconnected.
nD: 2 Hyper-Voxels, which touch in only one Hyper-Edge may be connected or disconnected.

Transition Elements



STEVE – Construction of a Single Tetrahedron

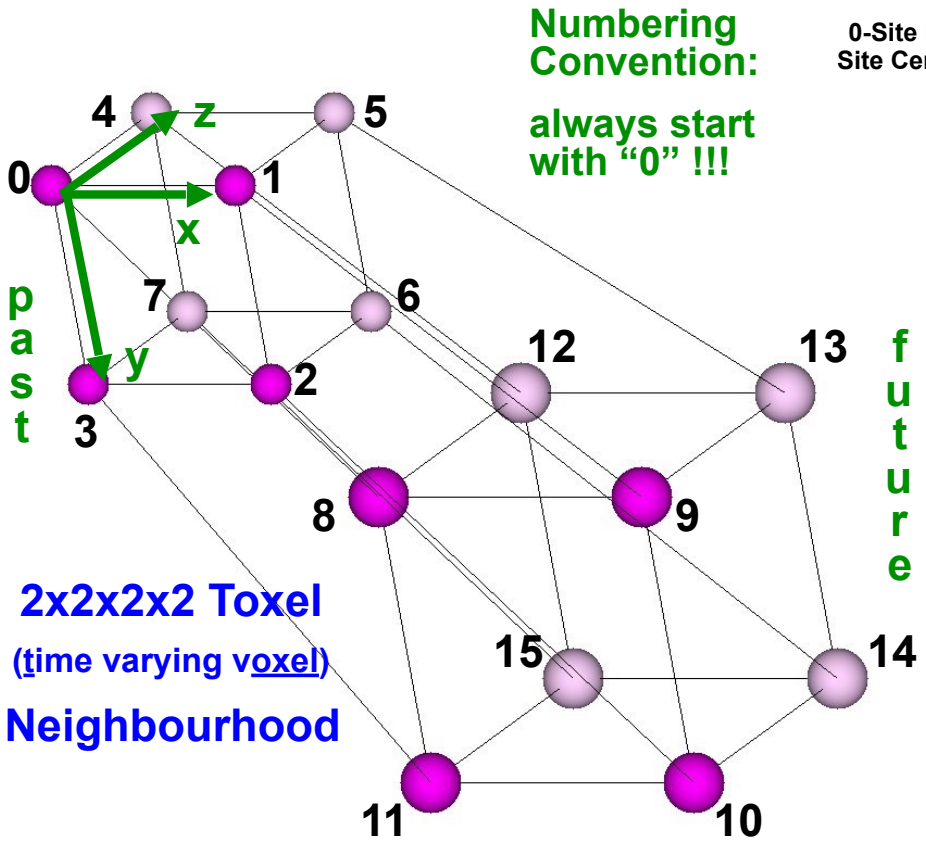


In 3D one has 24 initial Vector Pairs.

Since a Tesseract is bounded by eight 3-Cubes, in 4D one has 192 initial Vector Pairs.

STEVE's 4D Frame of Reference

NOTE: in the field of digital image processing, the y-direction is arranged from top to bottom.



Sample Input File: (W1 denotes ASCII)

```
W1
# CREATOR: FFFSTEVE (written by Bernd R. Schlei) Copyri
2 0.000000 1.000000 : tnum / tmin / tmax
2 0.000000 1.000000 : xnum / xmin / xmax
2 0.000000 1.000000 : ynum / ymin / xmax
2 0.000000 1.000000 : znum / zmin / zmax
1 0.100000 : # of columns / iso-value
0 0 0 0
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
0.000000e+00
2.550000e-01
```

0-Site Position:
Site Center ID: 0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

0-Site Positions (t/x/y/z) will be counted:
- from past to future (t)
- from left to right (x)
- from top to bottom (y)
- from front to back (z)

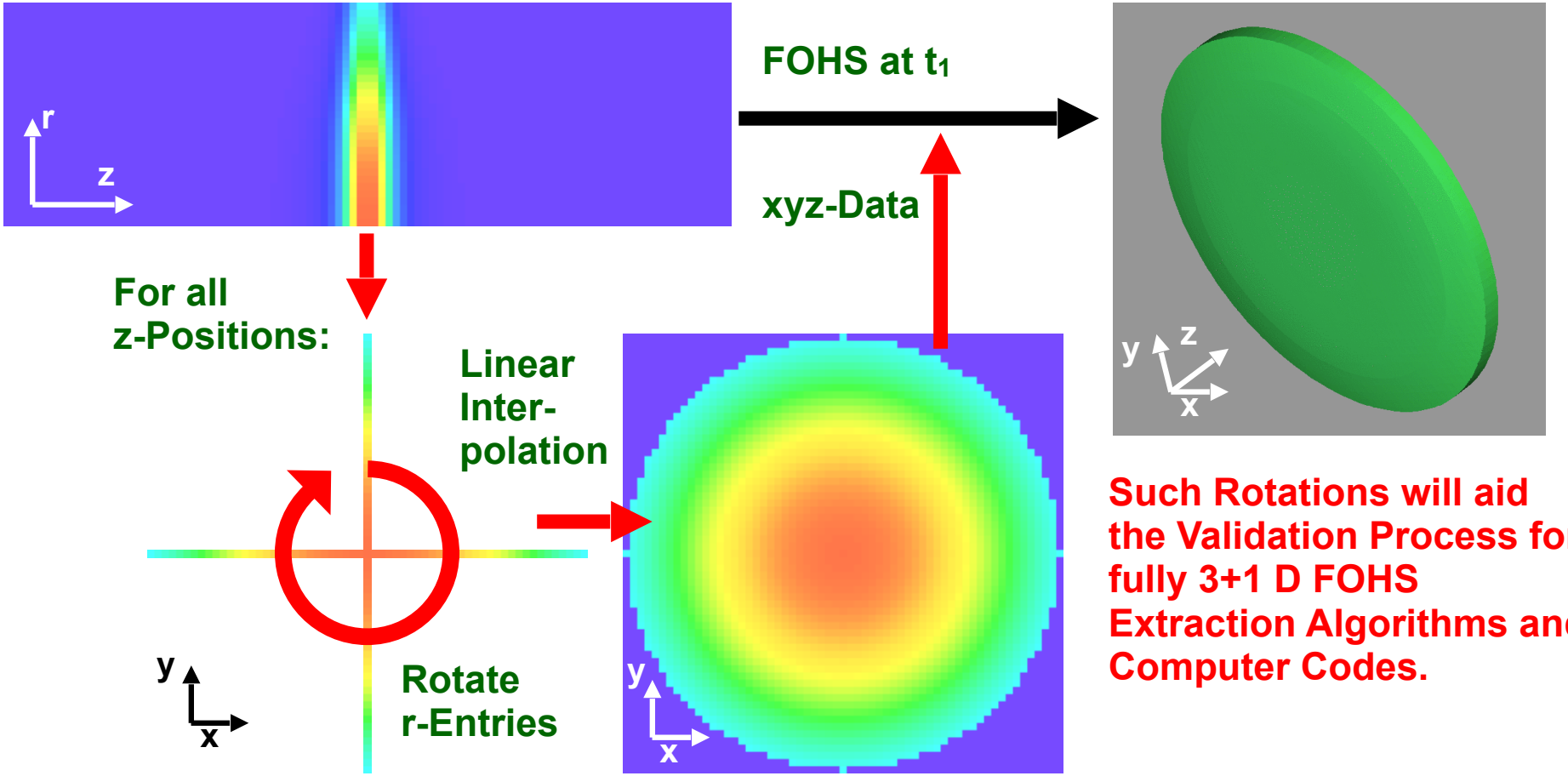
min/max values are measured always at the Site Centers !!!

More columns (# of columns>1) may be added for further field values at the given sites.

The first column will always be evaluated with respect to the above given iso-value.

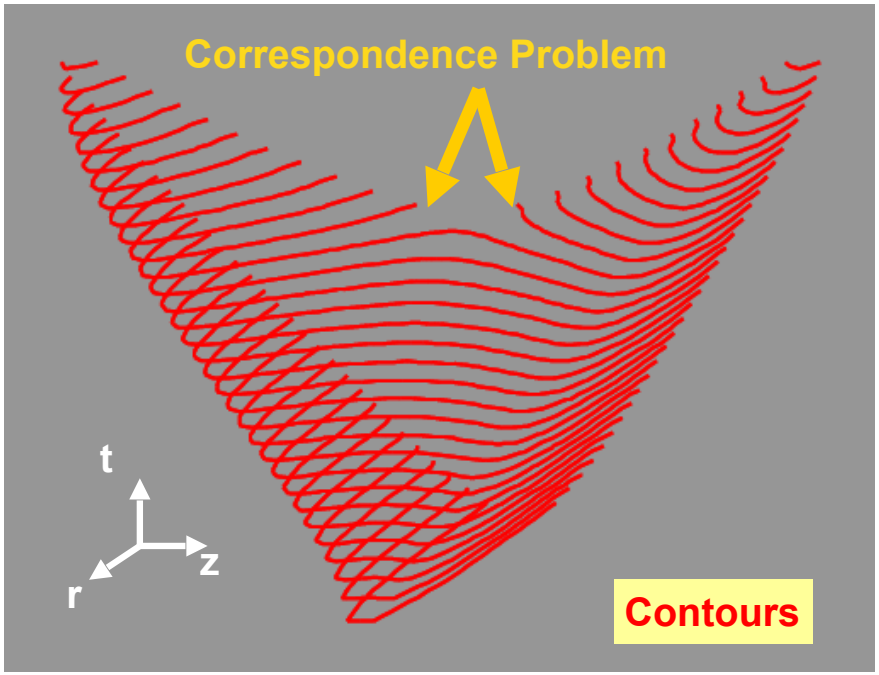
Rotation of 2+1 D Image Data and FOHS Intersections

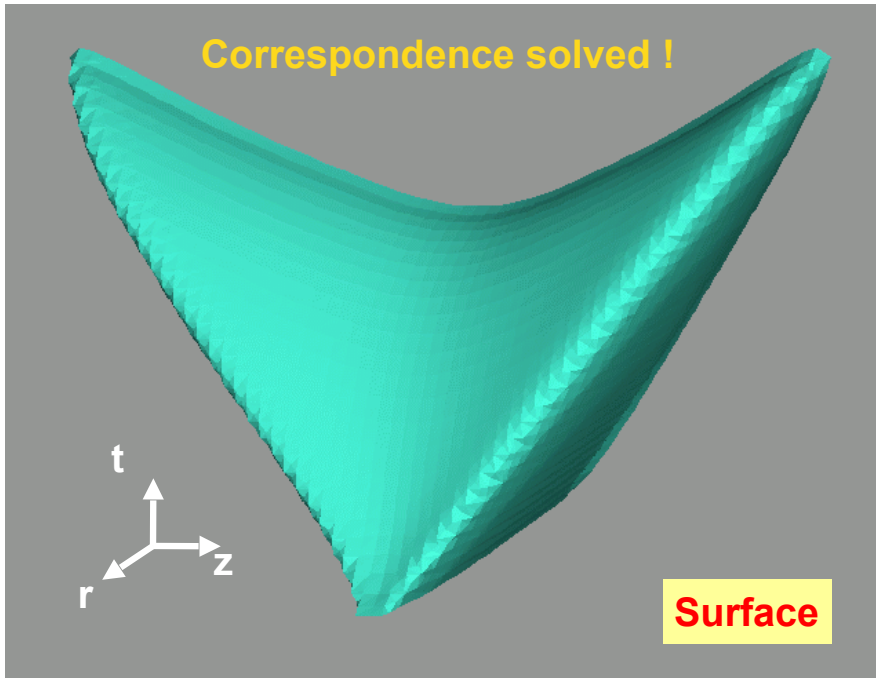
Rotation of cylinder symmetric 2+1 D Image Data at fixed Times results in 3+1 D Image Data. This enables the Extraction of 3D FOHS Intersections for fixed Times.



Such Rotations will aid the Validation Process for fully 3+1 D FOHS Extraction Algorithms and Computer Codes.

Blob-Pixel Contours vs. VESTA Surface



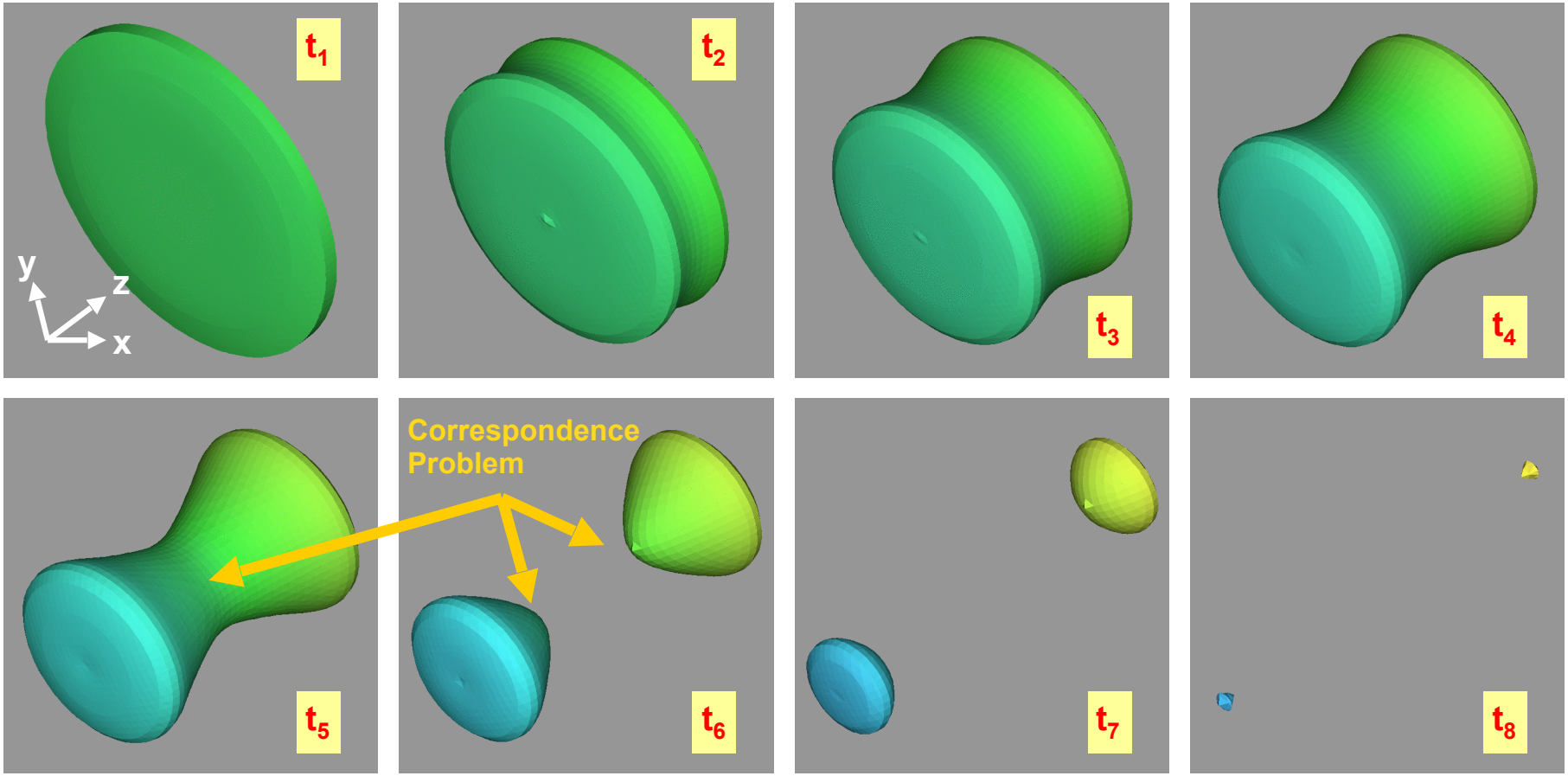


While Building Temporal Correlations between the Contours, one may encounter Correspondence Problems.

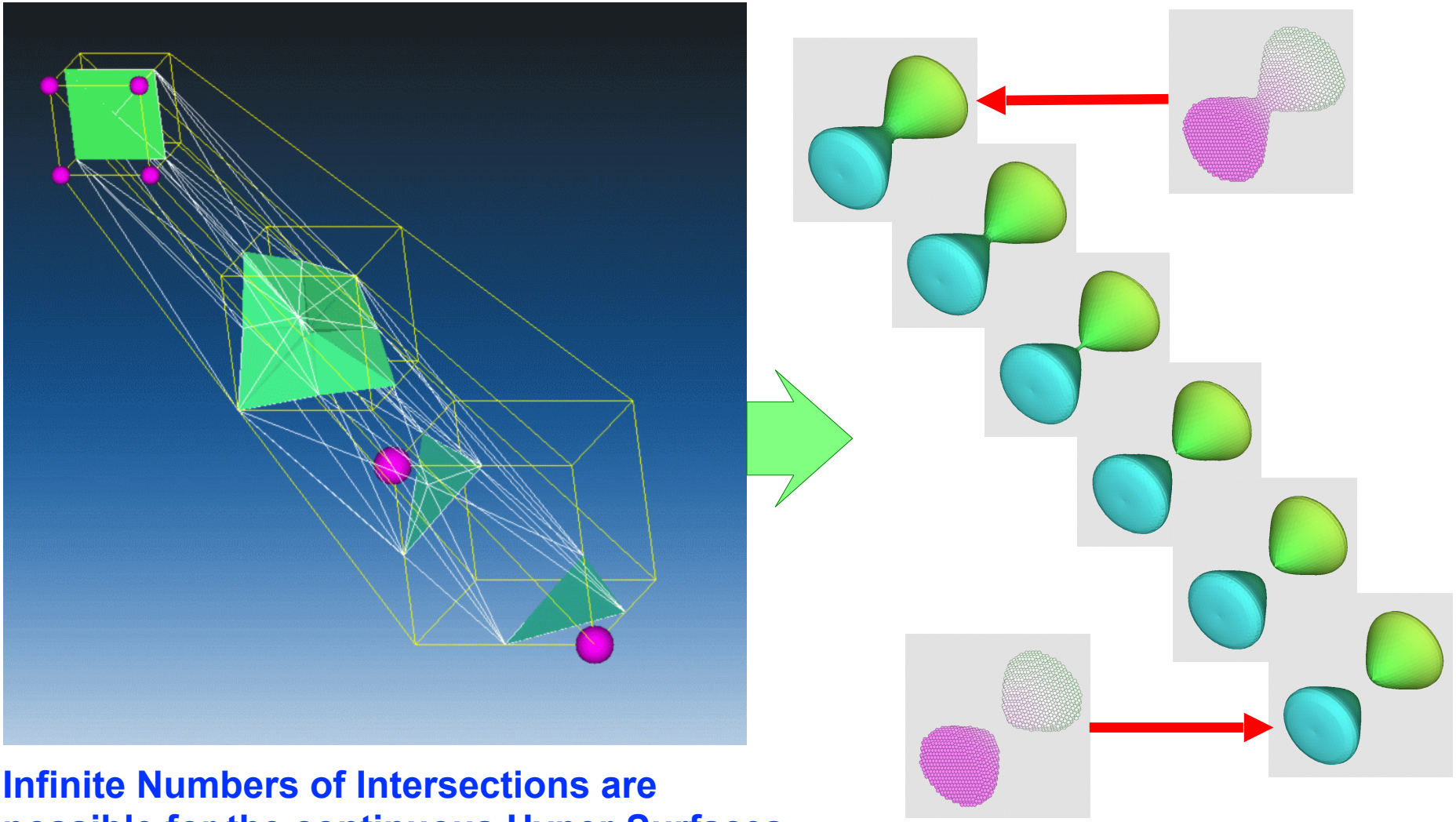
Instead, consider a Temporal Stack of Blob-Pixels (→ Voxels). Use only Voxel Faces, which separate the Shapes Interior from its exterior Regions, and construct VESTA cycles (→ Triangular Surface Tiling).

Discrete Time-Sequence of Isothermes

VESTA Rendering of FOHS from 2+1 D Hydrodynamics at fixed Times ($t_1 < \ldots < t_8$).

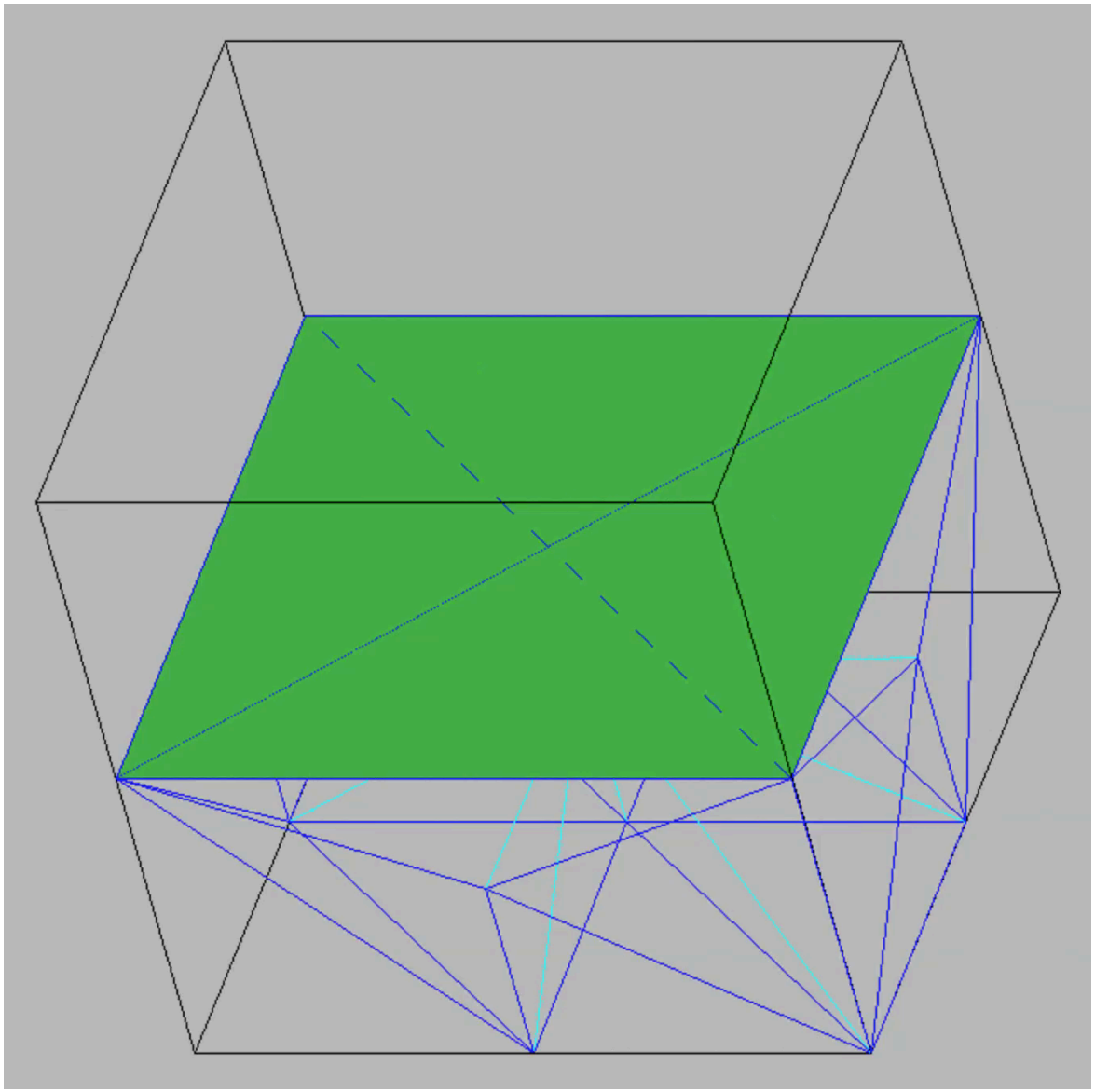


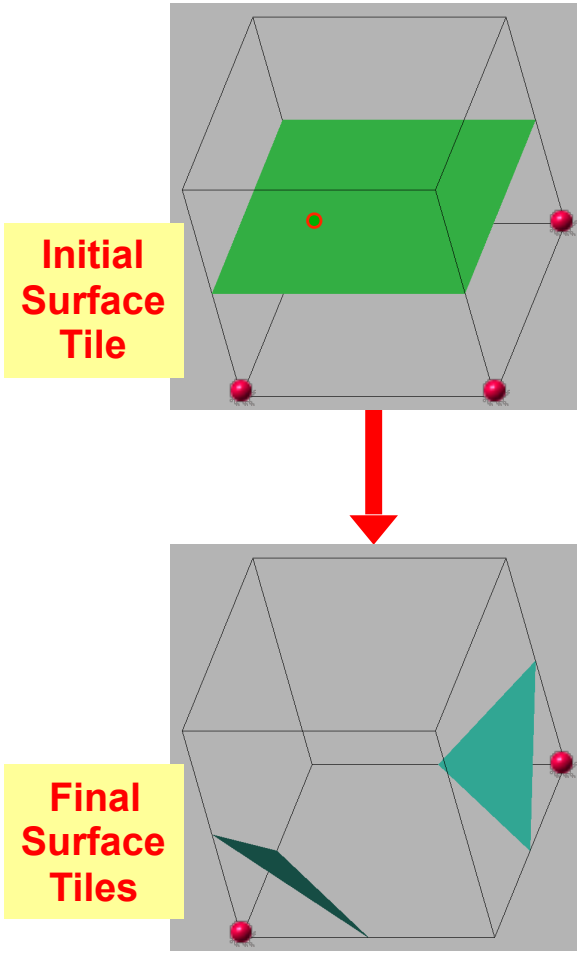
4D: STEVE — Space-Time-Enclosing Volume Extraction



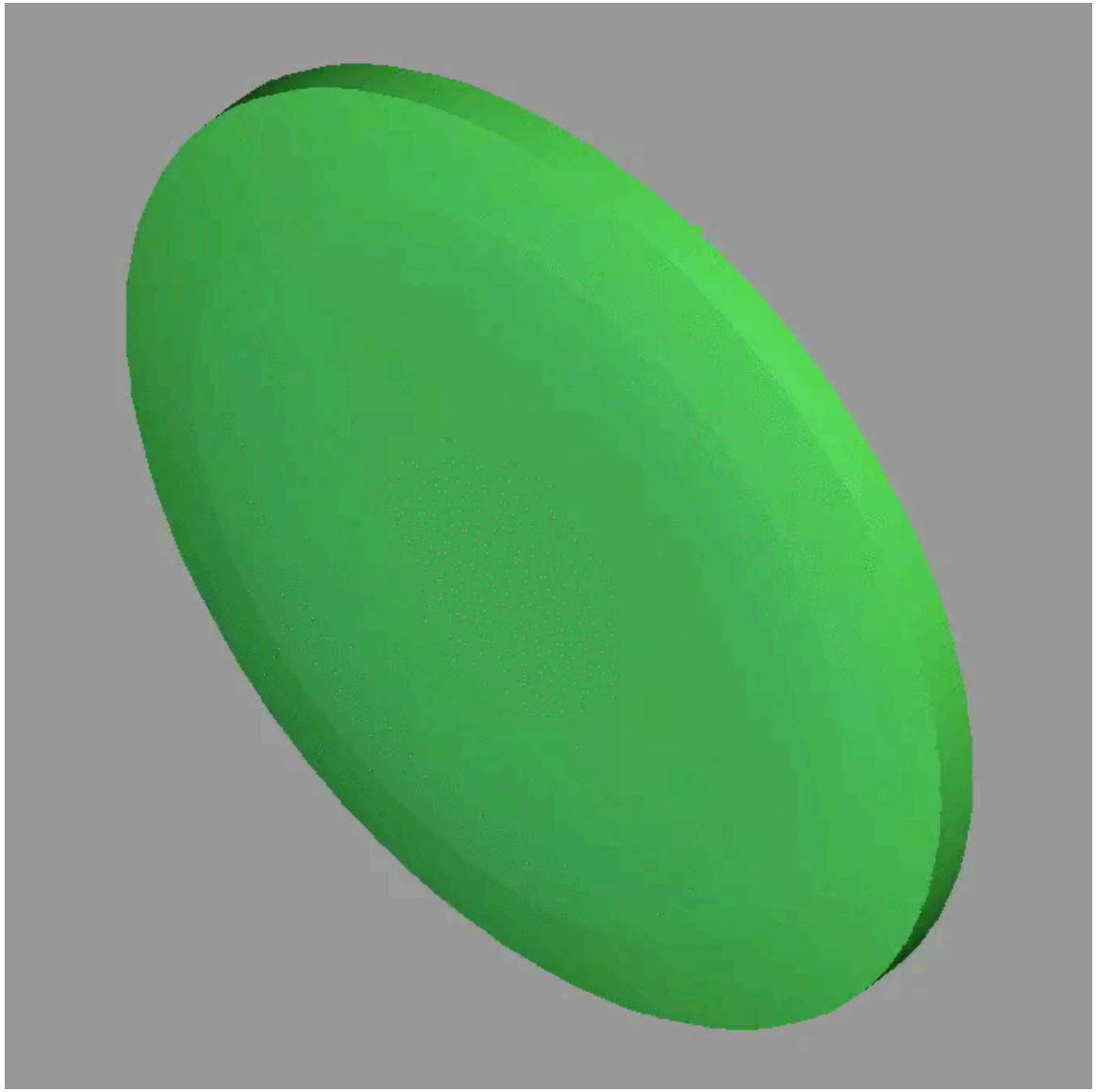
Infinite Numbers of Intersections are possible for the continuous Hyper-Surfaces.

Movie 1: Example of a Propagating Surface





Movie 2: Time-Sequence of (Continuous) FOHS Intersections



Definition of FOHS 4-Normals

L. D. Landau and E. M. Lifschitz, "The Classical Theory of Fields", Pergamon Press, 1951.

Freeze-Out Hyper-Surface Element 4-Normal: $d\sigma^\mu = \sqrt{-g}\, dS^\mu, \quad \mu = 0,1,2,3.$

$g = det(g^{ik})$, **with** g^{ik} **, i.e., the "Metric Tensor", and** $dS^i = -\frac{1}{6}\,\epsilon^{iklm}\, dS_{klm}$,

where ϵ^{iklm} **is the "Completely Antisymmetric Unit Tensor" of fourth rank.**

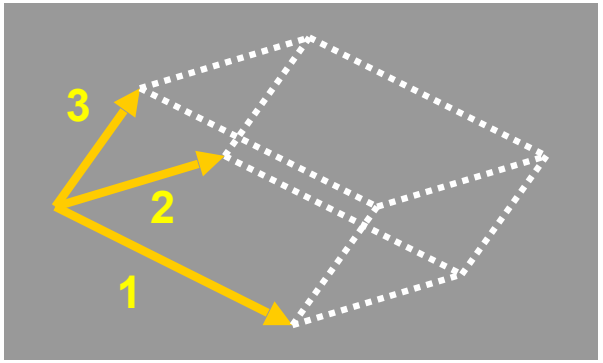
Conversely, we also have $dS_{klm} = \epsilon_{nklm}\, dS^n$.

The Projections of the Volume of a Parallelepiped are spanned by three 4-Vectors:

$dx_1^\mu,\ dx_2^\mu,\ dx_3^\mu, \quad \mu = 0,1,2,3.$

They are given by the Determinants

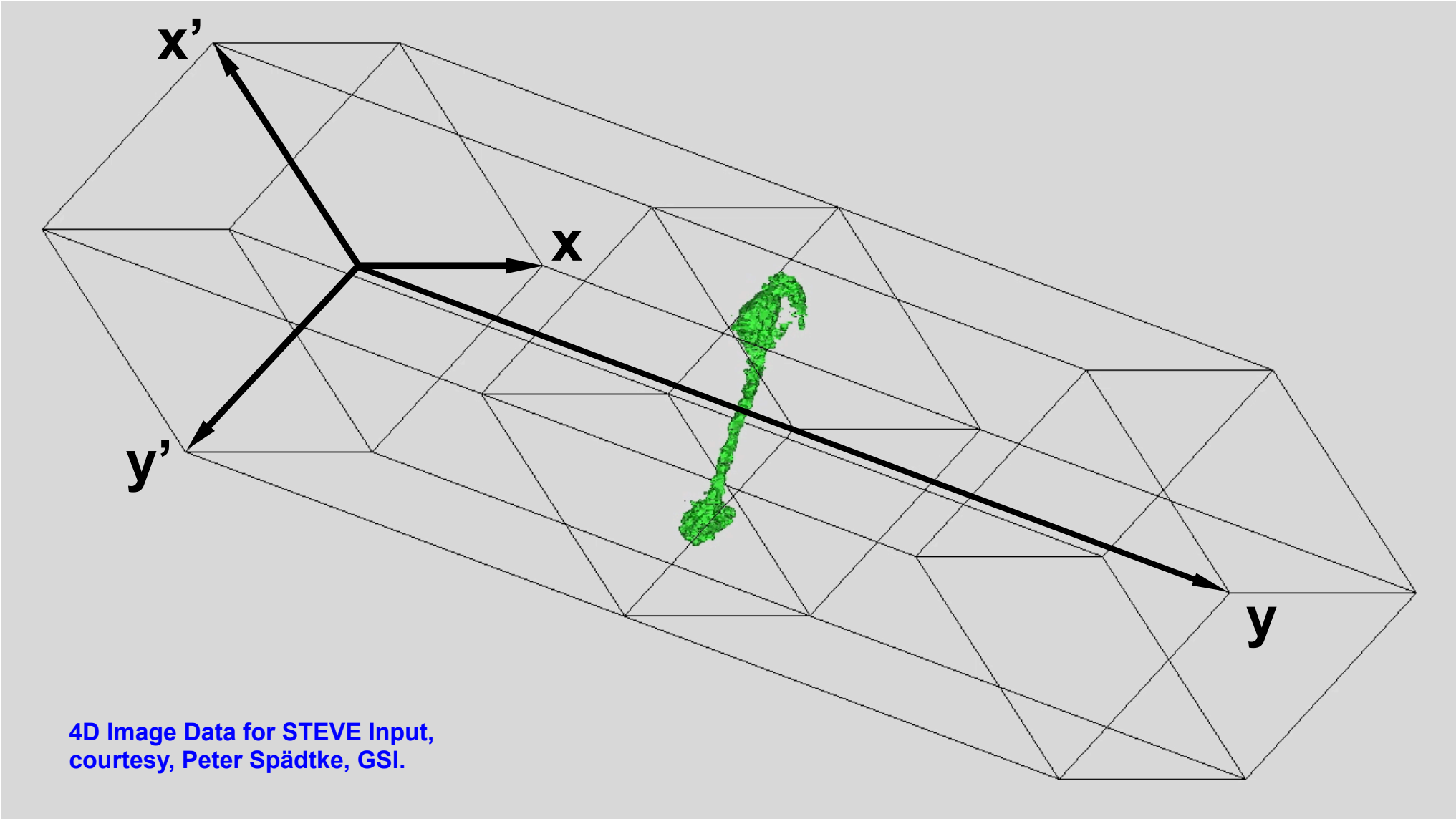
$$dS^{ikl} = \begin{vmatrix} dx_1^i & dx_2^i & dx_3^i \\ dx_1^k & dx_2^k & dx_3^k \\ dx_1^l & dx_2^l & dx_3^l \end{vmatrix}$$



$$\begin{aligned} d\sigma_0 &= \sqrt{-g}\, dS^{123} \\ d\sigma_1 &= \sqrt{-g}\, dS^{032} \\ d\sigma_2 &= \sqrt{-g}\, dS^{013} \\ d\sigma_3 &= \sqrt{-g}\, dS^{021} \end{aligned}$$

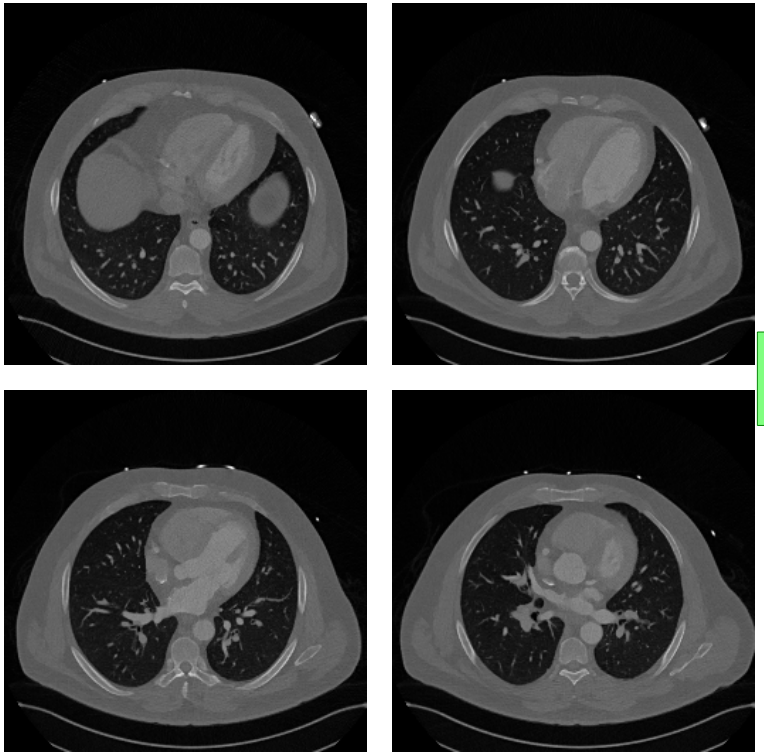
$d\sigma_\mu$ **points to the Exterior of an enclosed Space-Time Region.**

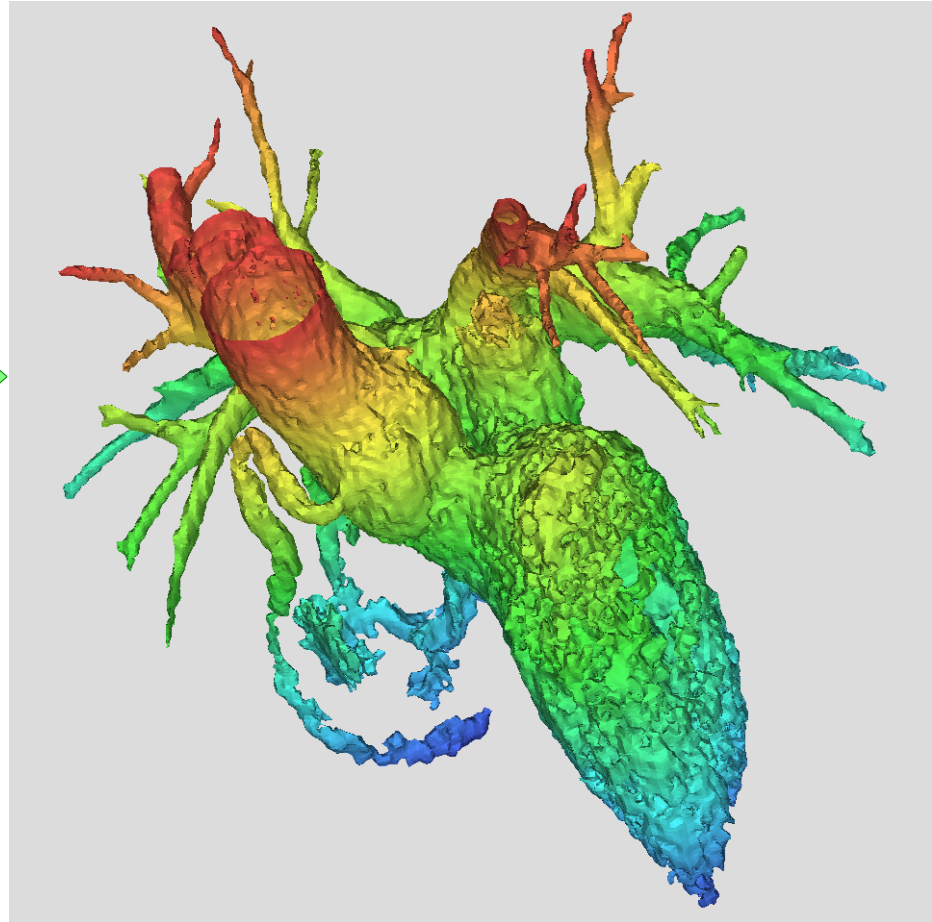
Movie 3: 3D Intersections of Envelope in 4D Transversal Trace-Space



Medical Image Data
4D-CT

3D: Surface Extraction for Data Sets from Computed Tomography (CT)





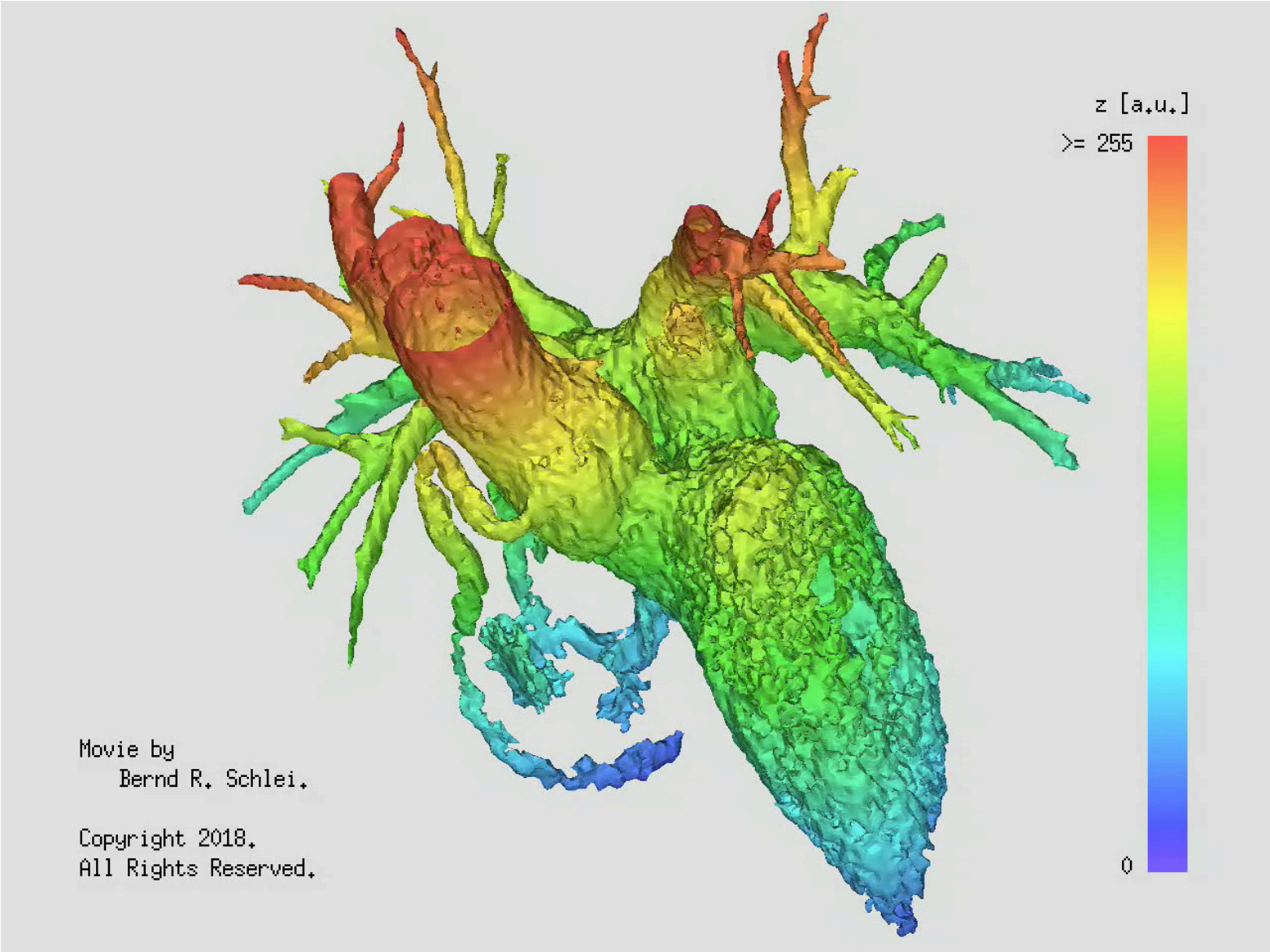
3+1D Image Data from Computed Tomography, courtesy, Christian Graeff et al., GSI.

A Series of 9 CT Images (256 x 256 x 150 voxels), 2-Byte Hounsfield Scale.

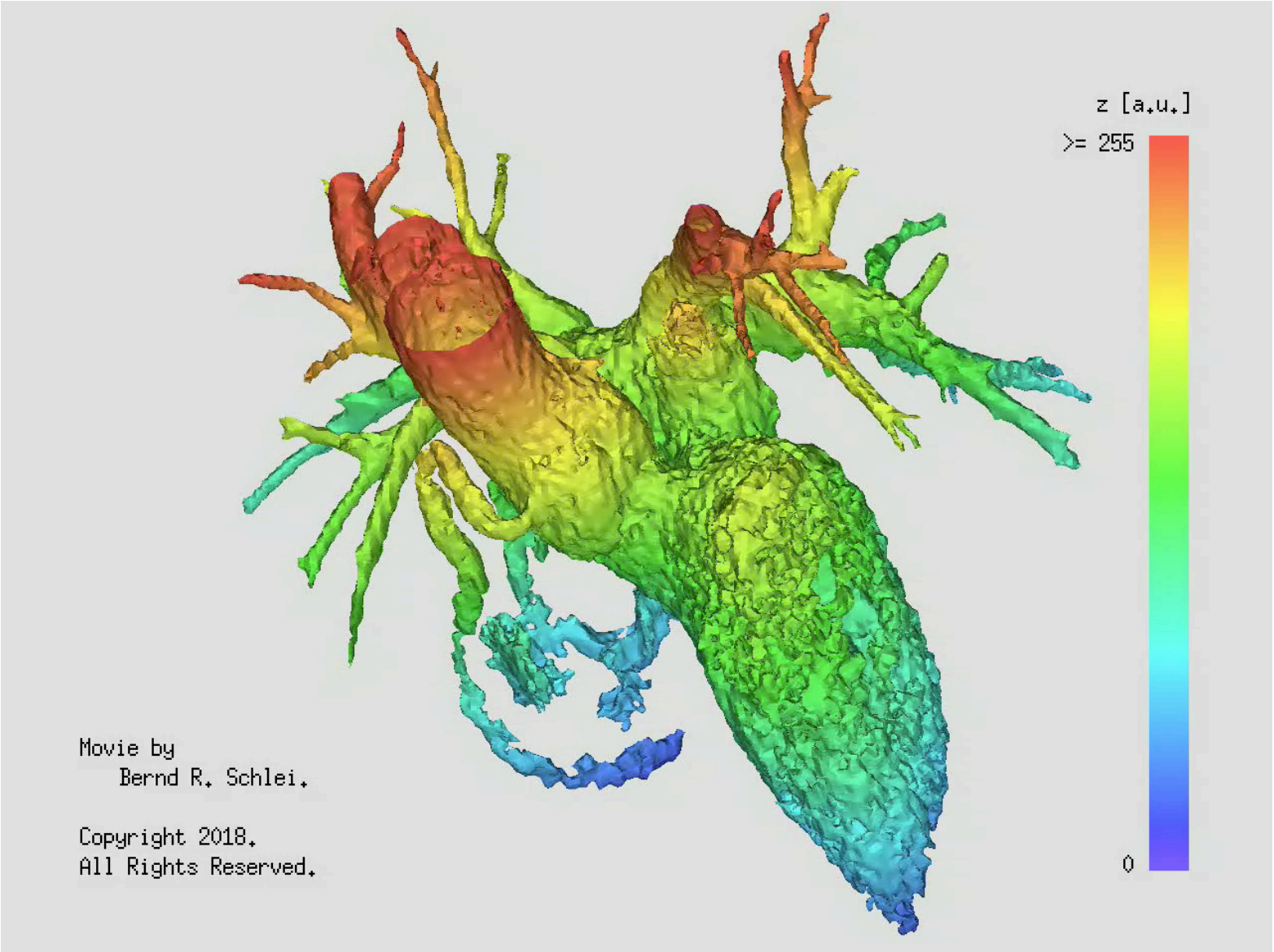
My Threshold: 250.0 HU = GL 123.

B. R. Schlei, "Flood fill segmentation for images from time-varying computed tomography," submitted to GSI Scientific Report 2017.

Movie 4: Time Sequence of 3D VESTA-Renderings from (Discrete) 4D-CT

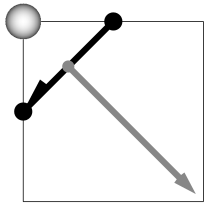


Movie 5: 3D Intersections of Continuous Hyper-Surface from (Discrete) 4D-CT

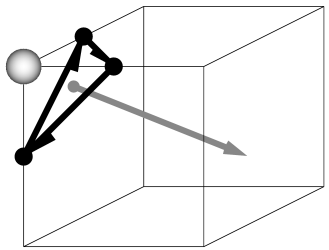


Normal N-Vectors of Hyper-Surface Elements

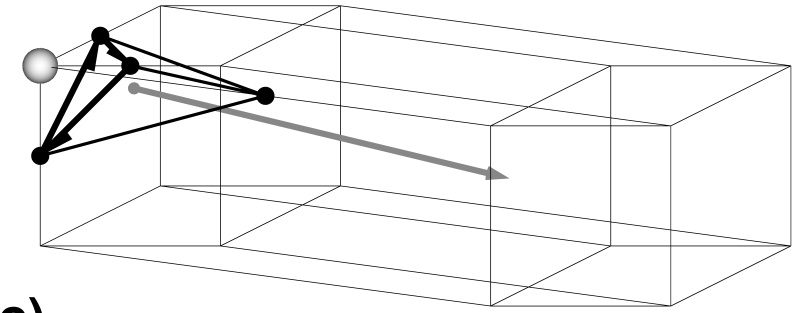
Normalized normal vectors (gray) that are attached to the centers of the oriented (N − 1)-simplices, which enclose the active N-dimensional picture element sites with ID 0 within the shown ND-cells: (a) in 2D (b) in 3D, and (c) in 4D, respectively.



(a)



(b)



(c)

E.g., covariant 4-Normal

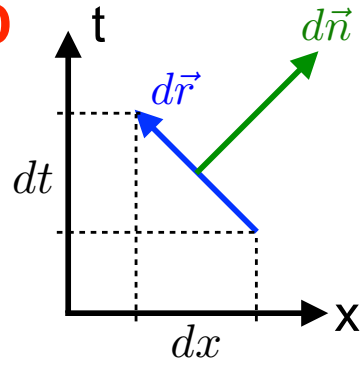
$$d\sigma_\mu(x^\mu)=(d\sigma_x, d\sigma_y, d\sigma_z, d\sigma_t)$$

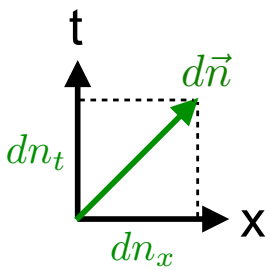
at contravariant Event

$$x^\mu=(x, y, z, t)$$

Towards Magnitudes of Velocities for a Hyper-Surface

2D



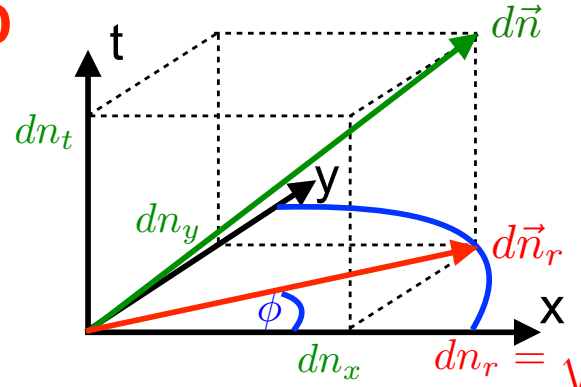


$$d\vec{r} = (dx, dt), \quad dx < 0,\ dt > 0$$

$$d\vec{n} = (dn_x, dn_t) = (dt, -dx)$$

$$v = \frac{dx}{dt} = \frac{-dn_t}{dn_x}$$

3D



$$dn_r = \sqrt{dn_x^2 + dn_y^2}$$

$$\frac{dr}{dt} = \frac{-dn_t}{dn_r}, \quad \frac{d\vec{r}}{dr} = (\cos\phi, \sin\phi) = \frac{1}{dn_r}(dn_x, dn_y)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\frac{d\vec{r}}{dr} = \frac{-dn_t}{dn_x^2 + dn_y^2}(dn_x, dn_y)$$

The Hyper-Surface in Spacetime is equivalent to a 3D Velocity Field.

4D

3-Velocity

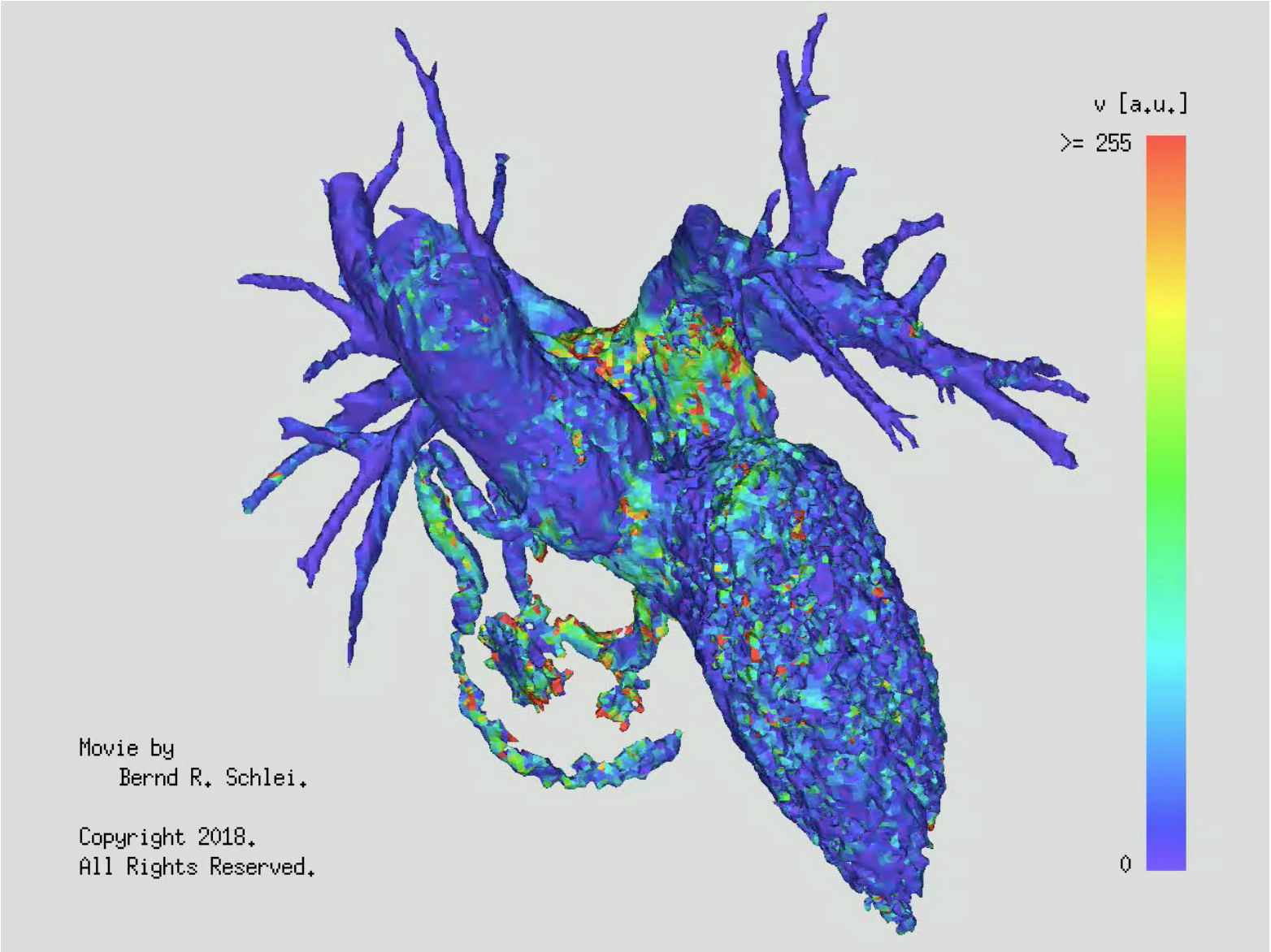
$$\boxed{\vec{v} = (v_x, v_y, v_z) \equiv \frac{-d\sigma_t}{d\sigma_x^2 + d\sigma_y^2 + d\sigma_z^2}(d\sigma_x, d\sigma_y, d\sigma_z)}$$

Magnitude

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

B. R. Schlei, "Visualizing velocity field strengths with hyper-surfaces in spacetime," submitted to GSI Scientific Report 2017.

Movie 6: Magnitudes of Velocities for a Hyper-Surface from 4D-CT



Summary and Outlook

Summary

- **Nowadays, more explicit considerations of Higher Dimensions may be useful.**
- **Higher dimensional Objects can be observed via Projections, Intersections, etc.**
- **We are capable of constructing Manifolds of Co-Dimension 1 very efficiently!**

	No. of Paths		No. of Templates	
3D	**(VESTA)**	24	**(MCA)**	256
4D	**(STEVE)**	192		65,536
5D		1920		4,294,967,296
6D		23,040		$\approx 1.845 \times 10^{19}$
7D		322,560		$\approx 3.403 \times 10^{38}$
8D		5,160,960		$\approx 1.158 \times 10^{77}$
9D		92,897,280		$\approx 1.341 \times 10^{154}$
10D		1,857,945,600		$\approx 1.798 \times 10^{308}$
11D		40,874,803,200		$\approx 3.232 \times 10^{616}$
***N*D**	$\prod_{k=2}^{N} 2k$		2^{2^N}	

Note: The estimated number of atoms in our universe is $\sim 10^{80}$.

- **Our Technology does NOT use Templates.**
- **Topological Ambiguities can be resolved in a purely DATA-DRIVEN Manner.**

- **We can handle Dimensions > 5 comparatively easy!**

- **This all has led to:**

B. R. Schlei, "Method for Hyper-Surface Construction in N Dimensions",

U.S. Patent 9,607,431 B2, Mar. 28, 2017,

E.P. Patent 2 715 673 B1, Nov. 22, 2017.

The United States of America

The Director of the United States Patent and Trademark Office

Has received an application for a patent for a new and useful invention. The title and description of the invention are enclosed. The requirements of law have been complied with, and it has been determined that a patent on the invention shall be granted under the law.

Therefore, this

United States Patent

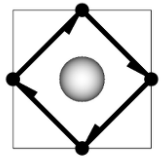
Grants to the person(s) having title to this patent the right to exclude others from making, using, offering for sale, or selling the invention throughout the United States of America or importing the invention into the United States of America, and if the invention is a process, of the right to exclude others from using, offering for sale or selling throughout the United States of America, or importing into the United States of America, products made by that process, for the term set forth in 35 U.S.C. 154(a)(2) or (c)(1), subject to the payment of maintenance fees as provided by 35 U.S.C. 41(b). See the Maintenance Fee Notice on the inside of the cover.

Michelle K. Lee

Director of the United States Patent and Trademark Office

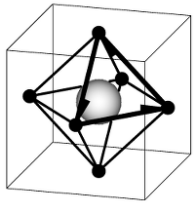
Outlook: roseN — RObust (hyper-)Surface Extraction in n dimensions

2D



4 Edges

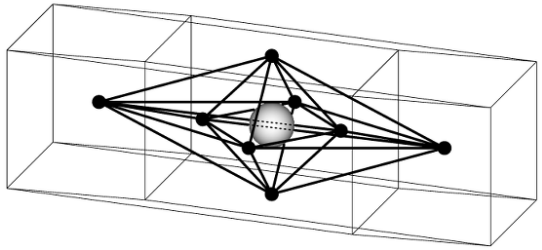
3D



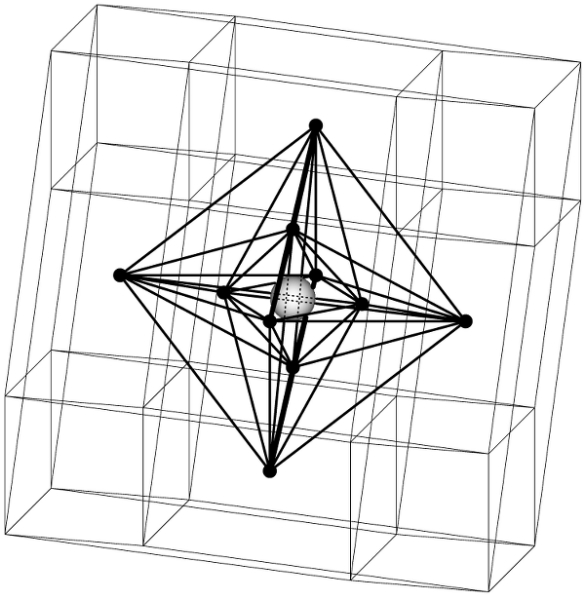
8 Triangles

4D

16 Tetrahedrons

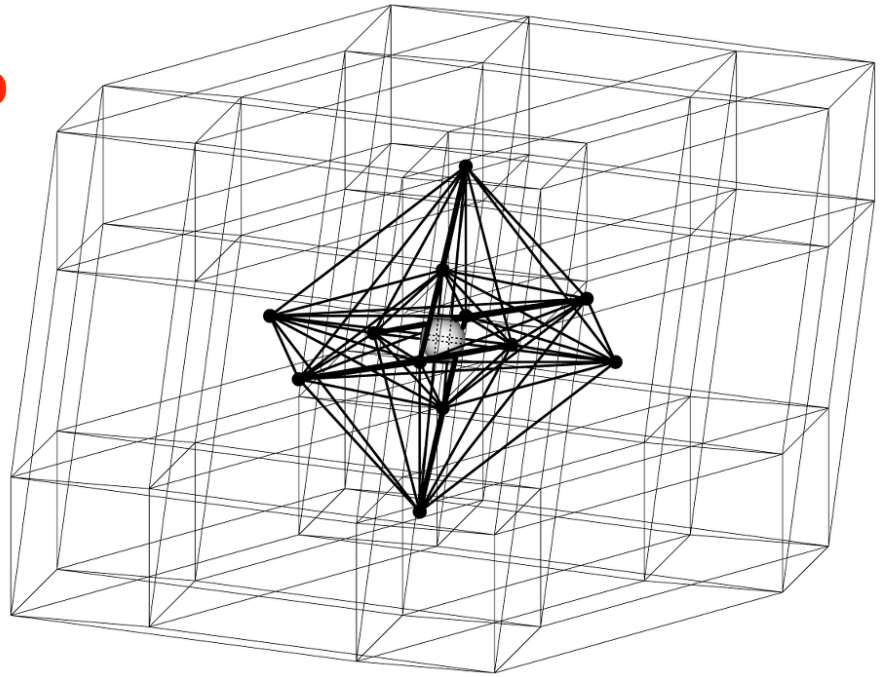


5D



32 Pentachorons (4-Simplices)

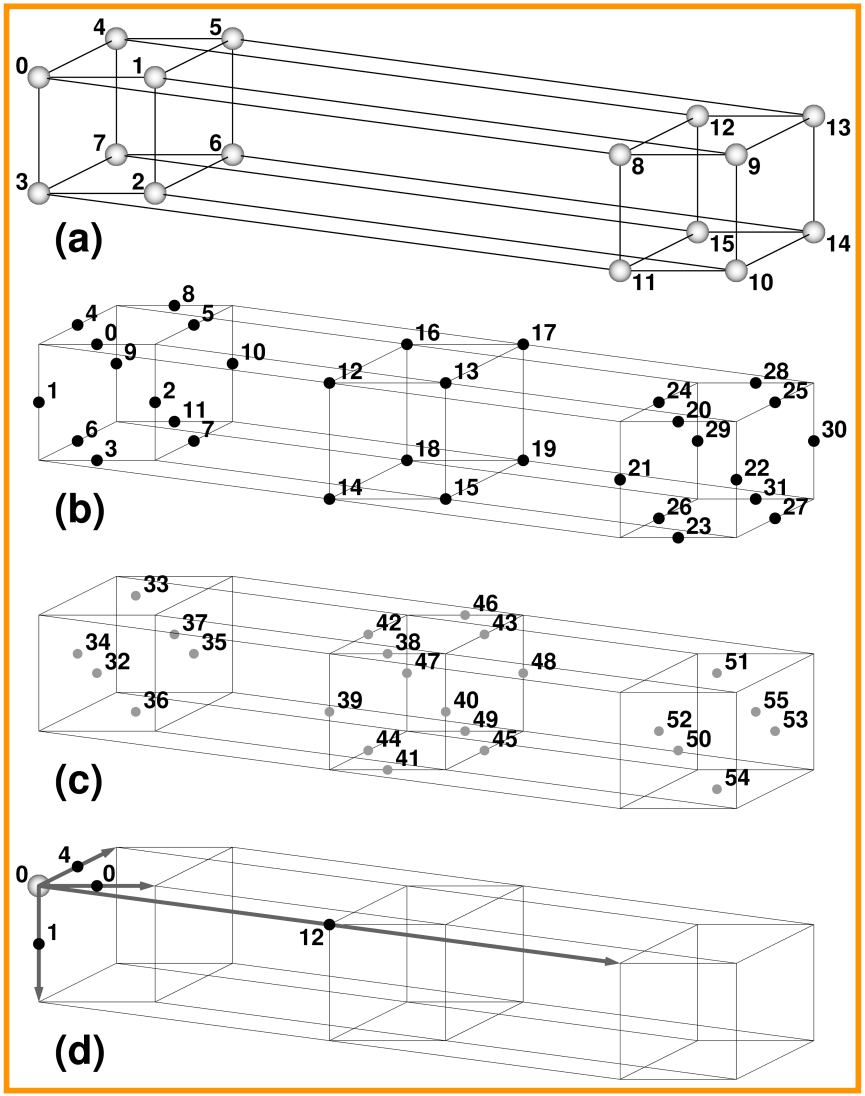
6D

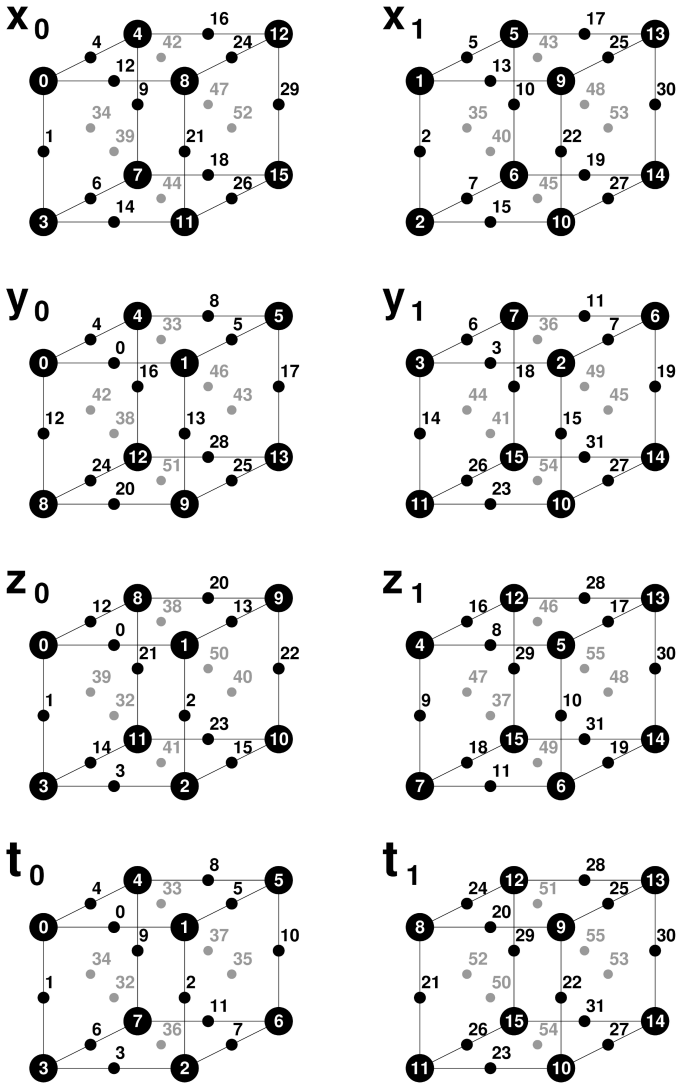


64 5-Simplices

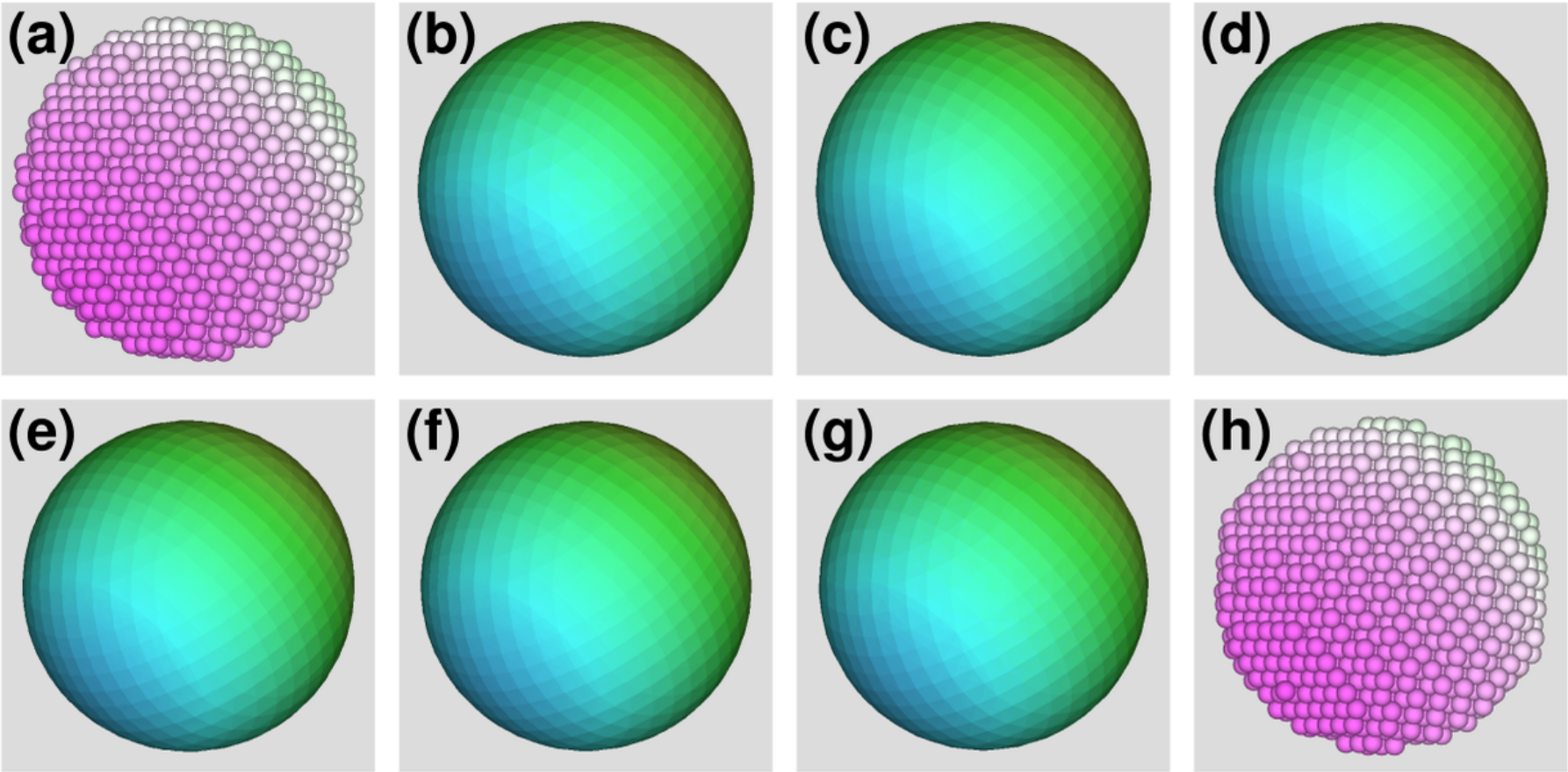
Backup Slides

Towards 4D: Tesseract and 3D Cubical Subspaces

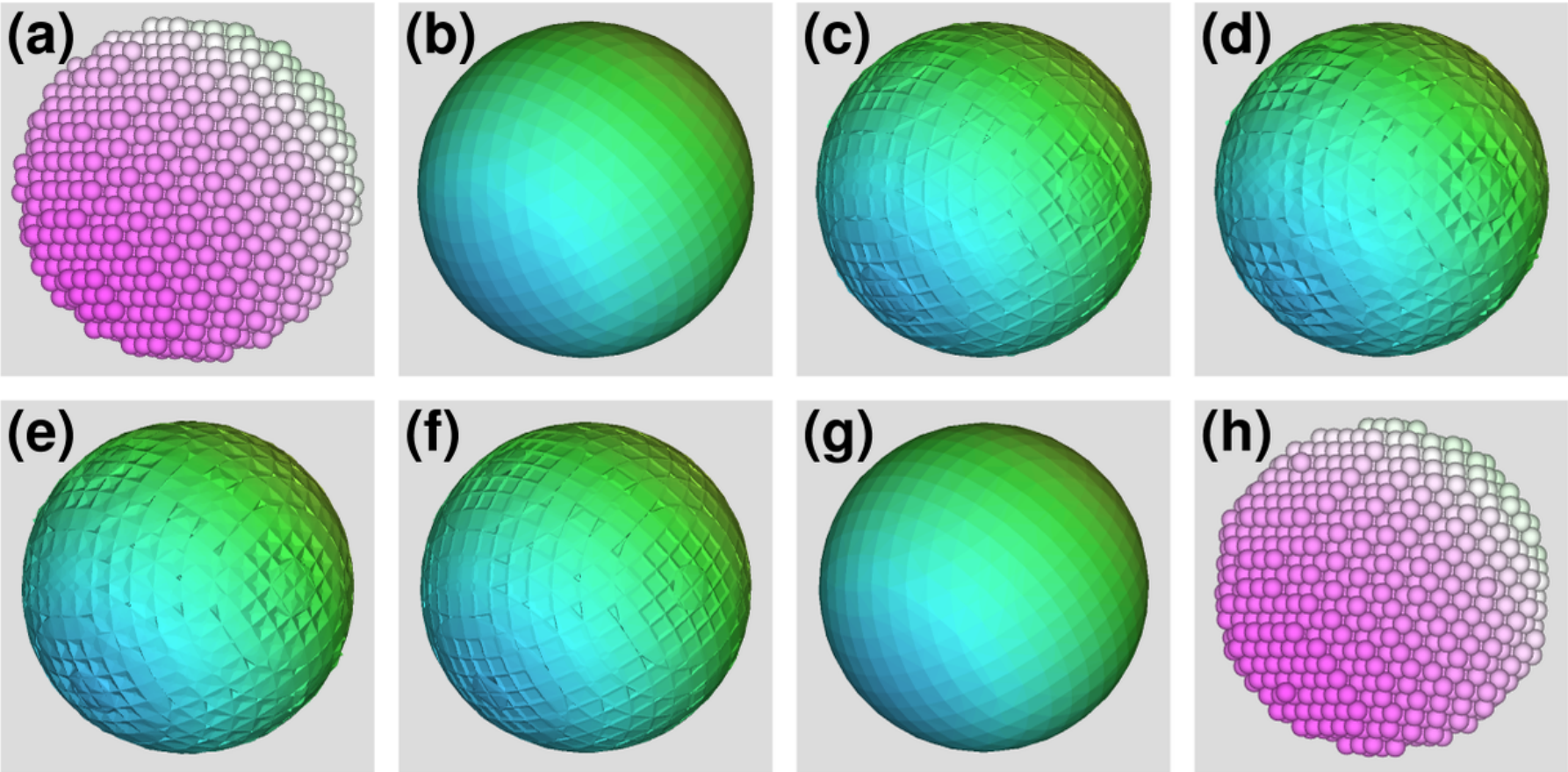




Intersections of a 4D Hypersphere with STEVE



Intersections of a 4D Hypersphere with MCA



NO-GO: Appearance of Numerical Artefacts!