What happens if Planck's constant changes?

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The quantum world is very sensitive to the value of Planck's constant h. For instance, as discussed by Yang [1], doubling the value of h would result in a radical change on the geometric sizes and apparent colors, the solar spectrum and luminosity, as well as the energy conversion between light and materials.

If, as it is believed, h had been steadily decreasing since the emergence of the early universe it would imply that there is an ongoing transition from quantum to classical, mixed states becoming pure, and pure states evolving into classical states. We would thus have an emergence of the classical from the quantum. Reversing this scenario in time could also imply that the early Universe was much more "quantum" than it is today. These hypotheses should of course be confirmed by cosmological experimental data.

Pao-Keng Yang: How does Planck's constant influence the macroscopic worlds? *Eur. J. Phys*. **37**, 055406 (2016).

Suppose Planck's constant h has undergone a relative small change of δ, thus becoming

$$h^{\delta} = (1 + \delta)h.$$

The Bohr radius of a hydrogen atom is given by $a_0 = 4\pi\varepsilon_0 \hbar^2 / m_e c^2$. Assuming that m_e and c are conserved this radius

$$a_0^{\delta} = (1 + \delta)^2 a_0 \approx (1 + 2\delta)a_0$$

hence the relative change of Bohr's radius is approximately 2δ. Extrapolating to other atoms, it is reasonable to assume that their size would also increase or decrease by a factor $1 + 2\delta$. Since all atoms would change approximately equally, we wouldn't be able to detect this change of size. However, having supposed that the mass of the Earth remains the same,its radius R_E would change by a factor $(1 + \delta)^4 \approx (1 + 4\delta)$ and become

$$R_E^{\delta} \approx R_E(1 + 4\delta).$$

Suppose for instance that $\delta = 2.5\%$; this leads to an increase of radius of the Earth into by 10%. Due to the conservation of angular momentum the spinning of the Earth would slow down, so that the length of a day would increase to approximately 27 hours; since the masses of the Earth and the Sun are not affected and there would be less than 365 days in a year! The acceleration $g \approx 9.81\ ms^{-2}$ of gravity would become

$$g^{\delta} = \frac{GM_E}{(R_E^{\delta})^2} \approx g(1 - 8\delta);$$

for $\delta = 2.5\%$ it would thus decrease by 20%; the density of the atmosphere near the surface of the Earth would thus become substantially smaller because of the weaker gravity.
Practical consequences!

Variable Planck's Constant?

Planck's constant is a central number for modern physics. To test whether Planck's constant is really constant, Mohageg and Kentosh turned to the same GPS systems that car drivers use to find their way home. GPS relies on the most accurate timing devices we currently possess: atomic clocks. These count the passage of time according to frequency of the radiation that atoms emit when their electrons jump between different energy levels. Kentosh and Mohageg looked through a year's worth of GPS data of seven highly stable GPS satellites and found that the corrections depended in an unexpected way on a satellite's distance above the Earth. After careful analysis of the data they obtained, Kentosh and Mohageg concluded that h is identical at different locations to an accuracy of seven parts in a thousand. Their results, which have been largely commented (and criticized) in the media, are however controversial.

J. Kentosh and M. Mohageg: Global positioning system test of the local position invariance of Planck's constant, *Phys. Rev. Lett.* **108**(11), 110801 (2012)

The fine structure constant

The variability of physical "constants" is a possibility that cannot be outruled and which has being an active area of research for some time in cosmology and astrophysics. Paul Dirac already suggested this possibility in 1937 ("Large Numbers Hypothesis") in a letter to *Nature* questioning Eddington's attempts to calculate the constants from scratch. That some constants of Nature could vary in space and time is a subject of fascination which has motivated numerous theoretical and experimental researches. Planck's constant also appears in the fine structure constant. Recent cosmological advances by John Webb, Barrow, Berengut, Flambaum *et al.* using Keck and VLT telescopes in Hawaii, have put an upper bound on the relative change of the fine structure constant $\alpha = e^2/4\pi\varepsilon_0\hbar c \approx 1/137$ at roughly 10^{-17} per year. Space-time variations of α in cosmology is a new phenomenon beyond the standard model of physics which, if proved true, must mean that at least one of the three fundamental constants $e, \hbar, c$ that constitute it must vary. This is a delicate issue, related to *choices of units*.

Dependence of the Wigner distribution on h

Let $\psi \in L^2(\mathbb{R}^n)$ and $z = (x, p)$. The textbook definition of the Wigner transform of ψ goes as follows.

$$W_\hbar\psi(x,p) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}py}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)d^ny.$$

This can be written as a *probability amplitude*:

$$W_\hbar\psi(z) = \left(\tfrac{1}{\pi\hbar}\right)^n \langle\psi|\widehat{R}(z)\psi\rangle \tag{1}$$

where $\widehat{R}(z_0)$ is Grossmann–Royer's reflection operator:

$$\widehat{R}(z_0)\psi(x) = e^{\frac{2i}{\hbar}p_0\cdot(x-x_0)}\psi(2x_0-x)$$

(NB: we have $\widehat{R}(z_0) = \widehat{T}(z_0)\widehat{R}(0)\widehat{T}(z_0)^{-1}$ where $\widehat{T}(z_0)$ is the usual Heisenberg displacement operator).

The Wigner transform of a state $|\psi\rangle$ plays the role of a "quasiprobability distribution"; under suitable integrability conditions on ψ it has the "right marginals"

$$\begin{aligned} \int W_\hbar\psi(x,p)d^n p &= |\psi(x)|^2 \\ \int W_\hbar\psi(x,p)d^n x &= |F_\hbar\psi(p)|^2 \end{aligned}$$

but it usually takes negative values, unless ψ is a Gaussian function (in particular, a coherent state). It also allows to calculate averages of observables:

$$\langle\widehat{A}\rangle_\psi = \int W_\hbar\psi(x,p)a(x,p)d^n x d^n p.$$

The Wigner transform is at the basis of the phase space picture of quantum mechanics (Wigner–Weyl–Moyal formalism).

The Wigner transform also plays a crucial role in the study of mixed quantum states (*i.e.* quantum states we only have an incomplete knowledge). By definition the density matrix of a mixed state is a convex sum

$$\widehat{\rho}_{\hbar} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$$

of projectors $|\psi_j\rangle\langle\psi_j|$ weighted by the probabilities λ_j. The datum of $\widehat{\rho}_{\hbar}$ is equivalent to that of its Wigner distribution

$$\rho_{\hbar}(x,p) = \sum_j \lambda_j W_{\hbar}\psi_j(x,p)$$

and ρ has again a statistical interpretation:

$$\langle \widehat{A} \rangle_{\rho} = \int \rho_{\hbar}\psi(x,p)a(x,p)d^n x d^n p.$$

First question

We ask: can $W_\hbar\psi$ be the Wigner transform of another function ψ' if we change the value of $\hbar$ to $\hbar'$? The answer is **NO** !!! Suppose indeed that $W_\hbar\psi = W_{\hbar'}\psi'$. Taking the marginals with respect to p we have

$$|\psi(x)|^2 = \int W_\hbar\psi(x,p)dp = \int W_{\hbar'}\psi'(x,p)dp = |\psi'(x)|^2$$

hence $\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle$. But we also have (Moyal identity)

$$\begin{aligned}\iint W_\hbar\psi(x,p)^2 d^n x d^n p &= \left(\tfrac{1}{2\pi\hbar}\right)^n \langle\psi|\psi\rangle^2 \\ \iint W_{\hbar'}\psi'(x,p)^2 d^n x d^n p &= \left(\tfrac{1}{2\pi\hbar'}\right)^n \langle\psi'|\psi'\rangle^2\end{aligned}$$

hence $W_\hbar\psi = W_{\hbar'}\psi'$ and $\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle$ imply that we must have $\hbar = \hbar'$:

No pure state remains pure if $\hbar$ changes!

Second question

Assume that we have determined a phase space distribution $\rho(x,p)$ (by quantum tomography, homodyne detection, or any other method). Once ρ is known we can reconstruct the quantum state using some suitable quantization procedure. Suppose that $\widehat{\rho}_{\hbar}$ and $\widehat{\rho}_{\hbar'}$ are density matrices corresponding to the same ρ but to different values h and h' of Planck's constant. Since

$$\begin{aligned}
\mathrm{Tr}(\widehat{\rho}_{\hbar}^2) &= (2\pi\hbar)^n \int \rho(x,p)^2 d^n x d^n p \\
\mathrm{Tr}(\widehat{\rho}_{\hbar'}^2) &= (2\pi\hbar')^n \int \rho(x,p)^2 d^n x d^n p
\end{aligned}$$

we must have

$$\mathrm{Tr}(\widehat{\rho}_{\hbar'}^2) = \left(\frac{\hbar'}{\hbar}\right)^n \mathrm{Tr}(\widehat{\rho}_{\hbar}^2).$$

The purity of a quantum state thus critically depends on the value of h; in particular if the new value $\hbar'$ is too large then $\widehat{\rho}_{\hbar'}$ will no longer represent a quantum state since it would lead to $\mathrm{Tr}(\widehat{\rho}_{\hbar'}^2) > 1$.

The mathematical problem

An arbitrary phase space distribution $\rho(x, p)$ is the Wigner distribution of a quantum state if and only if the associated quantization $\widehat{\rho}_{\hbar}$ has the three following properties:

- $\widehat{\rho}_{\hbar}$ is of trace-class and $\mathrm{Tr}(\widehat{\rho}_{\hbar}) = 1$;
- $\widehat{\rho}_{\hbar}$ is self-adjoint: $\widehat{\rho}_{\hbar}^{*} = \widehat{\rho}_{\hbar}$;
- $\widehat{\rho}_{\hbar} \geq 0$ that is $\langle \widehat{\rho}_{\hbar}\psi | \psi \rangle \geq 0$ for all ψ.

It is the positivity property $\widehat{\rho}_{\hbar} \geq 0$ which is very difficult to verify. There is no "easy" method available. We are running short of mathematics!

See however:

E. Cordero, M. de Gosson, F. Nicola: On the Positivity of Trace Class Operators https://arxiv.org/abs/1706.06171

Gaussians states

Let us now consider Gaussians of the type

$$\rho(z) = (2\pi)^{-n}\sqrt{\det \Sigma^{-1}}e^{-\frac{1}{2}\Sigma^{-1}z^2} \tag{2}$$

where Σ is a $2n \times 2n$ covariance matrix. We have $\rho \geq 0$ and $\int \rho(,pz)d^n x d^n p = 1$ hence the function ρ can always be viewed as a classical probability distribution. It is the $\hbar$-Wigner distribution of a quantum sate *if and only if* Σ is such that

$$\Sigma + \frac{i\hbar}{2}J \geq 0 \ \ , \ \ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

These conditions are *equivalent* to the indeterminacy principle in its Schrödinger–Robertson form:

$$\Delta p_j^2 \Delta x_j^2 \geq \Delta(x_j, p_j)^2 + \tfrac{1}{4}\hbar^2.$$

- The purity of the corresponding Gaussian state is

$$\mathrm{Tr}(\widehat{\rho}^2) = \left(\frac{\hbar}{2}\right)^n \det(\Sigma^{-1/2})$$

hence $\widehat{\rho}$ is a pure Gaussian state if and only if $\det(\Sigma) = (\hbar/2)^n$. Therefore:

- The purity of the corresponding Gaussian state is

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- *Any decrease of $\hbar$ leads to a loss of purity: the state becomes more and more mixed; indeterminacy increases;*

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hence $\widehat{\rho}$ is a pure Gaussian state if and only if $\det(\Sigma) = (\hbar/2)^n$. Therefore:

- *Any decrease of $\hbar$ leads to a loss of purity: the state becomes more and more mixed; indeterminacy increases;*
- *Any increase of $\hbar$ "purifies" the state until it reaches the critical value $2(\det \Sigma)^{1/n}$; thereafter it becomes a classical state: we are witnessing a transition from the quantum world to the classical world.*

General Mixed States

We are lacking almost altogether a classification of these state transitions outside the Gaussian case. Simple examples show that the set of admissible values of $\hbar$ can have an extremely intricate structure (in particular it can be discrete). There exist theoretical conditions (the "KLM conditions") which allow in principle to test for which values of $\hbar$ the function ρ is the Wigner distribution of a quantum state but these conditions are of little use outside the case of Gaussian distributions, or pure states. This is because checking them amounts to verify an *uncountable* infinite set of inequalities, which is impossible to implement in practice, even numerically.

M. de Gosson, Mixed quantum states with variable Planck constant, *Phys. Lett. A*, **381**(36), 25 September 2017

E. Cordero, M. de Gosson, F. Nicola: On the Positivity of Trace Class Operators https://arxiv.org/abs/1706.06171

The KLM conditions

Let $\sigma(z,z') = (z')^T Jz$ $(z = (x,p))$ be the symplectic form on phase space $\mathbb{R}^{2n}_z \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$. Let

$$\rho_\Diamond(z) = \int e^{i\sigma(z,z')}\rho(z')dz'$$

be the "symplectic Fourier transform" of a quasi-distribution (classical, or quantum) ρ. Assume that $\rho_\Diamond$ is continuous at the origin. Then $\widehat{\rho}_\hbar \geq 0$ if and only if for all $z_1, ..., z_N \in \mathbb{R}^{2n}$ the symmetric $N \times N$ matrix Λ with entries

$$\Lambda_{jk} = (e^{-\frac{i\hbar}{2}\sigma(z_j,z_k)}\rho_\Diamond(z_j - z_k))_{1\leq j,k\leq N}$$

is positive semi-definite: $\Lambda \geq 0$. This condition is due to Kastler, Loupias, and Miracle–Sole (ca. 1968). The proof is highly non trivial.

These conditions are very sensitive to the value of $\hbar$!

The set of all $\hbar$ for which $\widehat{\rho}_\hbar \geq 0$ is called the Wigner spectrum of ρ.

Entanglement

A mixed (or pure) quantum state is said to be *separable* (with respect to the partition (A, B) of the position and momentum variables) if there exist sequences of density matrices $\widehat{\rho}_j^A$ and $\widehat{\rho}_j^B$ and real probabilities α_j such that

$$\widehat{\rho}_\hbar = \sum_{j\in\mathcal{I}} \alpha_j \widehat{\rho}_j^A \otimes \widehat{\rho}_j^B . \tag{3}$$

If this is *not* the case, the state is said to be *entangled*. Conditions for separability and entanglement are not fully understood; there is a necessary condition (the PPT criterion), but we are lacking sufficient conditions outside the Gaussian case (and a few other examples). However since the very fact of the operator $\widehat{\rho}_\hbar$ to be sensitive to Planck's constant strongly suggests that separability and entanglement might also dependent on h. Work very much in progress! Any advance might lead to experimental setups allowing to test the variability of h!

M. de Gosson and Mohageg, On the Dependence of Quantum States on the Value of Planck's Constant: https://arxiv.org/abs/1612.05578).

The Symplectic Camel

(*Alias* Gromov's symplectic non-squeezing theorem, 1985).

> *"It is easier for a camel to go through the eye of a needle, than for a rich man to enter into the kingdom of God" (Mark* **10:25***).*

Key paper:

M. Gromov, *Pseudoholomorphic curves in symplectic manifolds.* Invent. Math. **82** (1985)).

The Symplectic Camel:

M. Gromov, *Pseudoholomorphic curves in symplectic manifolds.* Invent. Math. **82** (1985)).

Theorem

Let $B^{2n}(r) = \{z : |z| \leq r\}$ *and* $Z_j^{2n}(R) = \{z : x_j^2 + p_j^2 \leq R^2\}$*. If there exists a canonical transformation* f *such that* $f(B^{2n}(r)) \subset Z_j^{2n}(R)$ *then* $r \leq R$*.*

Dynamical illustration

The ball is being deformed by the action of a canonical transformation:

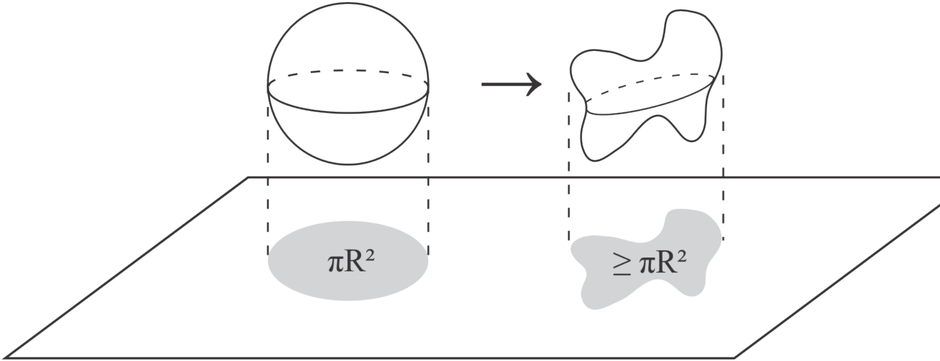

Fig.2

The projection is on plane of conjugate (canonical)

Choose now $R = \sqrt{\hbar}$; then the area of the projection on each plane of conjugate coordinates x_j, p_j remains at least $\pi(\sqrt{\hbar})^2 = \pi\hbar = \frac{1}{2}h$. The deformed ball is a "quantum blob", and is closely related to the uncertainty principle. Quantum blobs provide a coarse graining of phase space, better than the usual decomposition in cubic cells with volume $\sim \hbar^n$. The number $c = \pi\hbar$ is called the symplectic capacity of the quantum blob; it is a fundamental symplectic invariant in view of Gromov's theorem.

- M. de Gosson, The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? *Found. Phys.* 39(2), 194–214. (2009)
- M. de Gosson, Quantum blobs. *Found. Phys.* 43(4), 440–457 (2013)
- M. de Gosson and F. Luef, Symplectic Capacities and the Geometry of Uncertainty: the Irruption of Symplectic Topology in Classical and Quantum Mechanics. *Phys. Reps.* **484**, 131–179 (2009)

Symplectic Capacities

A symplectic capacity on $\mathbb{R}^{2n}$ assigns to every $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \geq 0$, or $+\infty$, satisfying:

- (SC1) **Monotonicity**. If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- (SC2) **Symplectic invariance**. If f is a canonical transformation then $c(f(\Omega)) = c(\Omega)$;
- (SC3) **Conformality**. If λ is a real number then $c(\lambda\Omega) = \lambda^2 c(\Omega)$; here $\lambda\Omega$ is the set of all points λz when $z \in \Omega$;
- (SC4) **Nontriviality**. $c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R))$.

Here $B^{2n}(R)$ is the ball $|x|^2 + |p|^2 \leq R^2$ and $Z_j^{2n}(R)$ the cylinder $x_j^2 + p_j^2 \leq R^2$. The existence of symplectic capacities (in fact infinitely many) follows from Gromov's theorem. When $n = 1$ all symplectic capacities coincide with area on connected and simply connected surfaces.

Symplectic Capacities

- Two distinguished symplectic capacities are:

$$c_{\min}(\Omega) = \sup_{f \in \mathrm{Symp}(n)} \{\pi R^2 : f(B^{2n}(R)) \subset \Omega\} \tag{4a}$$

$$c_{\max}(\Omega) = \inf_{f \in \mathrm{Symp}(n)} \{\pi R^2 : f(\Omega) \subset Z_j^{2n}(R)\}. \tag{4b}$$

  That they really are symplectic capacities follows from Gromov's theorem.

- We have

$$c_{\min}(\Omega) \leq c(\Omega) \leq c_{\max}(\Omega) \tag{5}$$

  for every symplectic capacity c. For $\lambda \in [0,1]$ the function $c = \lambda c_{\max}(\Omega) + (1-\lambda) c_{\min}(\Omega)$ is a symplectic capacity.

Ellipsoids

The construction of symplectic capacities is notoriously difficult (the fact that symplectic capacities exist is actually equivalent to Gromov's non-squeezing theorem). However they all agree on phase space ellipsoids: Let $\Omega = \{z : Mz \cdot z \leq R^2\}$ where M is a symmetric positive definite $2n \times 2n$ matrix. We have

$$c(\Omega) = \pi R^2 / \lambda_{\max} \tag{6}$$

for *every* symplectic capacity c; here $\lambda_{\max}$ is the largest symplectic eigenvalue of M. The symplectic eigenvalues are defined as follows: Since $M > 0$ we have $JM \sim M^{1/2} J M^{1/2}$ hence the eigenvalues of JM are of the type $\pm i\lambda_j$ with $\lambda_j > 0$. The symplectic eigenvalues are by definition $\lambda_1, ..., \lambda_n$.

Covariance matrix

We assume that ρ is a (quasi-)probability density on $\mathbb{R}^{2n}$. (In classical physics $\rho \geq 0$; in quantum mechanics ρ can take < 0 values: Wigner function, or convex sum thereof: mixed states). The covariance matrix of a random variable $Z = (X_1, ..., X_n; P_1, ..., P_n)$ is

$$\Sigma = \int_{\mathbb{R}^{2n}} \underbrace{(z - \langle z \rangle)(z - \langle z \rangle)^T}_{2n \times 2n \text{ matrix}} \rho(z) dz \tag{7}$$

where the vector $\langle z \rangle = \int_{\mathbb{R}^{2n}} z\rho(z)dz$ is the mean value of Z.

- We have $\Sigma = \Sigma^T$.
- In quantum mechanics Σ is always definite positive: $\Sigma > 0$ (Narcowich).

We write

$$\Sigma = \begin{pmatrix} \Delta(X,X) & \Delta(X,P) \\ \Delta(P,X) & \Delta(P,P) \end{pmatrix}$$

where $\Delta(X,X)$ etc. are $n \times n$ blocks with entries $\Delta(X_j, X_k)$.

Examples: In the case $n = 1$ we have

$$\Sigma = \begin{pmatrix} \Delta X^2 & \Delta\left(X, P_x\right) \\ \Delta\left(P_x, X\right) & \Delta P_x^2 \end{pmatrix}$$

and in the case $n = 2$

$$\Sigma = \begin{pmatrix} \Delta X^2 & \Delta\left(X,Y\right) & \Delta\left(X,P_x\right) & \Delta\left(X,P_y\right) \\ \Delta\left(Y,X\right) & \Delta Y^2 & \Delta\left(Y,P_x\right) & \Delta\left(Y,P_y\right) \\ \Delta\left(P_x,X\right) & \Delta\left(P_x,Y\right) & \Delta P_x^2 & \Delta\left(P_x,P_y\right) \\ \Delta\left(P_y,X\right) & \Delta\left(P_y,Y\right) & \Delta\left(P_y,P_x\right) & \Delta P_y^2 \end{pmatrix}.$$

Main observation

We have $(\Sigma + \frac{i\hbar}{2}J)^* = \Sigma + \frac{i\hbar}{2}J$ because $J^* = J^T = -J$. It follows that:

Theorem

For a quantum system the condition

$$\Sigma + \frac{i\hbar}{2}J \text{ is positive semi-definite} \tag{B}$$

is equivalent to the Robertson–Schrödinger inequalities

$$(\Delta P_j)^2(\Delta X_j)^2 \geq \Delta(X_j, P_j)^2 + \tfrac{1}{4}\hbar^2 \tag{A}$$

for $1 \leq j \leq n$.

Proof.

Linear algebra, using the principal minors of the matrix $\Sigma + \frac{i\hbar}{2}J$. □

Example

Assume that $n = 1$; then the condition

$$\Sigma + \frac{i\hbar}{2}J = \begin{pmatrix} \Delta X^2 & \Delta\left(X, P_x\right) + \frac{i\hbar}{2} \\ \Delta\left(P_x, X\right) - \frac{i\hbar}{2} & \Delta P_x^2 \end{pmatrix} \geq 0$$

is equivalent to

$$\det\left(\Sigma + \frac{i\hbar}{2}J\right) \geq 0$$

that is to the Robertson–Schrödinger inequality

$$\Delta X^2 \Delta P_x^2 \geq \Delta\left(X, P_x\right)^2 + \tfrac{1}{4}\hbar^2.$$

Covariance ellipsoid

The covariance ellipsoid of a quantum state is defined by
$\Omega_\Sigma = \{z = (x, p) : \frac{1}{2}\Sigma^{-1}z \cdot z \leq 1\}$.

Theorem

We always have $c(\Omega_\Sigma) \geq \pi\hbar$ and this statemernt is equivalent to the uncertaintry principle of Robertson–Schrödinger.

Thus we again hav a very strong dependence on Planck's constant: decreasing h leads to smaller covariance ellipsoids.

THANK YOU FOR YOUR KIND ATTENTION!