Relativistic thermodynamics
and
relativistic temperature

P. Ván

HAS, Wigner RCP, Department of Theoretical Physics,

BME, Department of Energy Engineering,

Montavid Thermodynamic Research Group

1. Thermodynamics and objectivity
2. Special relativistic thermodynamics
   – *Temperature of moving bodies.*
3. Relativistic fluids

Collaboration with T.S. Biró

Planck 2018, Budapest

About thermodynamics

– What are the First and the Second principles? Special or general?

> <u>Derived:</u> Part of statistical physics. Only macroscopic.
> Ideal background.
> <u>Fundamental:</u> The single universal theory in physics? Appears unexpectedly: Unruh temperature, black holes, emergent gravity…

– Obscure or clear? Quasimathematics in the elementary courses. Axioms galore. Processes?

Example 1 (T.S. Biró). Negative heat capacity of black holes.
Christodolou-Rovelli volume vs. thermodynamic extensivity.

Example 2 (S. Csabai). Doppler transformation of temperatures.
Does temperature a velocity?

CONSISTENT PHYSICS??

About the history of thermodynamics

Fourier, Théorie analytique de la chaleur (Paris, 1822)
Carnot, Reflections on the Motive Power of Fire (Paris, 1824)

There is no interaction in the next 50 years.

Process based axioms:
Clausius, laws of thermodynamics (1850), entropy (1865)
Planck, Vorlesungen über Thermodynamik
(1897-1926, 11 improved editions)
Fényes (1952), Gyarmati (1962), …, Truesdell-Bharatha (1977), …

UNIVERSAL – toward non-equilibrium thermodynamics

Thermostatic axioms:
Gibbs (1880-1900), Charatheodory (1909), Giles (1964),
Lieb-Yngvason (1999)

SPECIAL – toward statistical physics

Versions of the Second Law

(Uffink, 2001)

Version of second law	Applies only to cycle?	Time-reversal non-invariant?	Allows reversible processes?	Implies existence of irreversible processes?	Argues for universal irreversibility?
Carnot's theorem	yes	yes	yes	no	no
Clausius (1850)	yes	no	yes	no	no
Kelvin (1851)	yes	no	yes	no	no
Kelvin (1852)	no	yes	yes	yes	yes
Kelvin (1855)	yes	no	yes	no	no
Clausius (1865)	no	yes	yes	yes	yes
Clausius (1876)	yes	yes	yes	no	no
Planck (1897)	no	yes	yes	yes	yes
Gibbs (1875)	n.a.	n.a.	yes	no	no
Carathéodory (1909)	no	yes	yes	no	no
Lieb & Yngvason (1999)	no	no	yes	no	no

About objectivity

Invarariance or covariance? Transformation rules or spacetimes?

– *Weak relativity:* Change between inertial reference frames.
– *Strong relativity:* Change between arbitrary reference frames.
– *Absolute:* Independent of reference frames.
Spacetime concepts: Galilean relativistic, special relativistic, general relativistic spacetimes.

General relativity (the theory of gravitation) is absolute.
The terminology is misleading.

Galilean covariance: $\hat{t} = t, \hat{x}^i = x^i + t v^i \quad \rightarrow \quad x^a$

$$\begin{aligned} \hat{\rho} &= \rho, \\ \hat{p}^i &= p^i + \rho v^i, \\ \hat{e} &= e + p^i v_i + \rho \frac{v^2}{2}, \\ \hat{j}^i &= j^i + \rho v^i, \end{aligned}$$

$$\begin{aligned} \hat{P}^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ \hat{q}^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}. \end{aligned}$$

About the history of objectivity

Galileo: absolute time. Newton: absolute space.

Galilean weak relativity:

Lange 1885-6, Ueber das Beharrungsgesetz (2014), Mach (1888)

Galilean strong relativity in field theories:

Zaremba (1903), Jaumann* (1911), time derivatives

Truesdell-Noll (1965), rigid bodies, material frame indifference,...

e.g. Speziale (1991) turbulence

Lorentz, Poincare, Einstein: electrodynamics → special relativity

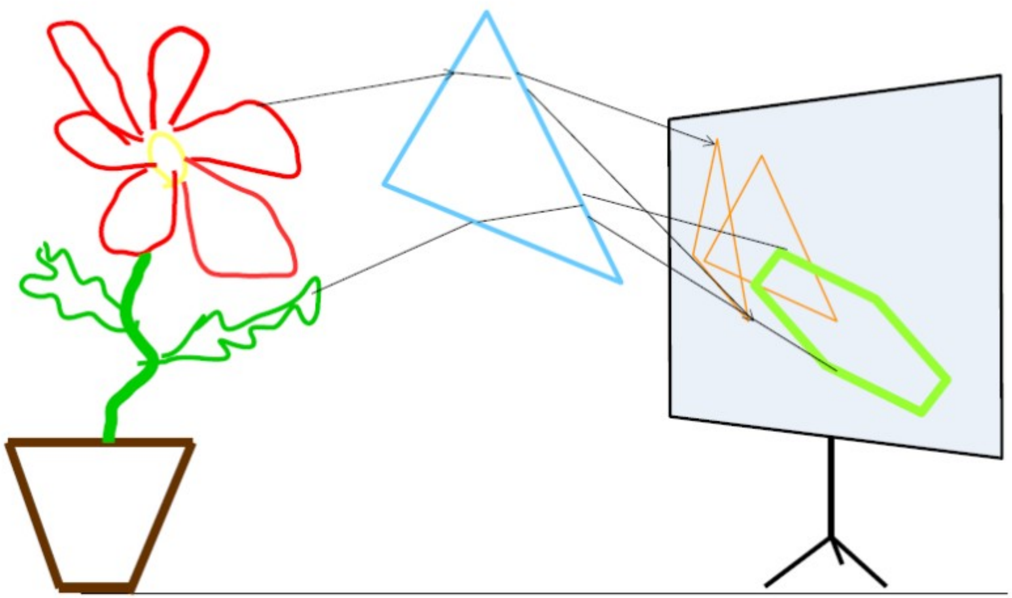
Minkowski: space-time, four vectors

Planck: thermodynamics + ...

Eckart (1940-48):

thermodynamics for Galilean and special relativistic *fluids*.

Model hierarchy: statics, body processes, field theory
Theory coexistence: gravitation, electrodynamics, …

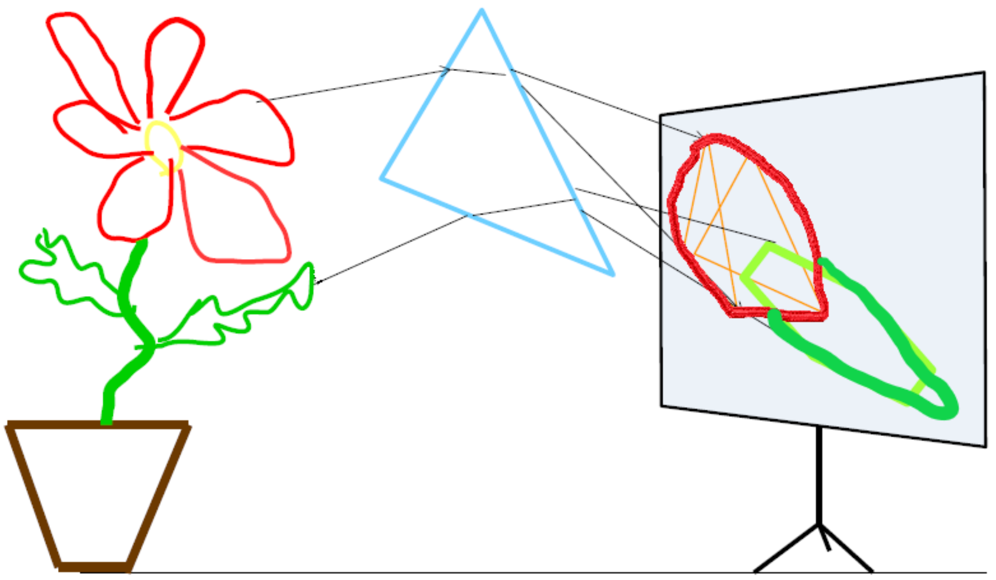


Thermodynamics is a consistency challenge.

Thermodynamics

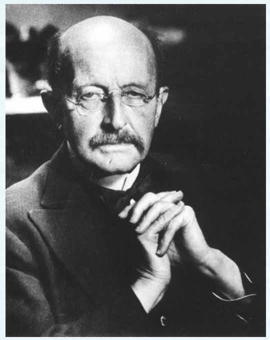
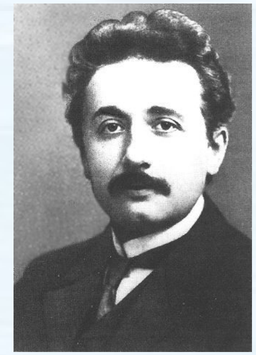
Model hierarchy: statics, body processes, field theory
Theory connection: quantum mechanics, electrodynamics, ...

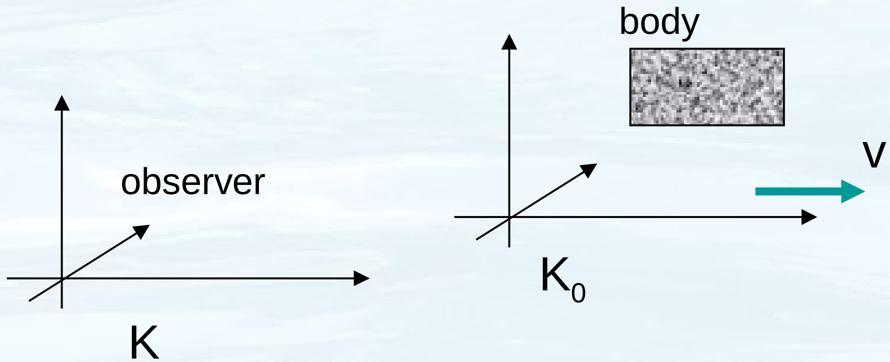
Universal and absolute?



Thermodynamics is a <u>consistency</u> challenge.

About the temperature of moving bodies

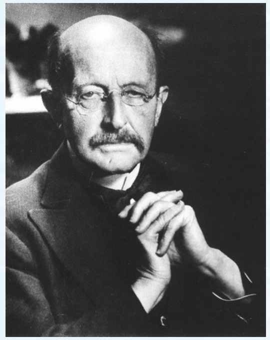
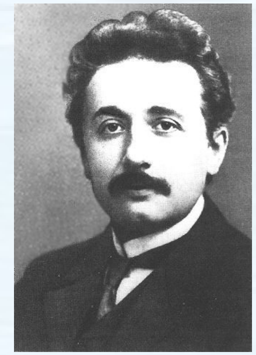
Planck  and Einstein  (1907)

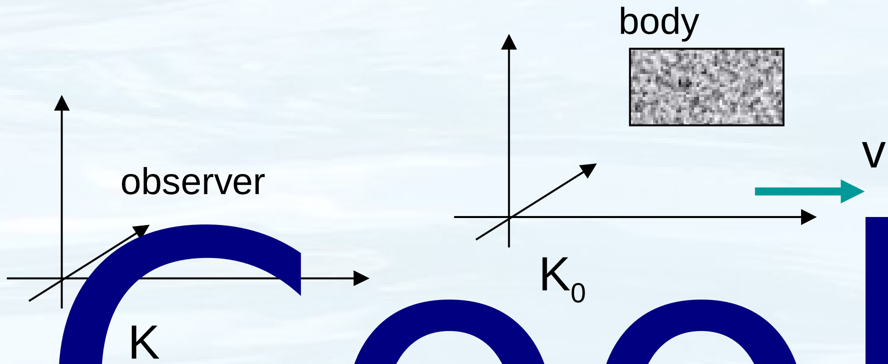


$$dE = TdS - pdV - vdG$$

Relativistic thermodynamics?

About the temperature of moving bodies

Planck  and Einstein  (1907)

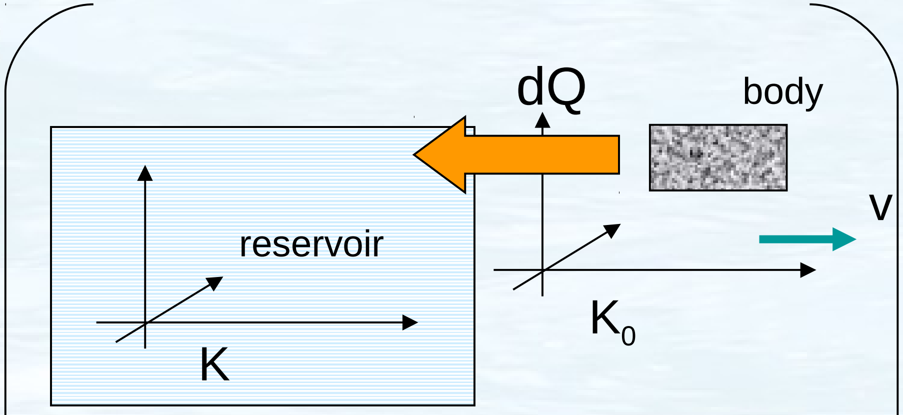


Cool

$$dE = TdS - pdV - vdG$$

Relativistic thermodynamics?

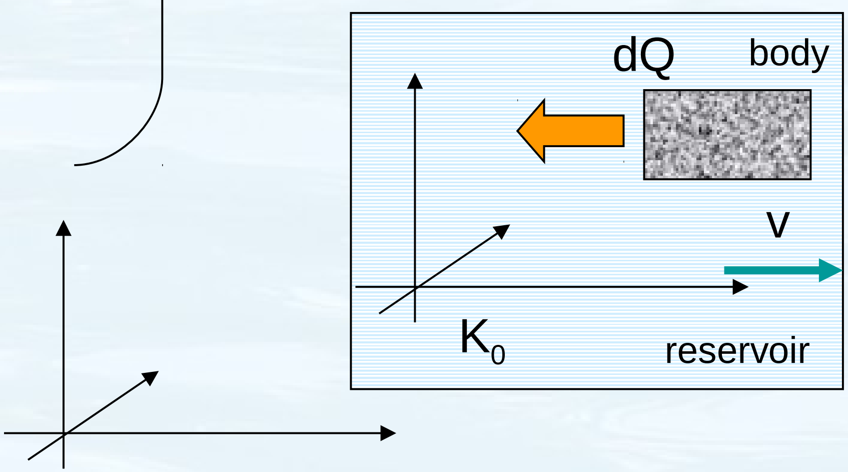
Rest frame arguments:
Ott (1963)



Planck-Einstein

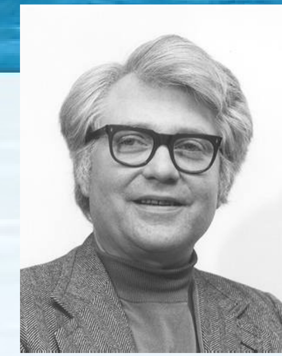
$$dE = TdS - pdV + vdG$$

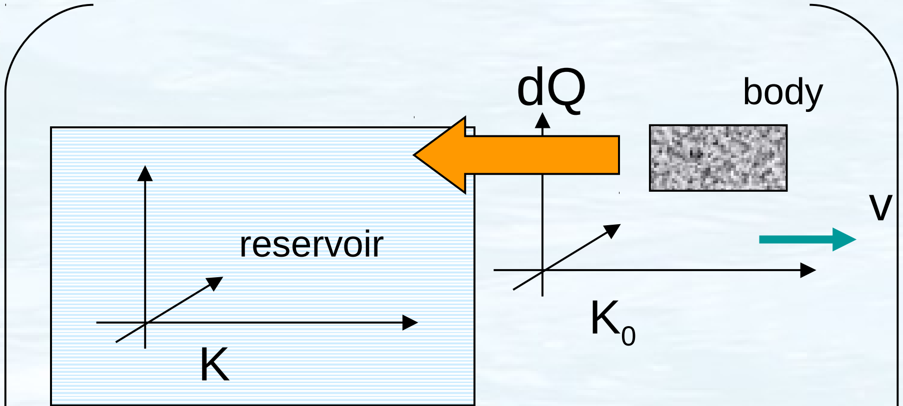
$$dE = TdS - pdV + \cancel{vdG}$$



Ott

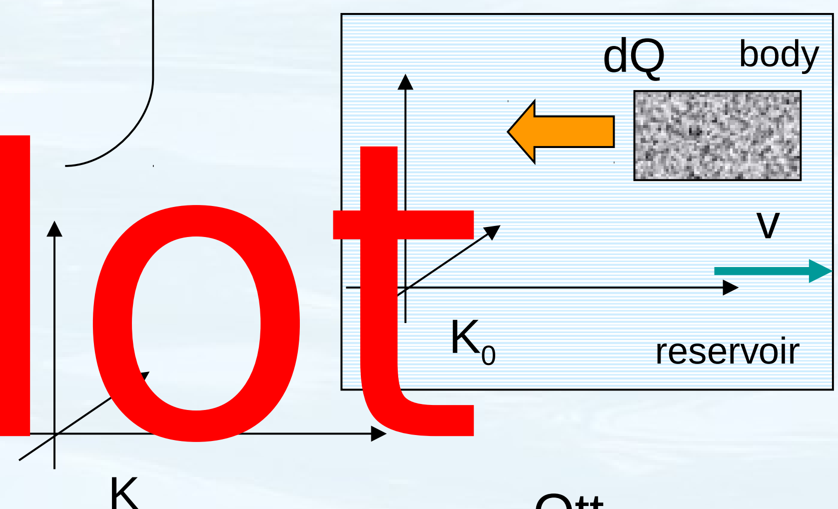
Rest frame arguments:
Ott (1963)





Planck-Einstein

$$dE = TdS - pdV + vdG$$

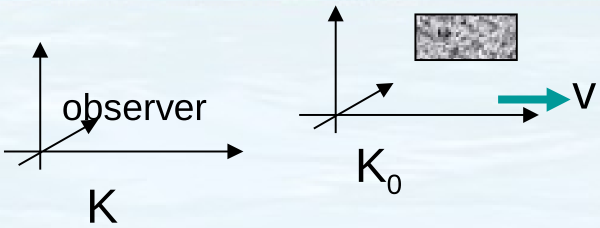


Ott

Hot

$$dE = TdS - pdV + \cancel{vdG}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$



Permanent discussion

~1963-70, Møller, von Treder, Israel, ter Haar, Callen, ..., renewed Dunkel-Talkner-Hänggi 2007, ...:

– new (?) arguments, no (re)solution.

– Planck-Einstein (1907): cooler $T = \frac{T_0}{\gamma}$

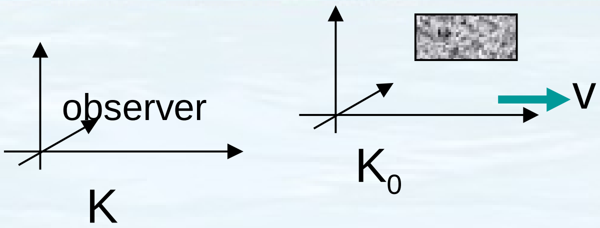
– Ott (1963) [Blanusa (1947)] : hotter $T = \gamma T_0$

– Landsberg (1966-67): equal $T = T_0$

– Costa-Matsas-Landsberg (1995): direction dependent

(Doppler) $T = \frac{T_0}{\gamma(1 - v\cos\alpha)}$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$



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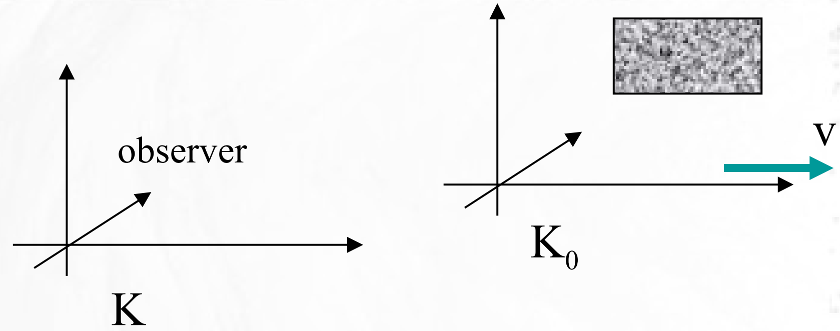
(Doppler) $T = \frac{T_0}{\gamma(1-v\cos\alpha)}$

?????

$$dE = TdS - pdV + v\,dG$$

translation work



$$dE = TdS - pdV + vdG$$

$$\gamma\, dE_0 = TdS_0 - p_0 \frac{dV_0}{\gamma} + \gamma\, v^2 dE_0$$

$$\left\{ \begin{array}{l} p = p_0 \\ dV = dV_0/\gamma \\ dS = dS_0 \\ dE = \gamma\, dE_0 \\ dG = \gamma\, v dE_0 \end{array} \right.$$

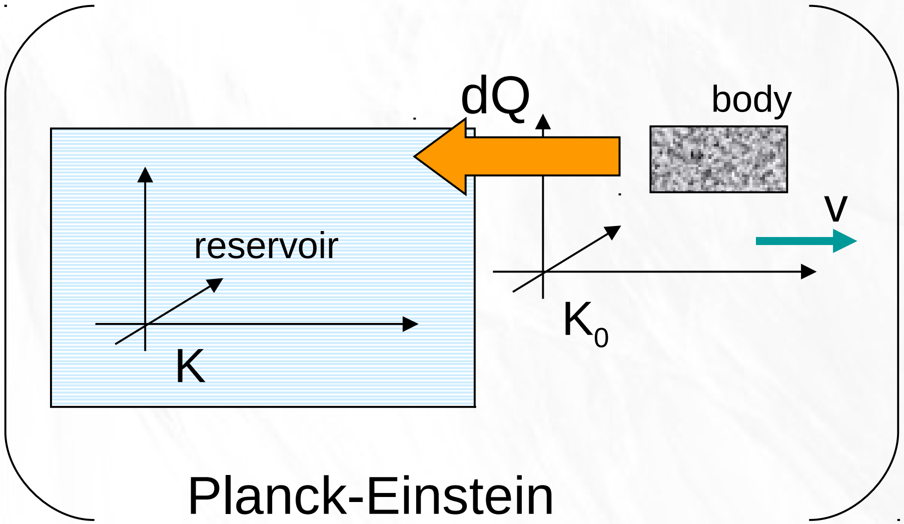
$$\frac{dE_0}{\gamma} = TdS_0 - p_0 \frac{dV_0}{\gamma}$$

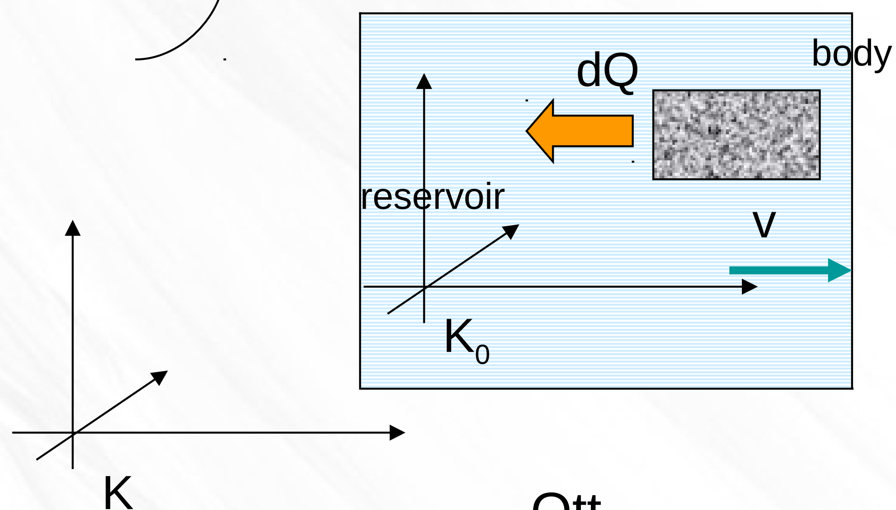
$$dE_0 = T_0 dS_0 - p_0 dV_0$$

$$T = \frac{T_0}{\gamma} = T_0 \sqrt{1 - v^2} < T_0$$

vector of reciprocal temperature

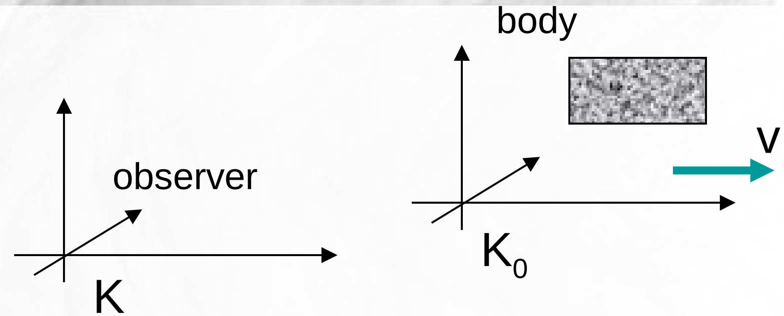
Ott (1963)


dQ
body
reservoir
v
K_0
K
Planck-Einstein


dQ
body
reservoir
v
K_0
K

Ott

$$dE = TdS - pdV \cancel{+ vdG}$$



There is no translation work

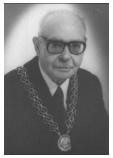
$$dE = TdS - pdV$$

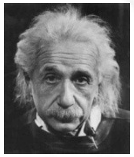
$$\gamma\, dE_0 = TdS_0 - \gamma^2 p_0 dV_0 / \gamma$$

$$\left\{ \begin{array}{l} p = \gamma^2 p_0 \\ dV = dV_0 / \gamma \\ dS = dS_0 \\ dE = \gamma\, dE_0 \end{array} \right.$$

$$dE_0 = T_0 dS_0 - p_0 dV_0 \longrightarrow \boxed{T = \gamma T_0}$$

Blanusa (1947)
Einstein (1952) (correspondence with Laue)





But: is the temperature a vector? Tensor?
discussion (~1963-70): new suggestions,
without resolution.

Landsberg

van Kampen

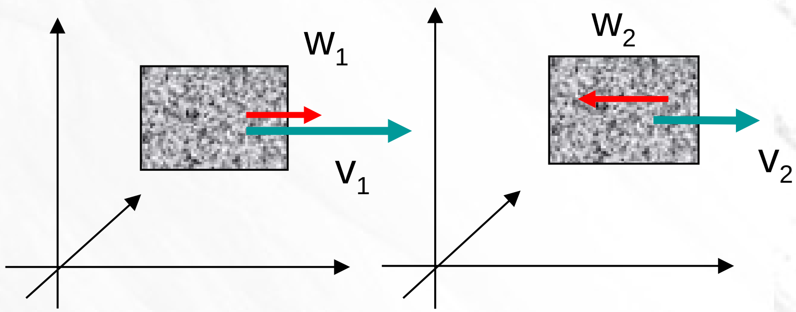
$$TdS = g_a dE^a = (u_a + w_a) dE^a$$

covariant formula

Temperature vector, or modified velocity?

equilibrium in 1+1 dimensions:

$$\frac{(u_1^a + w_1^a)}{T_1} = \frac{(u_2^a + w_2^a)}{T_2}$$



$$u^a = (\gamma, \gamma v), \quad w^a = (\gamma v w, \gamma w)$$

$$\frac{\gamma_1(1 + v_1 w_1)}{T_1} = \frac{\gamma_2(1 + v_2 w_2)}{T_2}$$

$$\frac{\gamma_1(v_1 + w_1)}{T_1} = \frac{\gamma_2(v_2 + w_2)}{T_2}$$

$\Rightarrow$

$$\frac{v_1 + w_1}{1 + v_1 w_1} = \frac{v_2 + w_2}{1 + v_2 w_2}$$

$$\frac{\sqrt{1 - w_1^2}}{T_1} = \frac{\sqrt{1 - w_2^2}}{T_2}$$

Transformation of temperatures

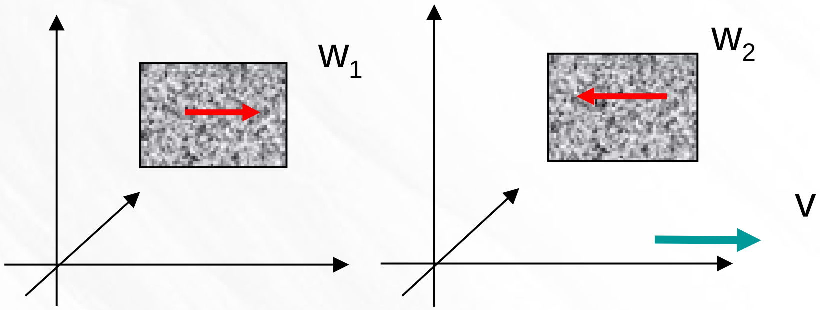
$$\frac{v_1 + w_1}{1 + v_1 w_1} = \frac{v_2 + w_2}{1 + v_2 w_2}, \quad \frac{\sqrt{1 - w_1^2}}{T_1} = \frac{\sqrt{1 - w_2^2}}{T_2}$$

Four velocities: v_1, v_2, w_1, w_2

Relative velocity

(Lorentz transformation)

$$v = \frac{v_2 - v_1}{1 - v_1 v_2}$$

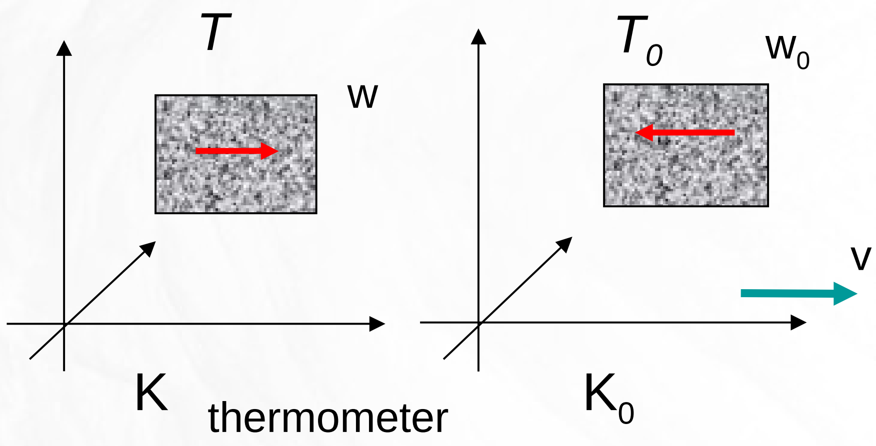


$$w_1 = \frac{v + w_2}{1 + v w_2}$$

$$\frac{T_1}{T_2} = \frac{\sqrt{1 - v^2}}{1 + v w_2}$$

general Doppler-like form!

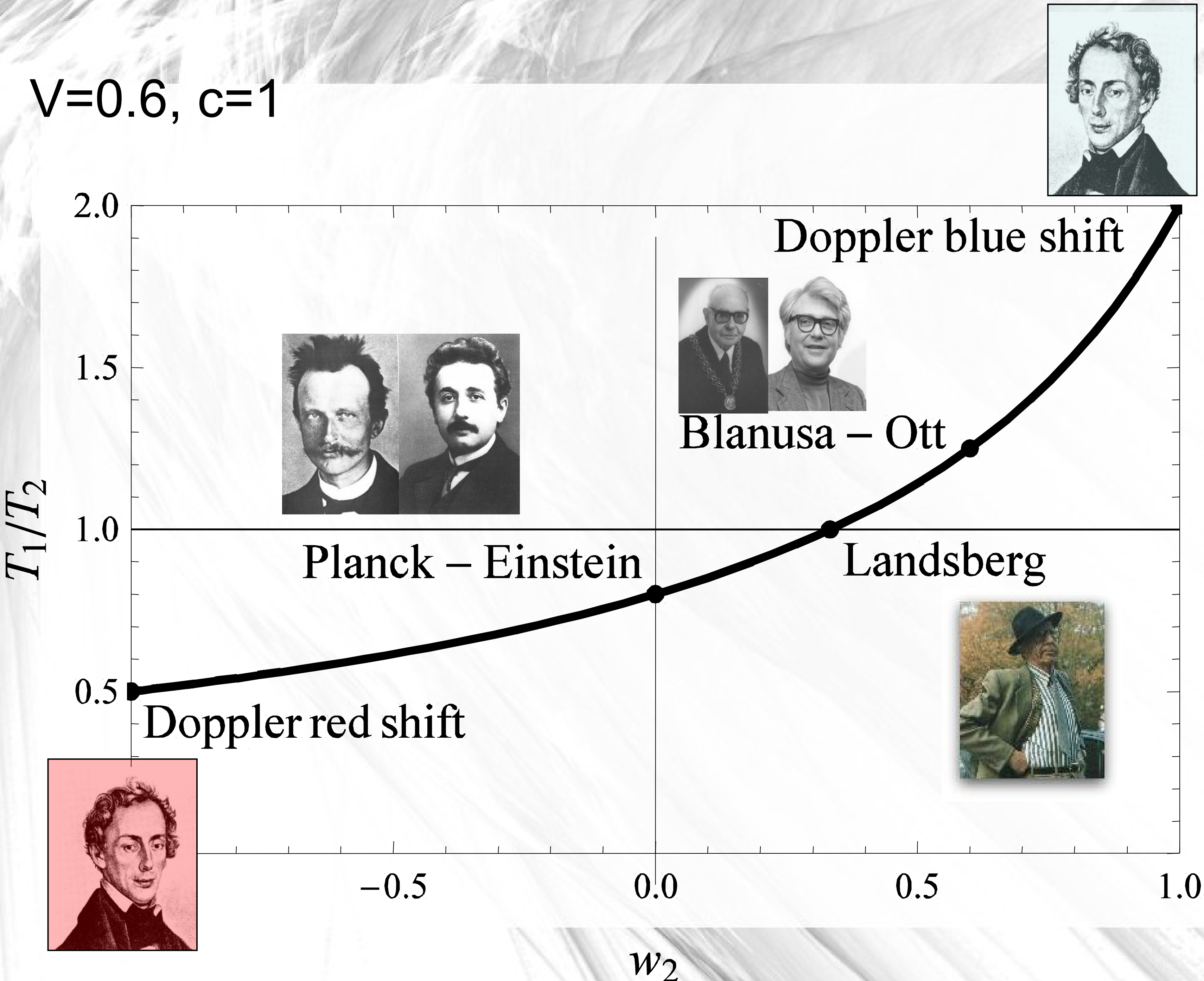
$$\frac{T}{T_0} = \frac{\sqrt{1-v^2}}{1+vw_0}$$



Special:

$w_0 = 0$	$T = T_0/\gamma$	*Planck-Einstein*
$w = 0$	$T = \gamma\ T_0$	*Ott*
$w_0 = 1,\ v > 0$	$T = T_0 \bullet$ *red*	*Doppler*
$w_0 = 1,\ v < 0$	$T = T_0 \bullet$ *blue*	*Doppler*
$w_0 + w = 0$	$T = T_0$	*Landsberg*

V=0.6, c=1



Doppler blue shift
Blanusa – Ott
Planck – Einstein
Landsberg
Doppler red shift
2.0
1.5
1.0
0.5
−0.5
0.0
0.5
1.0
T_1/T_2
w_2

What is w?

Thermodynamics of motion

=

thermomechanics?

Basic concepts:

$$T^{ab} = e u^a u^b + q^a u^b + q^b u^a + P^{ab},$$

energy-momentum density

$$N^a = n u^a + j^a.$$

particle number density

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

General, expressed by comoving splitting

$$T^{ab} = \begin{pmatrix} e & q^i \\ q^j & P^{ij} \end{pmatrix}, \qquad N^a = \begin{pmatrix} n \\ j^i \end{pmatrix}$$

$a, b \in \{0,1,2,3\}; \quad i, j \in \{1,2,3\}; \quad diag(1,-1,-1,-1)$

- u^a – *velocity field*
- e – *energy density*
- q^a – *momentum density or energy current??*
- P^{ab} – *pressure*
- n – *particle number density*
- j^a – *particle current*

Dissipative or perfect?

$$P^{ab} = -p\Delta^{ab} + \Pi^{ab} = (-p + \Pi)\Delta^{ab} + \pi^{ab}$$

$$q^a???$$

Basic balances:

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$

$$N^a = nu^a + j^a.$$

energy-momentum density

particle number density

general, expressed by comoving splitting

$$\partial_b T^{ab} = 0^a \quad \partial_a N = 0$$

$$\Delta^a_c \partial_b T^{cb} = h\dot{u}^a + \Delta^a_c \dot{q}^c + q^a \partial_b u^b + q^b \partial_b u^a - \Delta^{ab} \partial_b p + \Delta^a_c \partial_b \Pi^{cb} = 0^a$$

$$u_a \partial_b T^{ab} = \dot{e} + e\, \partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0$$

$$\partial_b N^b = \dot{n} + n\, \partial_a u^a + \partial_a j^a = 0$$

$$\dot{e} = u^a \partial_a e$$

momentum, energy and particle number balances, expressed by comoving, local rest frame quantitites

Dissipative and perfect

Constitutive theory:

Fields:

N^a	4
T^{ba}	10
u^a	3
	Σ17

j^a	3
q^a	3
Π^{ab}	6
	Σ 12

$$q^a u_a = j^a u_a = 0, \quad \Pi^{ba} u_a = \Pi^{ab} u_a = 0^b$$

Equations:

$\partial_a N^a = 0,$	1
$\partial_b T^{ab} = 0^a,$	4

N^a – *particle number density vector*
T^{ab} – *energy-momentum tensor*
u^a – *velocity field*

j^a – *particle current*
q^a – *energy current??*
Π^{ab} – *viscous pressure*

n, e, u^a – *basic fields*

Non-equilibrium thermodynamics, second law

The idea of Eckart:

$$u^a \partial_b T^{ab} = \dot{e} + e\,\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0$$

$$\partial_b N^b = \dot{n} + n\,\partial_a u^a + \partial_a j^a = 0$$

$$J^a = \beta q^a - \alpha j^a$$

$$ds + \alpha\, dn = \beta\, de$$

$$\partial_a S^a = \dot{s}(e,n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)\partial_i v_j + q^i \partial_i \frac{1}{T} \geq 0$$

$$\sigma_s = - j^a \partial_a \alpha - \beta \underbrace{\left(P^{ab} - p\delta^{ab}\right)}_{\Pi^{ab}} \partial_b u_a + q^a \left(\partial_a \beta + \beta \dot{u}_a\right) \geq 0$$

Eckart term

Stability

Generic stability: linear stability of homogeneous equilibrium

Instability of first order theories, e.g. Eckart
(Hiscock-Lindblom, 1985)
Stability of the Müller-Israel-Stewart theory
(Hiscock-Lindblom, 1983)
unphysical conditions

Divergence type theories – hyperbolicity, complicated
Gradient theories – no heat flow, stability?

Conceptual questions:

Does thermodynamics related to stability?

Separation of dissipative and ideal?

Why is the instability?

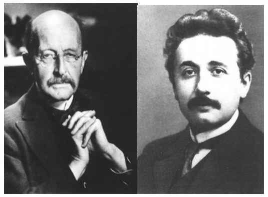
$$\sigma_s = -j^a \partial_a \alpha - \beta \underbrace{\left(P^{ab} - p\delta^{ab} \right)}_{\Pi^{ab}} \partial_b u_a + q^a \left(\partial_a \beta + \beta \dot{u}_a \right) \geq 0$$

acceleration

$$q^a = \lambda \Delta^a_b \left(\partial_b \beta + \beta \dot{u}^a \right)$$

$$\Delta^a_c \partial_b T^{cb} = h \dot{u}^a + \Delta^a_c \dot{q}^c + q^a \partial_b u^b + q^b \partial_b u^a - \Delta^{ab} \partial_b p + \Delta^a_c \partial_b \Pi^{cb} = 0^a$$

$$T^{ab} = \begin{pmatrix} e & q^i \\ q^j & P^{ij} \end{pmatrix}$$



momentum or heat?

Total or internal energy?

Killing the Eckart term is not enough.

Gibbs relations with motion

$$ds = \beta_a de^a, \quad \beta^a = \beta(u^a + w^a), \quad e^a = e u^a + q^a$$

$$\beta = \frac{1}{T}, \quad u^a w_a = 0, \quad u^a q_a = 0, \quad u^a u_a = 1$$

$$Tds = (u_a + w_a) d(e u^a + q^a) = de + w_a dq^a + (ew_a - q_a) du^a$$

$w_a = 0_a$	$de = T\,ds$	$s(e)$
$w_a = \frac{q_a}{e}$	$de + \frac{q_a}{e} d\,q^a = T\,ds$	$s(\lvert e^a \rvert)$
$w_a = \frac{q_a}{h}$	$de + \frac{q_a}{h} d\,q^a = T\,ds$	$s(e^a, u^a)$

balance compatible

Gibbs relations with motion

$$ds=\beta_a de^a, \quad \beta^a=\beta\left(u^a+w^a\right), \quad e^a=e u^a+q^a$$

$$\beta=\frac{1}{T}, \quad u^a w_a=0, \quad u^a q_a=0, \quad u^a u_a=0$$

$$T ds=\left(u_a+w_a\right) d\left(e u^a+q^a\right)=de+w_a dq^a+\left(e w_a-q_a\right) du^a$$

$$w_a=0_a \qquad de=T\,ds \qquad s(e)$$

$$w_a=\frac{q_a}{e} \qquad de+\frac{q_a}{e} d q^a=T\,ds \qquad s\left(\left|e^a\right|\right)$$

$$w_a=\frac{q_a}{h} \qquad de+\frac{q_a}{h} d q^a=T\,ds \qquad s\left(e^a, u^a\right)$$

Stable

balance compatible

States and EoS for motion

$T\,dS = dE - v_i\,dp^i$	Planck-Einstein, relative, body	$S(E, p^i)$
$T\,dS = dE$	Ott-Eckart, relative, body	$S(E)$
$T\,ds = (u_a + w_a)\,de^a$	covariant, local	$s(e^a)$
$T\,ds = de + \frac{q_i}{e}\,dq^i$	covariant, local, inertia?	$q^a = e\,w^a$
$T\,ds = de + \frac{q_i}{h}\,dq^i$	covariant, local, balance	$q^a = h\,w^a$

(Müller)-Israel-Stewart (1969-72):

$$S^a(T^{ab}, N^a) = \left(s(e,\ n) - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_b q^b - \frac{\beta_2}{2T}\pi^{bc}\pi_{bc} \right) u^a + \\ + \frac{1}{T}\left(q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b \right)$$

$$\partial_e s^{IS} \neq \frac{1}{T}$$

Thermodynamics and fluids:

– Stability of dissipative fluids is a benchmark.

– Fields are necessary to clarify space-time aspects.

– Objective (covariant) thermodynamics restores the generic stability of dissipative fluids.

– Further aspects: flow frames, ...

Thermostatics of motion? It is intrinsically non-equilibrium.

The extended Planckian program: universal non-equilibrium thermodynamics

Dissipative evolution in/for every nondissipative theory.

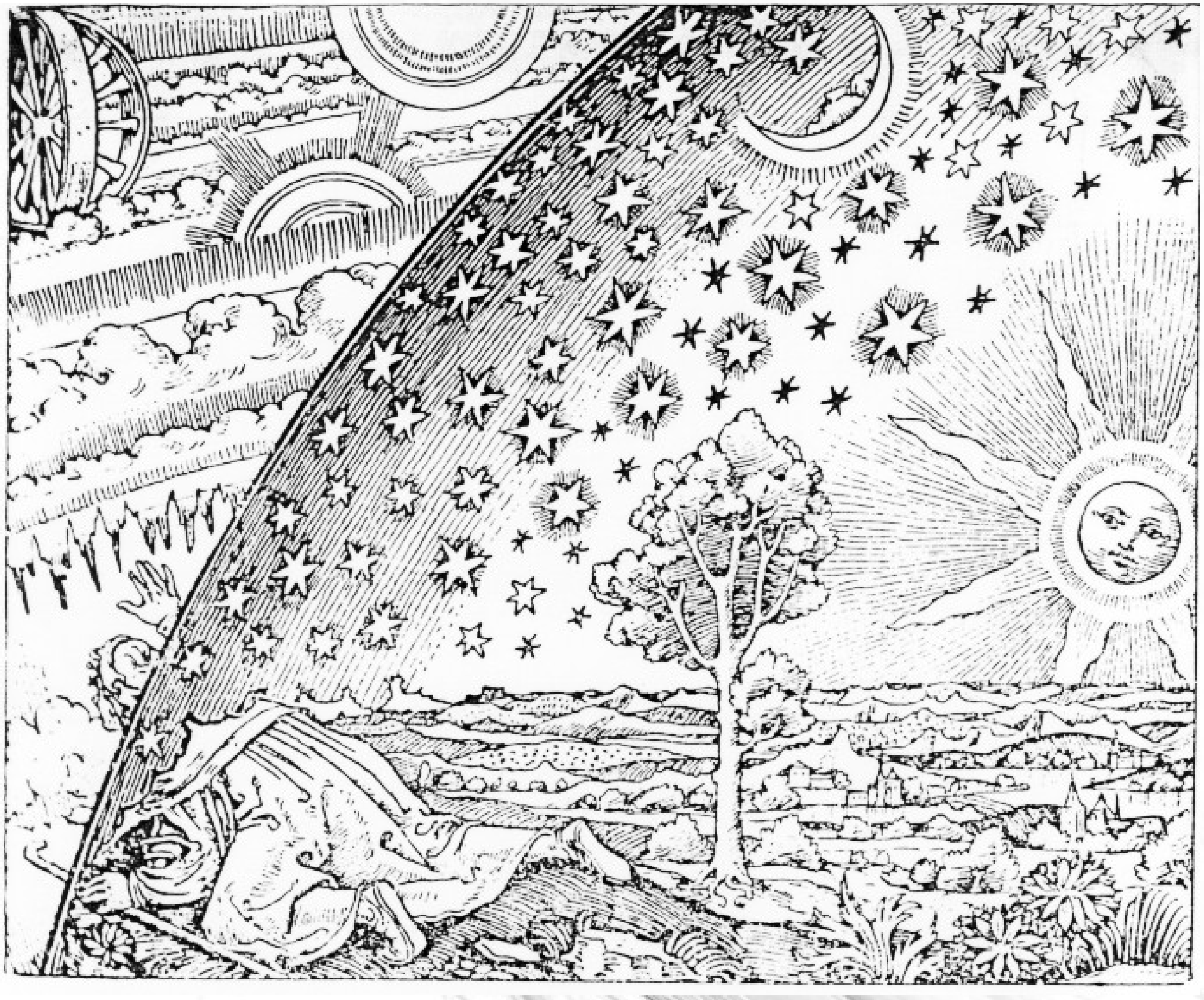
– Zeroth law: the existence of independent physical systems.
– Extensivity and additivity. Many nonextensive together are extensive.
– Weak nonlocality. E.g. phase fields, generalized mechanics.
– Constructed evolution. GENERIC, extended theories.
– Stability and Second Law.

Holistic approach: practical in complex media e.g. heat conduction

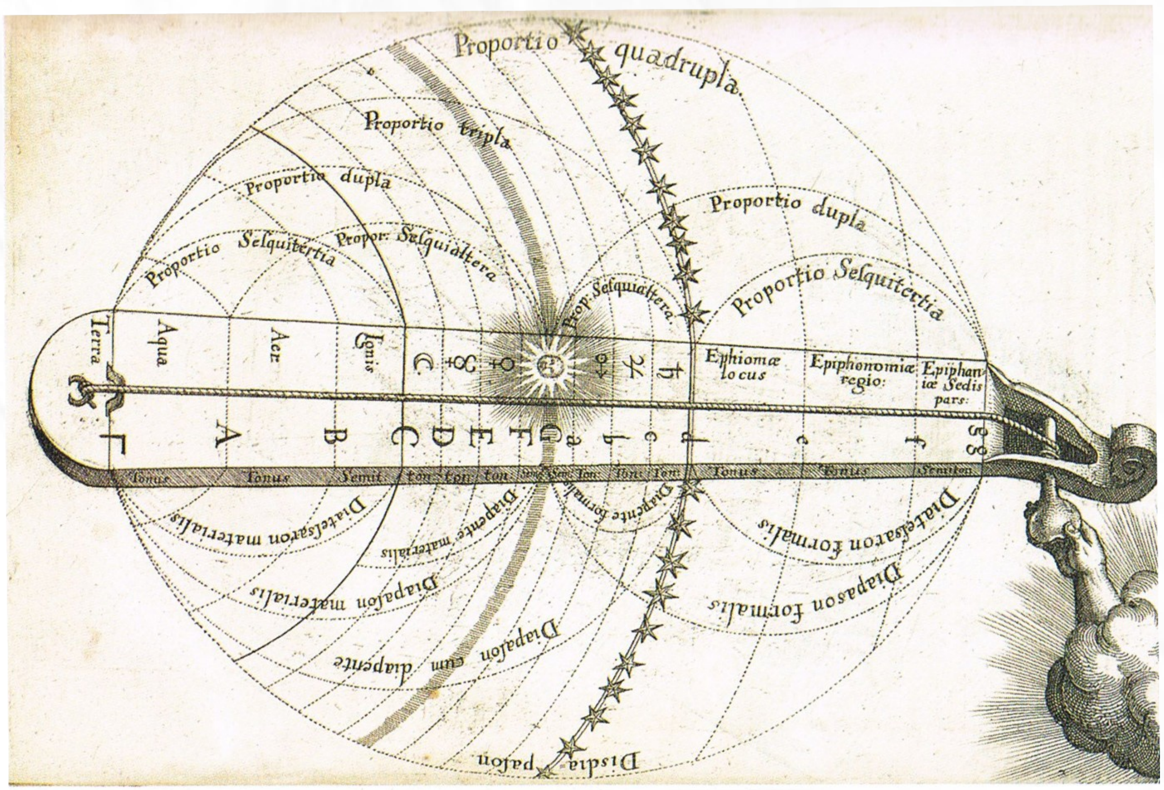
Does thermodynamics fundamental or not?

emergent gravitation (Verlinde), quantum phenomena...

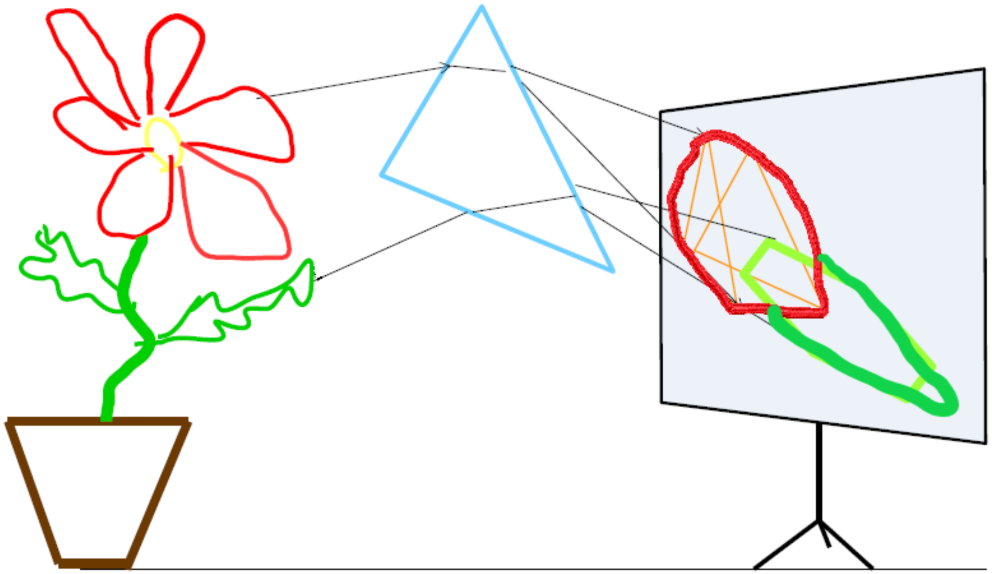
First principles and derived principles



Is there a perfect world?



Thank you for the attention!



Velocity – flow-frames

What is a fluid? What is moving?

Eckart (material) frame: $u^a = \frac{N^a}{\sqrt{-N^b N_b}} \rightarrow N^a = n u^a$

Landau-Lifshitz (energy) frame: $\hat{u}^a = \frac{E^a}{\sqrt{-E^b E_b}} \rightarrow T^{ab} = \hat{e}\hat{u}^a\hat{u}^b + \hat{P}^{ab}$

Thermometer frame: $\check{u}^a = \frac{\beta^a}{\sqrt{-\beta^b \beta_b}} \rightarrow \beta^a = \check{\beta}\check{u}^a$

Non-relativistic (Brenner), multicomponent.

Do we have a choice?

Causality

- infinite speed of signal propagation
- second order time derivatives
- hyperbolic system of equations

motivation of Extended Irreversible Thermodynamics (L. García-Colín)

Divergence type theories – finite speed is material

(Liu-Ruggeri-Müller, Geroch, Lindblom, Calzetta)

Physical:

Propagation speed of *continuum limit*.

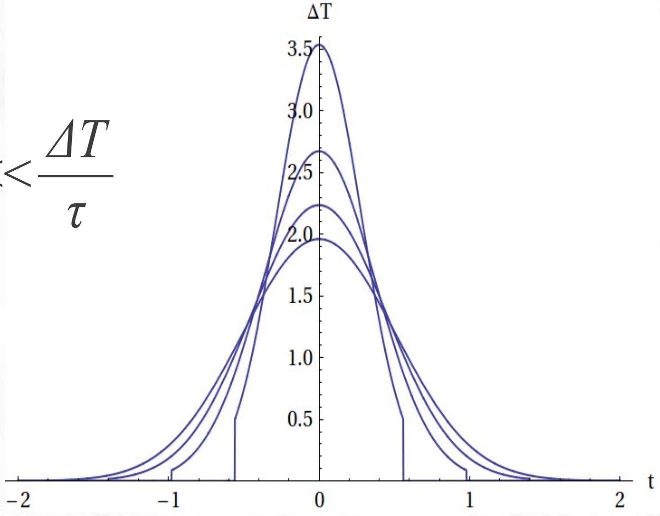
Propagation speed of observable signals.

Example:

$$\partial_t T = -\kappa \partial_x^2 T \qquad \partial_x T << \frac{\Delta T}{\xi}, \quad \partial_t T << \frac{\Delta T}{\tau}$$

water at room temperature:

$$v_{max} \approx \frac{\kappa}{\xi} = 14\, m/s$$



Stability conditions of the Israel-Stewart theory

(Hiscock-Lindblom 1983,87)

$$\Omega_1 = \frac{1}{e+p}\frac{\partial e}{\partial p}\Big|_{\frac{s}{n}} = \frac{T}{(e+p)\frac{\partial p}{\partial e}\Big|_n - n\frac{\partial p}{\partial n}\Big|_e} \geq 0,$$

$$\Omega_2 = \frac{1}{e+p}\frac{\partial e}{\partial(s/n)}\Big|_p \frac{\partial p}{\partial(s/n)}\Big|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

Conditons for the
- EOS
- IS coefficients
- both

\+ usual

$$\Omega_5 = \beta_0 \geq 0, \qquad \Omega_8 = \beta_2 \geq 0, \qquad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \qquad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2 T}\frac{\partial T}{\partial(s/n)}\Big|_n \geq 0,$$

$$\Omega_3 = (e+p)\left(1 - \frac{\partial p}{\partial e}\Big|_{\frac{s}{n}}\right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0, \qquad K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\frac{\partial T}{\partial n}\Big|_{s/n} \geq 0.$$

Stability conditions of the Israel-Stewart theory

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$$\Omega_1 = \frac{1}{e+p}\frac{\partial e}{\partial p}\Big|_{\frac{s}{n}} = \frac{T}{(e+p)\frac{\partial p}{\partial e}\Big|_n - n\frac{\partial p}{\partial n}\Big|_e} \geq 0,$$

Conditons for the
- EOS
- IS coefficients
- both

$$\Omega_2 = \frac{1}{e+p}\frac{\partial e}{\partial (s/n)}\Big|_p \frac{\partial p}{\partial (s/n)}\Big|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

+ usual

Eckart frame

$$\Omega_5 = \beta_0 \geq 0, \quad \Omega_8 = \beta_2 \geq 0, \quad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \quad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2 T}\frac{\partial T}{\partial (s/n)}\Big|_n \geq 0,$$

$$\Omega_3 = (e+p)\left(1 - \frac{\partial p}{\partial e}\Big|_{\frac{s}{n}}\right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0, \quad K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\frac{\partial T}{\partial n}\Big|_{s/n} \geq 0.$$

Thermodynamic relations – normalization

$$f_0(x,p)=e^{\alpha(x)-\beta_b(x)p^b}$$

$$N_0^a=\int p^a f_0$$

$$T_0^{ab}=\int p^b p^a f_0$$

Jüttner distribution AND Jüttner flow-frame

$$\alpha=\frac{\check{\mu}}{\check{T}},\ \beta_a=\frac{\check{u}_a}{\check{T}} \qquad f_0(x,p)=e^{\frac{\check{\mu}-\check{u}_b p^b}{\check{T}}}$$

When calculated frame independently, one obtains:

$$\beta^a=\beta(u^a+w^a) \qquad \check{u}^a=\frac{u^a+w^a}{\sqrt{1-w^2}} \qquad \check{T}=\frac{T}{\sqrt{1-w^2}},\ \check{\mu}=\frac{\mu}{\sqrt{1-w^2}},$$

$$f_0(x,p)=e^{\alpha-\beta_b p^b}=e^{\frac{\mu-(u_b+w_b)p^b}{T}}=e^{\frac{\check{\mu}-\check{u}_b p^b}{\check{T}}}$$

Energy-momentum density:

$$N_0^a = \int p^a f_0 = \check{n}\check{u}^a = nu^a + nw^a$$

$$\check{n} = n\sqrt{1-w^2} = 4\pi m^2 \check{T} K_2\left(\frac{m}{\check{T}}\right) e^{\frac{\check{\mu}}{\check{T}}}$$

$$T_0^{ab} = \int p^a p^b f_0 = \check{e}\check{u}^b\check{u}^a + \check{p}\check{\Delta}^{ab}$$

$$\check{e} = 3\check{n}\check{T} + m\check{n}\frac{K_1\left(\frac{m}{\check{T}}\right)}{K_2\left(\frac{m}{\check{T}}\right)}$$

$$T_0^{ab} = eu^b u^a + q^b u^a + u^b q^a - p\Delta^{ab} + \frac{q^b q^a}{e+p}$$

$$\check{p} = p, \quad e = \frac{\check{e} + p w^2}{1-w^2}, \quad q^a = (e+p) w^a$$

Heat flux:

$$I^a = q^a - \frac{e+p}{n} j^a = 0$$

Thermodynamics in arbitrary frames:

$$S^a + \alpha N^a - \beta_b T^{ab} = \Phi^a$$ objective/covariant starting point

$$\beta^a = \beta\left(u^a + w^a\right)$$

temperature vector

Constrained entropy inequality:

$$0 \leq \partial_a S^a + \alpha \partial_a N^a - \beta_b \partial_a T^{ab} = - N^a \partial_a \alpha - T^{ab} \partial_a \beta_b + \partial_a \Phi^a =$$

$$- n\dot{\alpha} + h\dot{\beta} + q^a\left(\beta \dot{w}_a\right) + \beta \dot{p} + \Pi^{ab} \partial_a u_b - j^a \partial_a \alpha + \ldots = \Sigma \geq 0$$

Thermodynamics:

a) $\Phi^a = p\beta^a$ matching $S_0^a + \alpha N_0^a - \beta_b T_0^{ab} = \beta^a p_0$

b) $ds + \alpha\, dn = \beta\left(de + w_a\, dq^a\right)$ $w^\mu = 0 \Rightarrow ds + \alpha dn = \beta de$

Entropy production:

$$0 \leq \Sigma = \left(nw^a - j^a\right)\partial_\mu \alpha + \left(q^a - hw^a\right)\left(\partial_a \beta + \beta \dot{u}_a\right) + \left(\Pi^{ab} - w^{(a}q^{b)}\right)\partial_a \beta_b + w^{(a}q^{b)}\partial_b\left(\beta\left(u_a - w_a\right)\right)$$

dynamic EOS: $\beta^a = \beta\left(u^a + w^a\right)$

thermometer dEOS/Jüttner: $w^a = 0$

$$ds + \alpha\, dn = \beta\, de$$

natural dEOS/Landau-Lifshitz: $w^a = q^a / e$

$$ds + \alpha\, dn = \beta_a dE^a = \beta\left(de + \frac{q_a}{e} dq^a\right) \quad s\left(E^a, n\right) = \hat{s}\left(E = -\sqrt{-E_a E^a}\right)$$

kinetic dEOS: $w^a = q^a / h$

$$ds + \alpha\, dn = \beta\left(de + \frac{q_a}{h} dq^a\right) \qquad s\left(E^a, u^a, n\right)$$

Thermodynamics defines local equilibrium and the flow frame.

Entropy production with kinetic dEOS:

$$\partial_a S^a = \Pi^{ab} \partial_b \beta_a + q^a \left(\partial_a \beta + \frac{\beta}{h} \left(h \dot{u}^a + \dot{q}^a + q^a \partial_b u^b + \partial_b \Pi^{ab} \right) \right) =$$

$$= \left(\Pi^{ab} - \frac{q^a q^b}{h} \right) \partial_a \beta_b \geqslant 0$$

$$\beta^a = \beta \left(u^a + w^a \right) = \tilde{\beta} \, \tilde{u}^a$$

$$\tilde{u}^a = \frac{\left(h u^a + q^a \right)}{\sqrt{h^2 - q^2}}$$

Prediction: heat conduction and viscosity are interdependent.

$$\partial_\mu S_0^\mu + \alpha \partial_\mu N_0^\mu - \beta_\nu \partial_\mu T_0^{\mu\nu} = 0$$

covariant Gibbs relation (Israel,1963)

Remark: $$\partial_\mu S^\mu + \alpha \partial_\mu N^\mu - \beta_\nu \partial_\mu T^{\mu\nu} = \sigma \geq 0$$

Lagrange multipliers – non-equilibrium

Rest frame quantities:

$$S^\mu = s u^\mu + J^\mu$$

$$N^\mu = n u^\mu + j^\mu$$

$$T^{\mu\nu} = u^\mu E^\nu + q^\mu u^\nu + P^{\mu\nu}$$

$$\boxed{E^\nu = e u^\nu + q^\nu}$$

$$u_\mu u^\mu = 1,\ \Delta^{\mu\nu} = \delta^{\mu\nu} - u^\mu u^\nu;$$

$$u_\mu J^\mu = 0, u_\mu j^\mu = 0;$$

$$u_\mu q^\mu = 0, u_\mu P^{\mu\nu} = P^{\mu\nu} u_\nu = 0.$$

$$\partial_\mu S^\mu + \alpha \partial_\mu N_0^\mu - \beta_\nu \partial_\mu T_0^{\mu\nu} =$$

$$\boxed{\dot{s} + \alpha \dot{n} - \beta_\mu \dot{E}^\mu} + \left(s + \alpha n - \beta_\mu E^\mu\right)\partial_\nu u^\nu -$$

$$\alpha \partial_\mu j^\mu + \beta_\nu \partial_\mu (q^\mu u^\nu + P^{\mu\nu}) = 0$$