Black hole horizon thermodynamics and Planck scale

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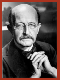
Outline of this talk

"The greatest enemy of knowledge is not ignorance, but the illusion of knowledge."

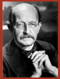


(Stephen William Hawking)

- Planck's idea of Planck scale
- Acceleration and energy variance
- Rindler trajectories in imaginary time
- Accelerated Doppler $\rightarrow$ Planck spectrum

Planck scale

from natural constants



We have **four** natural constants: $G, c, k, \hbar$.

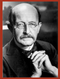
They connect:

1. c: length with time, and mass with energy
2. G: mass and length with energy
3. k: temperature with energy
4. $\hbar$: action scale: energy with time, momentum with length

try and catch: $M_P = \sqrt{\hbar c / G}$, $L_P = \sqrt{\hbar G / c^3}$

Planck scale

in a physical situation



In a Compton wavelength distance from source the Newton potential equals to the rest mass energy.

From this follows

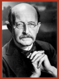
$$\frac{GMm}{r} = mc^2 \qquad \text{and} \qquad r = \frac{\hbar}{Mc} \tag{1}$$

This concludes again as

$$\frac{GM^2c}{\hbar} = c^2. \tag{2}$$

solution: $M = M_P = \sqrt{\hbar c/G}, \quad r = L_P = \sqrt{\hbar G/c^3}$

Planck's natural system of units



M. Planck, Über irreversible Strahlungsvorgänge, Sitz.Ber.Preuss.Akad.Wiss. 449-476 (1898)

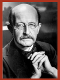
Wiens'law: $w = \frac{8\pi\nu^2}{c^3} b\nu \, \mathrm{e}^{-a\nu/T}$. From Planck's law limit: $b = h = 6.626 \times 10^{-27}$ erg sec, and $a = h/k = 4.798 \times 10^{-11}$ cm K.

1. length: $L_P = \sqrt{\hbar G/c^3} = 1.616 \times 10^{-33}$ cm
2. mass: $M_P = \sqrt{\hbar c/G} = 2.176 \times 10^{-5}$ g
3. time: $t_P = L_P/c = 5.392 \times 10^{-44}$ s
4. temperature: $T_P = M_P c^2/k = 1.417 \times 10^{32}$ K.

"These quantities preserve their natural meaning as long as the laws of gravitation, propagation of light in vacuum and both the two laws of heat theory remain valid, that is, being measured by most various intelligent beings unsing most different methods, they must always give the same value."

Hermitic operators

and their combinations



Let $A = A^\dagger$, $B = B^\dagger$ (be hermitic operators). For the sake of simplicity: $\langle A \rangle = \langle B \rangle = 0$.
Now $\Delta A^2 = \left\langle A^2 \right\rangle$ and $\Delta B^2 = \left\langle B^2 \right\rangle$.

We construct a combined (not hermitic) operator:

$$C \equiv \lambda A + \frac{i}{\lambda^*} B. \tag{3}$$

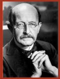
From this $C^\dagger = \lambda^* A - \frac{i}{\lambda} B$.

$$\begin{aligned} CC^\dagger &= |\lambda|^2 A^2 + iBA - iAB + \frac{1}{|\lambda|^2} B^2 \\ C^\dagger C &= |\lambda|^2 A^2 - iBA + iAB + \frac{1}{|\lambda|^2} B^2 \end{aligned} \tag{4}$$

$\left\langle CC^\dagger \right\rangle \geq 0$ **and** $\left\langle C^\dagger C \right\rangle \geq 0$.

An inequality

following from $\langle CC^\dagger \rangle \geq 0$ and $\langle C^\dagger C \rangle \geq 0$



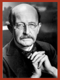
A consequence of eq.(4)

$$\frac{1}{2}\left(|\lambda|^2 \left\langle A^2 \right\rangle + \frac{1}{|\lambda|^2} \left\langle B^2 \right\rangle \right) \geq \left| \left\langle \frac{i}{2}[A, B] \right\rangle \right| \tag{5}$$

The minimum of the arithmetic mean is just the geometric one!

$$\Delta A \cdot \Delta B = \sqrt{\left\langle A^2 \right\rangle \left\langle B^2 \right\rangle} \geq \left| \left\langle \frac{i}{2}[A, B] \right\rangle \right| . \tag{6}$$

Minimal variance relations



Not "insecurity"

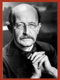
Note: $A_2 = A + a1$ and $B_2 = B + b1$ imply $[A_2, B_2] = [A, B]$ and $\Delta A_2^2 = \langle A_2^2 \rangle - \langle A_2 \rangle^2 = \langle A^2 \rangle$, same for B_2.

☺ The general result ☻

$$\Delta A \cdot \Delta B \geq \left| \left\langle \frac{i}{2} [A, B] \right\rangle \right| \tag{7}$$

- $\Delta x \cdot \Delta p \geq \left| \left\langle \frac{i}{2} \frac{\hbar}{i} \right\rangle \right| = \frac{\hbar}{2}.$ **Heisenberg**
- $\Delta E \cdot \Delta p \geq \left| \left\langle \frac{i}{2} [H, P] \right\rangle \right| = \frac{\hbar}{2} |\langle F \rangle|.$ **Force**
- $\Delta E \cdot \Delta x \geq \left| \left\langle \frac{i}{2} [H, Q] \right\rangle \right| = \frac{\hbar}{2} |\langle v \rangle|.$ **Velocity**

Quantum physics



lower bound for energy variance

For any pair of Hermitean opeartors: $\Delta A \cdot \Delta B \geq \left| \left\langle \frac{i}{2} [A, B] \right\rangle \right|.$

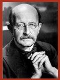
$\Delta P \cdot \Delta x \geq \frac{\hbar}{2}$ is well known.

$\boldsymbol{\Delta E \cdot \Delta P \geq \left| \frac{\hbar}{2} \langle F \rangle \right|}$ is *less* well known.

By constant g acceleration along a line:

$$\Delta E \cdot \Delta P \geq \frac{\hbar}{2} mg. \tag{8}$$

note: for $\langle F_i \rangle = 0$ motions the lower bound is zero.



Energy ”uncertainty”

photons in gravitational field

Use Schwarzschild metrics in weak field approximation:

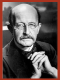
$$E = \hbar\omega \sqrt{1 - \frac{2GM}{c^2 r}} \approx \hbar\omega - \frac{GM}{r} \frac{\hbar\omega}{c^2}. \tag{9}$$

The weight of the photon is $m = \hbar\omega / c^2$ while its rest mass is zero. So $\langle E \rangle \approx \hbar\omega = mc^2$. Acceleration magnitude $g = GM/r^2$.

From dispersion relation $E = cP$; it holds to leading order that $\Delta E = c\Delta P$.

Our final result for a radially moving photon is ☺

$$\frac{\Delta E^2}{\langle E \rangle} \geq \frac{\hbar g}{2c}. \tag{10}$$



Constant acceleration on a line

in the comoving frame using $c = 1$ units

Velocity four-vector:

$$u^{\mu} = (\cosh\eta, \sinh\eta, 0, 0)\,. \tag{11}$$

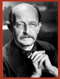
Acceleration four-vector:

$$\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}\eta}{\mathrm{d}\tau}\,(\sinh\eta, \cosh\eta, 0, 0)\,. \tag{12}$$

Its constant Minkowski-length be

$$\left\| \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} \right\|^2 = -\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2 = -g^2. \tag{13}$$

solution: $\eta = g\tau$.



Rindler trajectories

defined by constant comoving acceleration

One obtains

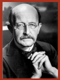
$$u^{\mu} = (\cosh(g\tau), \sinh(g\tau), 0, 0) \tag{14}$$

The Rindler trajectories are given in the $x^{\mu}(\tau)$ parametrization:

$$x^{\mu} = \left(\frac{1}{g} \sinh(g\tau), \frac{1}{g} (\cosh(g\tau) - 1), 0, 0 \right). \tag{15}$$

In the low acceleration limit we obtain **Galilei's** result:

$$x^{\mu} \approx \left(\tau, g \frac{\tau^2}{2}, 0, 0 \right). \tag{16}$$



Rindler trajectories in imaginary time

are periodic!

Consider $\tau = i\hbar\beta$ with $\beta = 1/kT$.

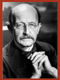
The Rindler trajectory becomes

$$x^{\mu} = \left(\frac{i}{g} \sin(\hbar\beta g), \frac{1}{g}(\cos(\hbar\beta g) - 1), 0, 0 \right). \tag{17}$$

Taken at the period, $g\tau = g(i\hbar\beta) = 2i\pi$, we determine the **Unruh** temperature:

Unruh temperature as an imaginary-time period ☺

$$1/\beta = kT = \frac{\hbar g}{2\pi c}. \tag{18}$$



Doppler effect

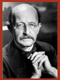
signals from moving source

Spectrum of a monochromatic source: $\delta(\nu - \omega)$

Spectrum of an inertially moving mono source: $\delta\left(\nu - \gamma(\omega - k \cdot v)\right)$

Spectrum of a free falling source on a line?

$$S(\nu) \ \sim \ \left| \mathcal{F}_{\tau}^{-1} \left(\mathrm{e}^{i(\omega t(\tau) - kx(\tau))} \right) \right|^2 . \tag{19}$$



Doppler effect

phase and amplitude along Rindler trajectories

The phase in the Fourier back-transformation: $\varphi = \omega(t - x)$ in $c = 1$ units.

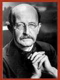
On a Rindler trajectory the retarded time:

$$t - x \;=\; \frac{1}{g}\left[1 - \mathrm{e}^{-g\tau}\right] \tag{20}$$

Here $g \to g/c$ is a frequency...

The Fourier amplitude becomes

$$A(\nu) \;=\; \mathrm{e}^{i\omega/g} \int\limits_{-\infty}^{+\infty} \mathrm{e}^{-i\frac{\omega}{g}\mathrm{e}^{-g\tau}}\, \mathrm{e}^{i\nu\tau}\, d\tau. \tag{21}$$



Doppler effect

integral over red-shift factor

The red-shift factor, $z = \frac{\mathrm{d}}{\mathrm{d}\tau}(t - x) = \mathrm{e}^{-g\tau}$ is a good variable.

Its limiting values are: $z(-\infty) = +\infty$ and $z(+\infty) = 0$. Differentials: $d\tau = -dz/(gz)$.

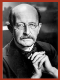
The complex amplitude becomes

$$A(\nu) \;=\; \mathrm{e}^{i\omega/g} \int\limits_0^\infty \frac{dz}{gz}\, \mathrm{e}^{-i\omega z/g}\, \mathrm{e}^{i\nu(-\frac{1}{g}\ln z)} \;=\; \frac{1}{g}\, \mathrm{e}^{i\omega/g} \left(i\frac{\omega}{g}\right)^{i\nu/g} \Gamma\left(-i\frac{\nu}{g}\right). \qquad (22)$$

Important: $i^{\,i\frac{\nu}{g}} \;=\; \mathrm{e}^{i\frac{\pi}{2}\cdot\frac{i\nu}{g}} \;=\; \mathrm{e}^{-\frac{\pi\nu}{2g}}$

Doppler effect

the observed intensity



With fixed sign of g there is **no time reversal**: $A(-\nu) \neq A(\nu)^*$.

The intensity:

$$|A(\nu)|^2 = \frac{1}{g^2}\, \mathrm{e}^{-\pi\frac{\nu}{g}}\, \Gamma\left(i\frac{\nu}{g}\right) \Gamma\left(-i\frac{\nu}{g}\right). \tag{23}$$

A property of Gamma functions:

$$\Gamma(ix)\,\Gamma(-ix) = \frac{1}{(-ix)}\,\Gamma(ix)\,\Gamma(1-ix) = \frac{i}{x}\,\frac{\pi}{i\sinh(\pi x)} = \frac{\pi}{x\sinh(\pi x)}. \tag{24}$$

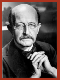
leads to

$$N(\nu) \equiv \frac{g\nu}{2\pi}\,|A(\nu)|^2 = \frac{1}{\mathrm{e}^{2\pi\nu/g} - 1}. \tag{25}$$

Compare with Planck's law: $kT = g\hbar/(2\pi c)$.

Time reversal

KMS relation



What about negative frequencies ("heated vacuum")?

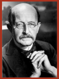
$$\frac{|A(-\nu)|^2}{|A(+\nu)|^2} = e^{\frac{2\pi\nu c}{g}} \tag{26}$$

Kubo-Martin-Schwinger for the numbers:

$$\frac{-N(-\nu)}{N(\nu)} = e^{\frac{2\pi\nu c}{g}} = 1 + \frac{1}{N(\nu)}. \tag{27}$$

KMS interpretation ☺

$$-N(-\nu) = 1 + N(\nu). \tag{28}$$



Pseudo-thermalization? Pseudo-Hydro?

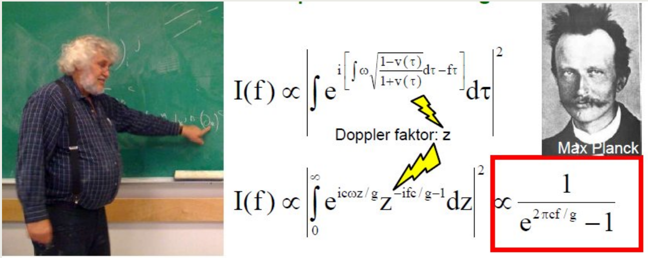
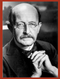


Figure: Unruh effect (1975), Hawking radiation (1975)

$$v = \tanh \xi, \quad u_i = (\cosh \xi, \sinh \xi), \quad a_i = (\sinh \xi, \cosh \xi) \frac{d\xi}{d\tau}$$

$$a_i a^i = -g^2, \quad \xi = \xi_0 - g\tau; \quad z = e^{-\xi}, \quad g\, d\tau = \frac{dz}{z}.$$



Near to the horizon

simplified treatment!

Spacetime of static, spherically symmetric black holes, $c = 1$

$$ds^2 \;=\; f(r)dt^2 - \frac{dr^2}{f(r)}$$

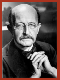
Horizon location: $f(R) = 0$. Expansion: $f(r) = f(R) + (r - R)f'(R) + \ldots$.

New variables 1.: $\varepsilon = r - R$, so $f(r) = \varepsilon f'(R) + \ldots$.
Use $dz^2 \;=\; dr^2/f(r) \;=\; d\varepsilon^2/\varepsilon f'(R)$.

$$\text{Solves :} \qquad z \;=\; \int \frac{d\varepsilon}{\sqrt{\varepsilon\, f'(R)}} \;=\; \frac{2}{\sqrt{f'(R)}}\,\sqrt{\varepsilon}.$$

Based on this $f(r) = \varepsilon f'(R) \;=\; f'(R)^2 z^2/4$ and

$$ds^2 \;=\; \frac{1}{4} f'(R)^2\, z^2 dt^2 \;-\; dz^2 \;=\; g^2 z^2 dt^2 \;-\; dz^2.$$



Near to the horizon

simplified treatment!

This spacetime looks as having hiperbolic "polar" coordinates.

New variables 2: $x^0 = z\sinh(gt)$, $x^1 = z\cosh(gt)$. With these

$$ds^2 = g^2 z^2 dt^2 - dz^2 = dx_0^2 - dx_1^2.$$

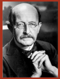
$g = c^2 f'(R)/2$ acceleration, g/c frequency.

Imaginary time: quantum solution e^{itH}, thermodynamics $\mathrm{e}^{-\beta H}$. Period: $g\tau/c = 2\pi$

The Hawking temperature for $f(r) = 1 - R/r$, with $R = 2GM/c^2$:

$$k_B T = \frac{\hbar g}{2\pi c} = \frac{\hbar c}{4\pi} f'(R) = \frac{\hbar c^3}{8\pi GM}.$$

Hawking temperature → Bekenstein entropy



and the Planck scale

From the definition of entropy ($k_B = 1$):

$$S = c^2 \int \frac{dM}{T} = \frac{8\pi G}{\hbar c} \int M dM = \frac{4\pi G}{\hbar c} M^2 = \frac{c^3}{G\hbar} \pi R^2.$$

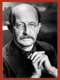
Planck length and Planck mass:

$$L_P \cdot M_P = \hbar / c, \qquad L_P / M_P = G / c^2.$$

The horizon is a spherical surface, $A = 4\pi R^2$, therefore

$$S = \frac{1}{4} \frac{A}{L_P^2}.$$

Black Hole Entropy = Horizon Area / 4, in Planck units.



A little problem with the HB entropy

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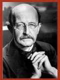
$$S(E) \sim A \sim R^2 \sim M^2 \sim E^2$$

Wrong e.o.s. curvature: $S''(E) > 0$, one needs $S''(E) < 0$ for stable thermo.

"Re-make": $S(E, V)$

But: no hair theorem $\Rightarrow$ may depend only on $E = M$.

Question: *does another variable, $V(M)$, exist*?



Invariant volume

Christodoulou – Rovelli

A radiating black hole $\to$ evaporates.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\sigma T^4 \cdot A$$

Especially: $A \sim M^2$, $T \sim 1/M$, $\to$

$$\frac{\mathrm{d}M}{\mathrm{d}t} \sim -\frac{1}{M^2}.$$

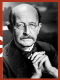
Solution: $M^3(t) = M^3(0) - \alpha t$.

Time for total evaporation from M to 0: $t = M^3/\alpha$.

The invariant volume:

$$V \sim t \cdot A \sim M^5 \sim R^5.$$

Black hole thermodynamics with two variables



advantages and a macula

Ansatz: $S(E,V) \; = \; kE^a V^b \; \sim \; M^2$ whenceforth $a+5b=2$ (holographic principle).

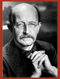
Temperature and pressure: the derivatives:

$$\begin{aligned} \frac{\partial S}{\partial E} &= akE^{a-1}V^b = \frac{1}{T}, \\ \frac{\partial S}{\partial V} &= bkE^a V^{b-1} = \frac{p}{T}. \end{aligned}$$

From here the pressure $p \; = \; \frac{b}{a}\frac{E}{V}$, so $p \; = \; b\,\varepsilon/a$. For the Hawking radiation $b/a = 1/3$.

This leads to the solution $\boxed{a = 3/4, b = 1/4}$.

Euler homogenity ($a+b=1$), stability, but $S = 8S_{BH}/3$.



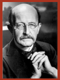
Is the Unruh temperature measurable?

Unruh temperature in Planck units: $\boxed{T = \frac{g}{2\pi}}$

Unruh temperature in ordinary units: $\boxed{k_B T = \frac{\hbar}{c} \frac{g}{2\pi}}$

Small for Newtonian gravity: $g = GM/R^2$, therefore $k_B T = Mc^2/2\pi \cdot L_P^2/R^2$. On Earth' surface we have $k_B T \approx 10^{-19}$ eV ($10^{-16}\times$ room temperature) .

Perceivable for heavy ion collisions: $g = c^2/L = mc^3/\hbar$ for stopping in a Compton wavelength. For a proton of mass $m = 940$ MeV we have $k_B T = mc^2/2\pi \approx 150$ MeV.



Quantum uncertainty may hindrance

the mesurement of Unruh temperature

We obtained the quantum variance limnit for photons moving in constant force field:

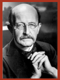
$$\frac{\Delta E^2}{\langle E \rangle} \geq \frac{\hbar g}{2c} = \pi \, T_{\rm Unruh}. \tag{29}$$

On the other hand a thermal (Bose distributed, approximately Boltzmann) spectrum delivers

$$\frac{\Delta E^2}{\langle E \rangle} \approx T. \tag{30}$$

Can one ever see this thermal variance?

Summary



- Planck scale is physical
- Occurs in energy uncertainty for accelerated photons
- Sets the imaginary time period in BH physics
- Occurs as ”temperature” due to smeared Doppler effect