Wigner's theorem on quantum mechanical symmetry transformations

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1. The Stern–Gerlach experiment
2. A mathematical model for the S–G experiment
3. Metric resolving sets
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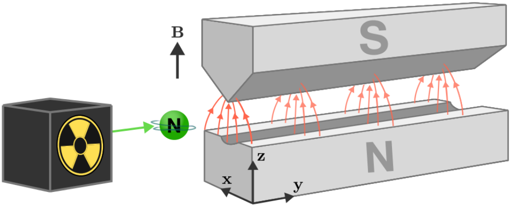
The Stern–Gerlach experiment

• A neutron exiting from a hot nuclear oven would follow a straight trajectory at constant speed, unless some force act on it.

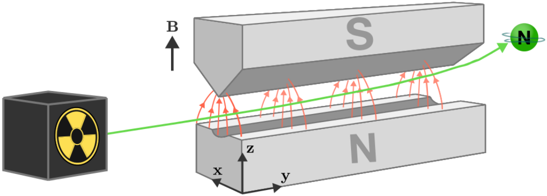
• However, here we arranged two permanent magnets of opposite polarity in order to generate a magnetic field B which is perpendicular to the neutron's path.



• How do we expect the magnetic field should deflect the path of the neutron?

We would expect no deflection.

But this is not what happens!



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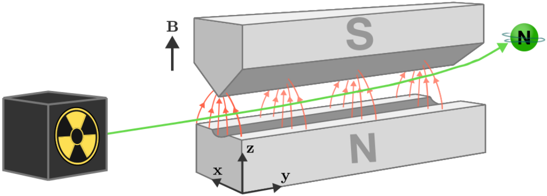
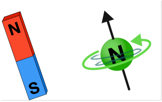
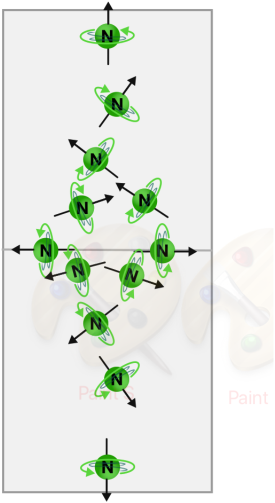


Uncharged objects can sometimes have magnetic moments, e.g. if they are spinning around an axis.

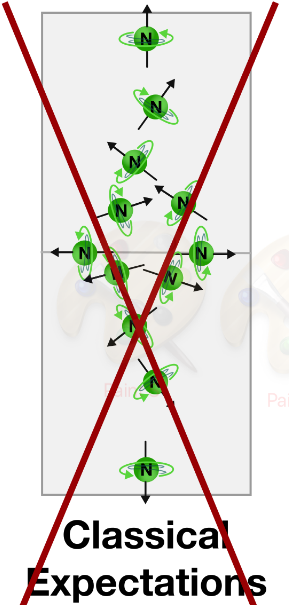


Classical expectation vs. What really happens



Classical Expectations

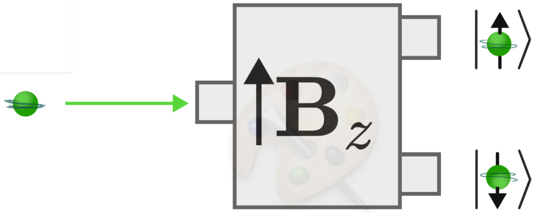
Classical expectation vs. What really happens



Classical expectation vs. What really happens

Quantum Surprises

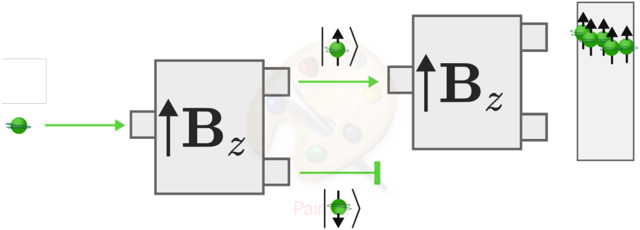
The SG Analyser



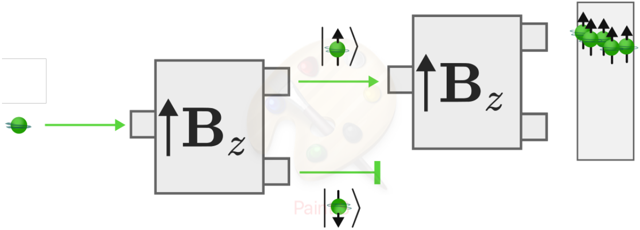
Input

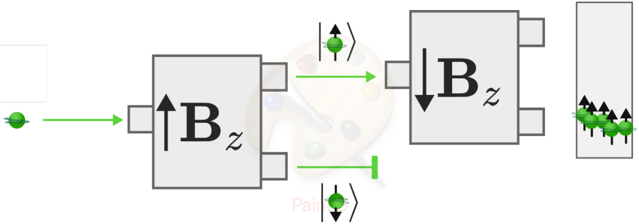
Possible Outputs

Two SG Analyser with parallel magnetic fields

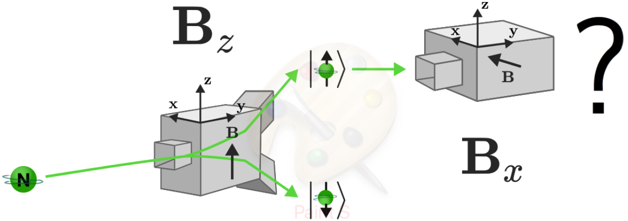


Two SG Analyser with parallel magnetic fields

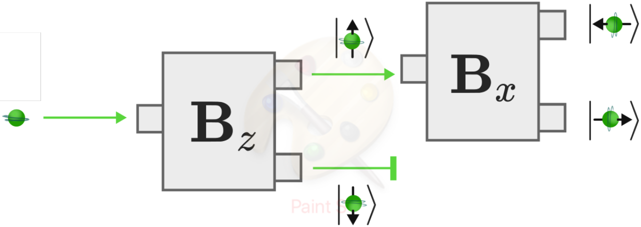


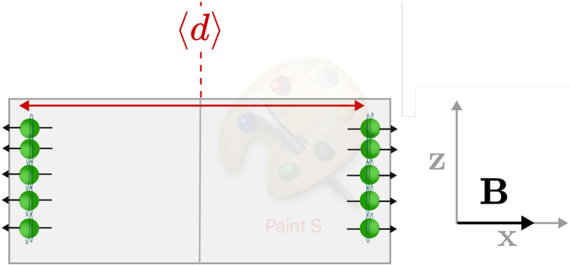


Two SG Analyser with perpendicular magnetic fields



Two SG Analyser with perpendicular magnetic fields





Two SG Analyser with one magnetic field along z, and another with angle θ

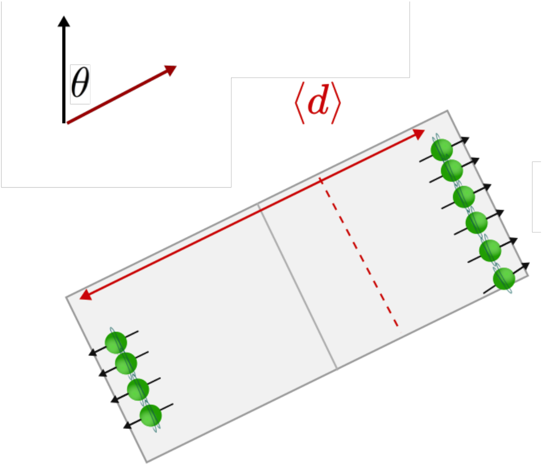


Figure: The empirical distribution is roughly $\{\sin^2\frac{\theta}{2}, \cos^2\frac{\theta}{2}\}$.

A mathematical model for the Stern–Gerlach experiment

• $\mathbb{C}^2$ – two-dimensional complex Hilbert space.

• $[\vec{u}]$ – the one-dimensional subspace generated by $\vec{u}$, where we implicitly assume that $\|\vec{u}\| = 1$.

• $Proj(\mathbb{C}^2)$ – Projective space over $\mathbb{C}^2$, i.e.

$$Proj(\mathbb{C}^2) = \{[\vec{u}] \colon \|\vec{u}\| = 1\}$$

This is the set of all *possible spin states*.

• $\mathbf{P}[\vec{u}]$ – the rank-one projection with range $[\vec{u}]$.

• $\mathcal{P}_1(\mathbb{C}^2)$ – the set of all rank-one projections, i.e.

$$\mathcal{P}_1(\mathbb{C}^2) = \{\mathbf{P}[\vec{u}] \colon \|\vec{u}\| = 1\}.$$

Of course, there is a natural one-one correspondence:

$$\mathcal{P}_1(\mathbb{C}^2) \ni \mathbf{P}[\vec{u}] \quad \longleftrightarrow \quad [\vec{u}] \in Proj(\mathbb{C}^2).$$

• If the neutron has spin *up* (↑) or *down* (↓) along the z direction, then its state is

$$\mathbf{P}[(1,0)] \text{ or } \mathbf{P}[(0,1)], \text{ respectively.}$$

• If the spin is *right* (→) or *left* (←) along the x direction, then its state is

$$\mathbf{P}\left[\tfrac{1}{\sqrt{2}}(1,1)\right] \text{ or } \mathbf{P}\left[\tfrac{1}{\sqrt{2}}(1,-1)\right], \text{ respectively.}$$

• If the spin is *out* (·) or *in* (×) in the y direction, then its state is

$$\mathbf{P}\left[\tfrac{1}{\sqrt{2}}(1,i)\right] \text{ or } \mathbf{P}\left[\tfrac{1}{\sqrt{2}}(1,-i)\right], \text{ respectively.}$$

Note that the above pairs are orthogonal pairs, and that the angle between e.g. the *up* (↑) and *out* (·) states is precisely $\frac{\pi}{4}$, since

$$\left|\left\langle (1,0); \tfrac{1}{\sqrt{2}}(1,i)\right\rangle\right| = \cos\tfrac{\pi}{4}.$$

• **Bloch representation:** in general, if the spin points into the $(\sin 2\theta \cos\nu, \sin 2\theta \sin\nu, \cos 2\theta)$ direction, then its state is

$$\mathbf{P}\left[\left(\cos\theta, e^{i\nu}\sin\theta\right)\right].$$

Notice that this has the following angle-doubling property:

$$\sphericalangle\Big((\sin 2\theta_1 \cos\nu_1, \sin 2\theta_1 \sin\nu_1, \cos 2\theta_1); \\ (\sin 2\theta_2 \cos\nu_2, \sin 2\theta_2 \sin\nu_2, \cos 2\theta_2)\Big) \\ = 2 \cdot \measuredangle\Big(\left[\left(\cos\theta_1, e^{i\nu_1}\sin\theta_1\right)\right]; \left[\left(\cos\theta_2, e^{i\nu_2}\sin\theta_2\right)\right]\Big).$$

• If the spin is prepared in the state $\mathbf{P}[\vec{u}]$ and we *measure* the spin in the direction $\mathbf{P}[\vec{v}]$, then the original state changes, namely, there are two possible outcomes:

1. either it changes to state $\mathbf{P}[\vec{v}]$ with probability $|\langle \vec{u}, \vec{v} \rangle|^2$,
2. or it changes to state $I - \mathbf{P}[\vec{v}]$ with probability $1 - |\langle \vec{u}, \vec{v} \rangle|^2$.

• $\mathrm{Tr}\, \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}]$ – transition probability

An easy calculation gives the following:

$$\mathrm{Tr}\, \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}] = |\langle \vec{u}, \vec{v} \rangle|^2 = 1 - \|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\|^2$$

for all $\|\vec{u}\| = \|\vec{v}\| = 1$, hence the transition probability is expressed a function of a very natural metric: the operator norm.

More generally

- H – a complex Hilbert space.

- $\mathcal{P}_1 := \mathcal{P}_1(\mathcal{H})$ – the set of all rank-one projections, which corresponds to the set of all *pure quantum states*.

- $\mathbf{P}[\vec{u}]$ – the rank-one projection with range $[\vec{u}]$, where $\|\vec{u}\| = 1$ is implicitly assumed.

- $\operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}]$ – transition probability, i.e. if our quantum system is in state $\mathbf{P}[\vec{u}]$, and we make a measurement whether it is in the state $\mathbf{P}[\vec{v}]$, then
  1. either it changes to state $\mathbf{P}[\vec{v}]$ with probability $\operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}]$,
  2. or it changes to state $\mathbf{P}[\vec{w}]$ with probability $1 - \operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}]$, where $\vec{w}$ is the unit vector that is orthogonal to $\vec{v}$ and lies in the subspace spanned by $\vec{u}, \vec{v}$.

Easy calculation gives

$$\operatorname{Tr} \mathbf{P}[\vec{u}]\mathbf{P}[\vec{v}] = |\langle \vec{u}, \vec{v} \rangle|^2 = 1 - \|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\|^2.$$

Theorem (E.P. Wigner, 1932(!); 1963–1964)

Let $\varphi\colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H})$ *be a bijective map such that*

$$\operatorname{Tr}\left(\mathbf{P}[\vec{u}] \cdot \mathbf{P}[\vec{v}]\right) = \operatorname{Tr}\left(\varphi(\mathbf{P}[\vec{u}]) \cdot \varphi(\mathbf{P}[\vec{v}])\right) \quad (\|\vec{u}\| = \|\vec{v}\| = 1). \quad \text{(W)}$$

Then there is a unitary or antiunitary operator $\mathbf{U}\colon \mathcal{H} \to \mathcal{H}$ *such that*

$$\varphi(\mathbf{P}[\vec{u}]) = \mathbf{U} \cdot \mathbf{P}[\vec{u}] \cdot \mathbf{U}^* = \mathbf{P}[\mathbf{U}\vec{u}] \qquad (\|\vec{u}\| = 1).$$

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- In fact, (W) is an **isometriness** condition:

$$\|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\| = \|\varphi(\mathbf{P}[\vec{u}]) - \varphi(\mathbf{P}[\vec{v}])\| \qquad (\|\vec{u}\| = \|\vec{v}\| = 1).$$

- Re-phrasing for **vectors**: If $\phi\colon H \to H$ satisfies

$$|\langle\vec{u},\vec{v}\rangle| = |\langle\phi(\vec{u}),\phi(\vec{v})\rangle| \quad (\vec{u},\vec{v}\in H),$$

then we have

$$\phi(\vec{u}) = \tau(\vec{u})\cdot\mathbf{U}\vec{u} \qquad (\vec{u}\in H)$$

where $\tau\colon H\to\mathbb{C}$, $|\tau(\vec{u})| = 1$ $(\vec{u}\in H)$.

• Let us emphasise that (W) is the only property we assume about φ, so it is **not** assumed that there is an underlying linear or antilinear map which generates φ, this is a *consequence*.

• Wigner's theorem is one of the important steps towards obtaining the general Schrödinger equation:

$$H\vec{v}(t) = i\frac{d}{dt}\vec{v}(t).$$

Very much recommended paper:

B. Simon, Quantum dynamics: from automorphism to Hamiltonian, *Studies in Mathematical Physics, Essays in honor of Valentine Bargmann*, eds. E.H. Lieb, B. Simon, A.S. Wightman, Princeton Series in Physics, Princeton University Press, Princeton, 327–349, 1976.

freely available from:
http://www.math.caltech.edu/SimonPapers/R12.pdf

The theorem which we will prove

Theorem (E.P. Wigner, non–bijective)

Let $\varphi\colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H})$ *be an isometry, i.e.*

$$\left\|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\right\| = \left\|\varphi(\mathbf{P}[\vec{u}]) - \varphi(\mathbf{P}[\vec{v}])\right\| \qquad (\|\vec{u}\| = \|\vec{v}\| = 1). \quad \text{(W)}$$

Then there is a linear or antilinear isometry $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ *such that*

$$\varphi(\mathbf{P}[\vec{u}]) = \mathbf{W} \cdot \mathbf{P}[\vec{u}] \cdot \mathbf{W}^* = \mathbf{P}[\mathbf{W}\vec{u}] \qquad (\|\vec{u}\| = 1).$$

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In the sequel it will be very important to keep in mind the following:
If $\varphi(\mathbf{P}[\vec{u}]) = \mathbf{P}[\vec{a}]$ and $\varphi(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{b}]$, then we have

$$|\langle \vec{u}, \vec{v} \rangle| = |\langle \vec{a}, \vec{b} \rangle|.$$

Metric resolving sets

Definition

Let (X, d) be a metric space and $D, R \subseteq X$. We say that R is a **resolving set** for D if for any two points $x_1, x_2 \in D$ whenever

$$d(x_1, y) = d(x_2, y) \qquad (\forall\, y \in R)$$

is satisfied, then

$$x_1 = x_2.$$

- Note that R does not have to be a subset of D.

- Throughout this talk we will assume that $\underline{\dim \mathcal{H} = \aleph_0}$. Fix an ONB: $\{\vec{e}_j\}_{j=1}^{\infty}$. For $j \in \mathbb{N}$ and $\vec{v} \in H$, $\|\vec{v}\| = 1$ we set

$$v_j := \langle \vec{v}, \vec{e}_j \rangle.$$

- The set

$$D := \{\mathbf{P}[\vec{v}] \colon v_j \neq 0,\ \forall\, j\}$$

is clearly **dense** in $\mathcal{P}_1(\mathcal{H})$ with respect to the operator norm.

Lemma

The set

$$R = \{\mathbf{P}[\vec{e}_j]\}_{j=1}^{\infty} \bigcup \left\{ \mathbf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right], \mathbf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right] \right\}_{j=1}^{\infty}$$

resolves D.

- Observe that $R \cap D = \emptyset$.

Proof of Wigner's theorem

in the separable infinite dimensional case

Based on the following paper:
Gy. P. Gehér, An elementary proof for the non-bijective version of Wigner's theorem, *Phys. Lett. A* **378** (2014), 2054–2057.

There is an ONS $\{\vec{f_j}\}_{j=1}^{\infty}$ such that

$$\mathbf{P}[\vec{f_j}] = \varphi(\mathbf{P}[\vec{e_j}]) \qquad (\forall\, j).$$

Define

$$\mathcal{H}' := \vee\{\vec{f_j}\}_{j=1}^{\infty}.$$

$\mathrm{ran}\varphi \subseteq \mathcal{P}_1(\mathcal{H}')$: If we have $\varphi(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}]$, then

$$|v_j| = |\langle \vec{w}, \vec{f_j} \rangle| \qquad (\forall\, j)$$

thus, by Parseval's identity $\vec{w} \in \mathcal{H}'$ and

$$\varphi(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H}').$$

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$$\varphi(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H}').$$

We modify φ so that each $\mathbf{P}[\vec{e}_j]$ is fixed:

Define the following linear isometry:

$$\mathbf{V}\colon \mathcal{H} \to \mathcal{H}' \subseteq \mathcal{H}, \quad \mathbf{V}\vec{e}_j = \vec{f}_j \qquad (\forall\, j).$$

The map $\varphi_1(\cdot) := \mathbf{V}^*\varphi(\cdot)\mathbf{V}$ obviously satisfies (W). Moreover,

$$\varphi_1(\mathbf{P}[\vec{e}_j]) = \mathbf{V}^*\varphi(\mathbf{P}[\vec{e}_j])\mathbf{V} = \mathbf{V}^*\mathbf{P}[\vec{f}_j]\mathbf{V} = \mathbf{P}[\mathbf{V}^*\vec{f}_j] = \mathbf{P}[\vec{e}_j].$$

Therefore

- $\varphi_1(\mathbf{P}[\vec{e_j}]) = \mathbf{P}[\vec{e_j}] \ \ (\forall\, j \in \mathbb{N})$
- $\varphi_1(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}] \Longrightarrow |v_j| = |w_j| \ \ (\forall\, j \in \mathbb{N})$;
- $\varphi_1(D) \subseteq D$;

Therefore

- $\varphi_1(\mathbf{P}[\vec{e}_j]) = \mathbf{P}[\vec{e}_j] \;\; (\forall\, j \in \mathbb{N})$
- $\varphi_1(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}] \implies |v_j| = |w_j| \;\; (\forall\, j \in \mathbb{N})$;
- $\varphi_1(D) \subseteq D$;

Also notice that $\exists \;\; |\delta_{j+1}| = |\varepsilon_{j+1}| = 1$ such that

$$\varphi_1\left(\mathbf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\frac{\vec{e}_j - \delta_{j+1}\,\vec{e}_{j+1}}{\sqrt{2}}\right]$$
$$\varphi_1\left(\mathbf{P}\left[\frac{\vec{e}_j + i\,\vec{e}_{j+1})}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\frac{\vec{e}_j + i\varepsilon_{j+1}\,\vec{e}_{j+1}}{\sqrt{2}}\right] \qquad (\forall\, j \in \mathbb{N}).$$

Applying (W) for the above yields $\sqrt{2} = |1 + i\delta_{j+1}\overline{\varepsilon_{j+1}}|$, and consequently,

$$\delta_{j+1} \in \{-\varepsilon_{j+1}, \varepsilon_{j+1}\} \qquad (\forall\, j \in \mathbb{N}).$$

We modify φ_1 so that every element of R stays fixed.

Define $\varphi_2(\cdot) := \mathbf{U}\varphi_1(\cdot)\mathbf{U}^*$, where

- if $\varepsilon_2 = \delta_2$, then $\mathbf{U}$ is the ***unitary*** operator with

$$\mathbf{U}\vec{e}_j = \overline{\Pi_{k=2}^{j}\delta_k} \cdot \vec{e}_j,$$

- if $\varepsilon_2 = -\delta_2$, then $\mathbf{U}$ is the ***antiunitary*** operator with

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$$\mathbf{U}\vec{e}_j = \Pi_{k=2}^{j}\delta_k \cdot \vec{e}_j.$$

In addition to the previous properties, ϕ_2 also satisfies

- $\varphi_2\left(\mathbf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right] \qquad (\forall\, j \in \mathbb{N});$
- $\varphi_2\left(\mathbf{P}\left[\frac{\vec{e}_1 + i\vec{e}_2}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\frac{\vec{e}_1 + i\vec{e}_2}{\sqrt{2}}\right] \qquad (j = 1);$
- $\varphi_2\left(\mathbf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right) \in \left\{\mathbf{P}\left[\frac{\vec{e}_j - i\vec{e}_{j+1}}{\sqrt{2}}\right], \mathbf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right\} \quad (j > 1).$

$\varphi_2\big|_R = \mathrm{Id}_R$:

Assume otherwise, then there exists a **first** $j > 1$ such that

$$\varphi_2\left(\mathbf{P}\left[\tfrac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\tfrac{\vec{e}_j - i\vec{e}_{j+1}}{\sqrt{2}}\right]$$

Claim: Then we have

$$\varphi_2(\mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + v_{j+1}\vec{e}_{j+1}]) = \mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + \overline{v_{j+1}}\vec{e}_{j+1}]$$

for all $t > 0, v_{j-1} \neq 0, v_{j+1} \neq 0, |v_{j-1}|^2 + t^2 + |v_{j+1}|^2 = 1.$

Proof: it is a rather easy calculation. □

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But this is a contradiction, since if

$$\vec{x} = \frac{-1}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{1}{\sqrt{2}}\vec{e}_{j+1}, \quad \vec{y} = \frac{i}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{i}{\sqrt{2}}\vec{e}_{j+1},$$

then

$$\sqrt{2}/4 = |i/4 + 1/4 - i/2| = |i/4 + 1/4 + i/2| = \sqrt{10}/4.$$

Therefore indeed φ_2 is the identity mapping on R, hence on D, and therefore on $\mathcal{P}_1$, and we easily calculate

$$\varphi(\mathbf{P}[\vec{u}]) = \mathbf{W}\mathbf{P}[\vec{u}]\mathbf{W}^*$$

where $\mathbf{W} = \mathbf{V}\mathbf{U}^*$. $\square$

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Some remarks:

- The finite dimensional case can be proved in a very similar way, even with some simplifications.
- The non-separable case can be proven as a consequence of the separable case (technical).
- A similar, but somewhat simpler proof works for the real case.