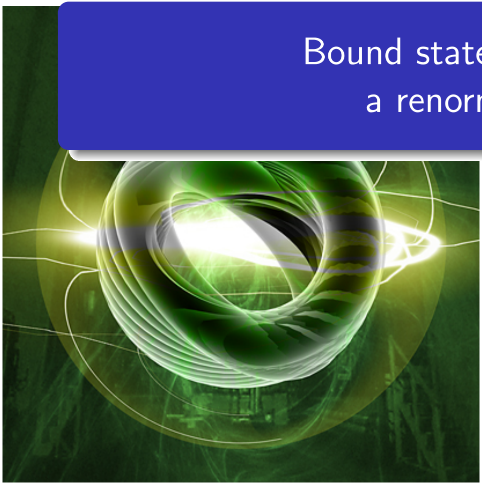
Bound states in quantum field theory: a renormalization group study



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AJ, A. Patkós, arXiv:1811.09418
AJ, A. Patkós, arXiv:1902.06492

Outlines

1. Introduction
2. Equations for the relativistic bound states
3. Computing the scattering amplitude
   - Dyson-Schwinger resummation
   - Functional Renormalization Group (FRG) equations
4. Bound states in NJL model
5. Conclusions

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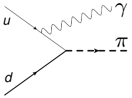
What are bound states?

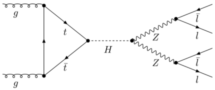
- Nonrelativistic case: Schrödinger-equation with $\hat{H}$ Hamilton-operator

$$\hat{H}\Psi = E\Psi$$

  Interpretation: $\Psi(\mathbf{x}_1, \ldots \mathbf{x}_n)$ wave function of the constituents, E is the energy of the bound state (discrete spectrum)

- Particle physics: Feynman diagrams for particle creation





  Interpretation: bound state is a particle with definite mass (and eventually lifetime)

What is the common language of atomic and particle physics?

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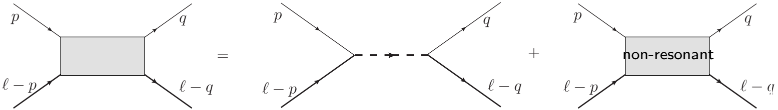
Poles of the scattering amplitudes

Subdiagram: long lifetime $\equiv$ **pole behaviour** of the scattering amplitude

$\Rightarrow$ 2-fermion $\psi + \chi \to \psi + \chi$ scattering



Mathematically: effective (1PI) action for the fermions, integrating out all other fields

$$\Gamma_{4-fermion} = \int_p \left[\psi_p^\dagger \mathcal{K}_p^{(\psi)} \psi_p + \chi_p^\dagger \mathcal{K}_p^{(\chi)} \chi_p\right] + \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \lambda^\ell_{p\alpha\gamma,q\beta\sigma} \psi^\dagger_{p\alpha} \psi_{q\beta} \chi^\dagger_{\ell-p,\gamma} \chi_{\ell-q,\sigma},$$

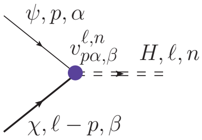
where λ is the 2-2 scattering amplitude. Bound states are poles of λ.

We should use hadrons, not quark scattering amplitudes...

Ansatz:

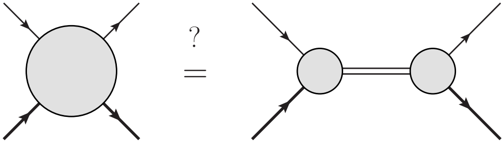
$$\Gamma_{eff} = \int_p \left[\psi_p^\dagger \mathcal{K}_p^{(\psi)} \psi_p + \chi_p^\dagger \mathcal{K}_p^{(\chi)} \chi_p + \sum_n H_{pn}^\dagger \mathcal{K}_{pn}^{(H)} H_{pn}\right] + \\ + \sum_n \int_{p\ell} \left[H_{\ell n} \psi_{p\alpha}^\dagger v_{p\alpha\beta}^{\ell n} \chi_{\ell-p,\beta}^* + H_{\ell n}^\dagger \chi_{\ell-p,\alpha}^T v_{p\alpha\beta}^{\dagger \ell n} \psi_{p,\beta}\right].$$

- original fermion fields
- many auxiliary fields $H_{\ell n}$ with momentum ℓ
- Yukawa couplings $v_{p\alpha\beta}^{\ell n}$ (symmetries!)



Equivalence of 4-fermion amplitudes

to avoid double counting: the two actions should provide the same 4-fermion amplitude



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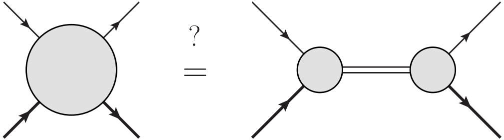

$$\Rightarrow \quad \lambda^{\ell}_{p\alpha\gamma,q\beta\sigma} = -\sum_n v^{\ell n}_{p\alpha\gamma} G^{(H)}_{\ell n} v^{\dagger \ell n}_{q\sigma\beta}$$

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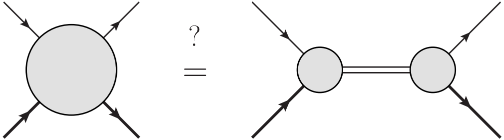

$$\Rightarrow \quad \lambda^{\ell}_{p\alpha\gamma,q\beta\sigma} = -\sum_n v^{\ell n}_{p\alpha\gamma} G^{(H)}_{\ell n} v^{\dagger \ell n}_{q\sigma\beta}$$

It is fulfilled if $v^{\ell n}_{p\alpha\gamma}$ is the "eigenvector" of $\lambda^{\ell}_{p\alpha\gamma,q\beta\sigma}$. Suppressing lower indices

$$\boxed{\boldsymbol{\lambda}^{\ell}\mathbf{v}^{\ell n} = -G^{(H)}_{\ell n}\mathbf{v}^{\ell n}}$$

formally equivalent to the **Schrödinger-equation**!

The relativistic analogue of the Schrödinger-equation

$$\boxed{\boldsymbol{\lambda}^{\ell}\mathbf{v}^{\ell n} = -G^{(H)}_{\ell n}\mathbf{v}^{\ell n}} \quad \text{or} \quad \boxed{(\boldsymbol{\lambda}^{\ell})^{-1}\mathbf{v}^{\ell n} = -\mathcal{K}^{(H)}_{\ell n}\mathbf{v}^{\ell n}}$$

- role of the Hamiltonian $\hat{H} \to \boldsymbol{\lambda}^{\ell}$: 4-fermion proper vertices
- role of the energy levels $E \to \mathcal{K}^{(H)}_{\ell n}$: momentum dependent kernel/self-energy
- role of the (two-particle) wave function $\Psi \to \mathbf{v}^{\ell n}$: momentum dependent Yukawa-couplings
  index is not just space, but space and time (and other internal indices)
- generalizabe to n-particle bound states!
- **BUT** while $\hat{H}$ is known, $\boldsymbol{\lambda}$ must also be calculated!

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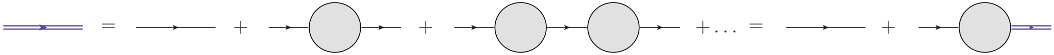
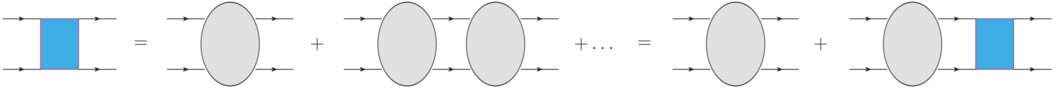
Dyson-Schwinger resummation

1-particle propagator: DS series



where the bubble means self-energy (1PI) diagrams

2-particle propagation: DS series (Bethe-Salpeter series)



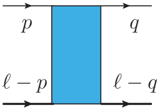
where the bubble means fermion-2PI diagrams

Model: QED with 2 fermion species

$$\mathcal{L} = \bar{\psi}(\gamma_\mu(i\partial^\mu - eA^\mu) - m_\psi)\psi + \bar{\chi}(\gamma_\mu(i\partial^\mu + eA^\mu) - m_\chi)\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

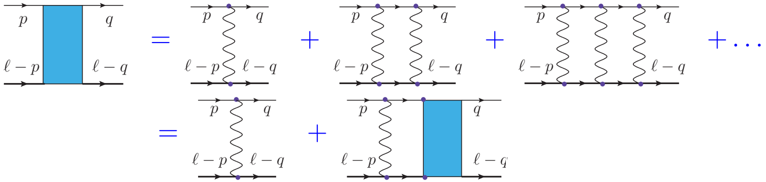
2-2 scattering amplitude:

$$\lambda^{\ell}_{p\alpha\gamma,q\beta\sigma} =$$

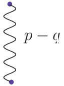
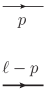


$$= -\left\langle \hat{\psi}^{\dagger}_{\alpha p}\hat{\psi}_{\beta q}\hat{\chi}^{\dagger}_{\gamma,\ell-p}\hat{\chi}_{\sigma,\ell-q}\right\rangle\Big|_{amputated}$$

Ladder resummation:



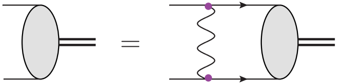
For matrix notation use multi-indices

Diagram elements: $\mathbf{V}_{ab} =$  $= \gamma^{\mu}_{\alpha\beta} G^{\gamma}_{\mu\nu}(p-q)\gamma^{\nu}_{\gamma\delta}$, $\qquad \mathcal{G}_{ab} =$  $\times\delta(p-q)$.

Ladder resummation: $\boldsymbol{\lambda} = \mathbf{V} - \mathbf{V}\mathbf{G}\boldsymbol{\lambda} \quad \Rightarrow \quad \boxed{\boldsymbol{\lambda} = (1+\mathbf{V}\mathbf{G})^{-1}\mathbf{V}}$

Bound states: λ has poles when **VG** has a -1 eigenvalue

$$\mathbf{V}\mathbf{G}\mathbf{v} = -\mathbf{v} \qquad \text{Bethe-Salpeter equation}$$



spinless particles: $V_k = \frac{1}{k^2}$, $\mathcal{G} = \frac{1}{(p^2 - m^2)((\ell - p)^2 - M^2)}$, then the BS-equation:

$$v^{(\ell)}(p) = -ie^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(p-q)^2} \frac{1}{(q^2 - m^2)((\ell - q)^2 - M^2)} v^{(\ell)}(q).$$

nonrelativistic case:

- omit retardation, then $\frac{1}{(p-q)^2} \to \frac{1}{(\mathbf{p} - \mathbf{q})^2} \quad \Rightarrow \quad v(\not{p}_0, \mathbf{p})$.
- q_0 integral can be performed: $\int_{q_0} \frac{1}{q^2 - m^2} \frac{1}{(\ell - q)^2 - M^2} \sim \frac{i}{\bar{E} + \frac{\mathbf{q}^2}{2m_{red}}}$,

$\ell = (\ell_0, 0)$, $\bar{E} = m + M - \ell_0$, $m_{red}^{-1} = m^{-1} + M^{-1}$

Nonrelativistic approximation

We arrive at $$v(\mathbf{p}) = 4\pi\alpha \int \frac{d^3q}{(2\pi)^3} \frac{v(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2(\bar{E}+\frac{\mathbf{q}^2}{2m_{red}})}$$

Introducing $v(\mathbf{q}) = (\bar{E}+\mathbf{q}^2/2m_{red})\Psi(\mathbf{q})$ we find:

$$(\bar{E}+\frac{\mathbf{p}^2}{2m_{red}})\Psi(\mathbf{p}) = 4\pi\alpha \int \frac{d^3q}{(2\pi)^3} \frac{\Psi(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2}$$

And finally, with inverse Fourier-transformation:

$$\left(\frac{\triangle}{2m_{red}} + \frac{\alpha}{r}\right)\Psi(\mathbf{r}) = \bar{E}\Psi(\mathbf{r})$$

Nonrelativistic BS-equation is equivalent to the Schrödinger-equation

wave function $\sim$ Bethe-Salpeter amplitude

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The Functional Renormalization Group

Idea of RG: gradual dressing of the interactions with quantum/statistical fluctuations: we start with the classical action in UV and reveal the 1PI action in the IR.

Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{STr}\, \partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1},$$

where

- Γ_k is the effective action at scale k, $\Gamma_{k=\Lambda} = S_{cl}$
- $R_k(p)$ is the quadratic regulator kernel

  $$S \to S - \int \frac{d^4 p}{(2\pi)^4} \Phi^*(p) R_k(p) \Phi(p)$$

  vanishes in the IR ($R_{k\to 0}(p) = 0$) and stops fluctuations in the UV ($R_{k=\Lambda}(p) \sim \infty$)
- technically: one-loop expression

Exact equation!

FRG approach in bound state study

FRG is an exact equation

$\Rightarrow$ must give an account for the bound states, too.

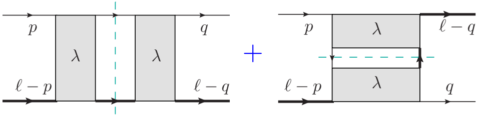
Precedent works

- [Ellwanger94]: FRG eq. for 4-point function
- [Gies02]: local approximation, generalized Hubbard-Stratonovich transformations, (partial) fixed points
- [Flörchinger09,10]: generalized Hubbard-Stratonovich transformation for composite operators
- [Diehl10]: FRG equations to study BEC-BCS crossover
- [Rose16]: bound states in SSB Φ^4 model with BMW.
- [Alkofer18]: dynamical hadronization with DSE+FRG methods.

Ansatz: nonlocal 4-fermion interaction instead of gauge interaction
(integrate out photons from QED)

$$\Gamma_{4-fermion} = \int_p \left[\psi_p^\dagger \mathcal{K}_p^{(\psi)} \psi_p + \chi_p^\dagger \mathcal{K}_p^{(\chi)} \chi_p\right] + \int_{p,q,\ell} \lambda^\ell_{p\alpha\gamma,q\beta\sigma} \psi^\dagger_{p\alpha} \psi_{q\beta} \chi^\dagger_{\ell-p,\gamma} \chi_{\ell-q,\sigma}.$$

FRG equation in graphical representation

$\partial_k \lambda^\ell_{p\alpha\gamma,q\beta\sigma} =$ 

s-channel approximation: $\partial_k \boldsymbol{\lambda} = -\boldsymbol{\lambda}(\partial_k \boldsymbol{\mathcal{G}})\boldsymbol{\lambda} \quad \Rightarrow \quad \partial_k \boldsymbol{\lambda}^{-1} = \partial_k \boldsymbol{\mathcal{G}}$

Solution:

$$\boxed{\boldsymbol{\lambda}_k = (1 + \mathbf{V}\boldsymbol{\mathcal{G}}_k)^{-1}\mathbf{V}.}$$

Same as in the BS case! (Ellwanger, Wetterich)

FRG equation for the bound states

Ansatz: auxiliary fields with nonlocal Yukawa interaction:

$$\Gamma_{eff} = \int_p \left[\psi_p^\dagger \mathcal{K}_p^{(\psi)} \psi_p + \chi_p^\dagger \mathcal{K}_p^{(\chi)} \chi_p + \sum_n H_{pn}^\dagger \mathcal{K}_{pn}^{(H)} H_{pn} \right] + \sum_n \int_{p\ell} \left[H_{\ell n} \psi_{p\alpha}^\dagger v_{p\alpha\beta}^{\ell n} \chi_{\ell-p,\beta}^* + \mathrm{h.c.} \right].$$

- we must not use direct FRG equations! – they do not reproduce BS-equations
- use FRG of the 4-fermion theory to derive equations for the representing modes!
- logics:

$$\Gamma_{eff}[k,\psi,H] \xrightarrow{H-elimination} \Gamma_{4-fermion}[k,\psi] \xrightarrow{\mathbf{FRG}} \Gamma_{4-fermion}[k-dk,\psi] \xrightarrow{H-representation} \Gamma_{eff}[k-dk,\psi,H].$$

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- k-derivative of the eigenvalue equations yields:

$$\partial_k G_{\ell n}^{(H)} = \mathbf{v}_{\ell n}^\dagger (\partial_k \boldsymbol{\lambda}^\ell)\,\mathbf{v}_{\ell n}, \qquad \partial_k \mathbf{v}_{\ell n} = \sum_{m\neq n} \frac{\mathbf{v}_{\ell m}^\dagger (\partial_k \boldsymbol{\lambda}^\ell)\,\mathbf{v}_{\ell n}}{G_{\ell n}^{(H)} - G_{\ell m}^{(H)}} \mathbf{v}_{\ell m}.$$

(like QM pert. th.)

use FRG equation for λ to make these equations explicit

Complete FRG equations with crossing symmetry

In BS approximation (s-channel) the matrix elements are simple

$$\partial_k \mathcal{K}_{\ell n}^{(H)} = -\mathbf{v}_{\ell n}^{\dagger}(\partial_k \boldsymbol{\mathcal{G}}^{\ell})\mathbf{v}_{\ell n}, \qquad \partial_k \mathbf{v}_{\ell n} = \sum_{m\neq n} \frac{\mathbf{v}_{\ell m}^{\dagger}(\partial_k \boldsymbol{\mathcal{G}}^{\ell})\mathbf{v}_{\ell n}}{\mathcal{K}_{\ell m}^{(H)} - \mathcal{K}_{\ell n}^{(H)}} \mathbf{v}_{\ell m}.$$

Complete treatment with crossing symmetry is more complicated

$$\partial_k \mathbf{v}_{\ell n} = \sum_{m\neq n} \frac{G_{\ell m}^{(H)} G_{\ell n}^{(H)} \mathbf{v}_{\ell m}^{\dagger}(\partial_k \boldsymbol{\mathcal{G}}^{\ell})\mathbf{v}_{\ell n} - L_{mn}^{\ell}}{G_{\ell n}^{(H)} - G_{\ell m}^{(H)}} \mathbf{v}_{\ell m}$$

where $(m' = (m, \ell + r - q),\ n' = (n', \ell + r - p))$

$$L_{mn}^{\ell} = v_{p\alpha\gamma}^{m\ell *} v_{p\alpha\gamma'}^{m'} G_{m'}^{(H)} v_{r\beta'\sigma}^{m' *} (\partial_k \mathcal{G})_{r,\beta'\sigma',\alpha'\gamma'}^{\ell+2r-p-q} v_{r\alpha'\gamma}^{n'} G_{n'}^{(H)} v_{q\beta\sigma'}^{n' *} v_{q\beta\sigma}^{n\ell}.$$

$\Rightarrow$ Technically involved, but theoretically correct.

Could we make some simplifications?

Nonrelativistic limit

Scale dependent kernel

- Regularize the classical 2-fermion propagator
  $p_k^2 = \Theta(p-k)p^2 + \Theta(k-p)k^2 \quad \Rightarrow \quad \mathcal{G}_{k,pq} = \frac{\delta_{pq}}{\bar{E}+p_k^2}$

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- **FRG** equation for $\boldsymbol{\lambda}_k^{-1}$ (4-fermion interaction)

$$\partial_k \boldsymbol{\lambda}_k^{-1} = \partial_k \boldsymbol{\mathcal{G}}_k = -\frac{2k}{(\bar{E}+k^2)^2}\mathbf{P}_k,$$

where $\mathbf{P}_{k,pq} = \Theta(k-p)\delta_{pq}$ projector.

Initial condition: $\boldsymbol{\lambda}_{k\to\infty} = \mathbf{V}$

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  where $\mathbf{P}_{k,pq} = \Theta(k-p)\delta_{pq}$ projector.

  Initial condition: $\boldsymbol{\lambda}_{k\to\infty} = \mathbf{V}$
- Solution

$$\boldsymbol{\lambda}_k(\bar{E}) = (1+\mathbf{V}\boldsymbol{\mathcal{G}}_k)^{-1}\mathbf{V}$$

  $\boldsymbol{\lambda}_k^\dagger = \boldsymbol{\lambda}_k$, and $\boldsymbol{\lambda}_k(\bar{E}=\bar{E}_n)\to\infty$ singular.

Nonrelativistic limit

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Eigenmode representation

Represent $\boldsymbol{\lambda}_k$ with its eigensystem

$$\boldsymbol{\lambda}_k^{-1} = -\sum_n \mathcal{K}_{k,n} \mathbf{x}_{k,n} \otimes \mathbf{x}_{k,n}^\dagger.$$

Differential equation for the eigensystem:

$$\partial_k \mathcal{K}_{k,n} = -\mathbf{x}_{k,n}^\dagger \partial_k \boldsymbol{\mathcal{G}}_k \mathbf{x}_{k,n},$$

$$\partial_k \mathbf{x}_{k,n} = \sum_m \frac{-\mathbf{x}_{k,m}^\dagger \partial_k \boldsymbol{\mathcal{G}}_k \mathbf{x}_{k,n}}{\mathcal{K}_{k,n} - \mathcal{K}_{k,m}} \mathbf{x}_{k,m}.$$

Initial conditions at large k (denote $n \to \mathbf{r}_0$)

$$x_{n,\mathbf{p}} = e^{i\mathbf{p}\mathbf{r}_0}, \quad \mathcal{K}_n = \frac{1}{V(\mathbf{r}_0)}$$

are eigenvectors/values of the inverse potential!

Numerical study

Numerical study

Discretization for s-channel (1D) with $\eta(p) = pv(|\mathbf{p}|)$

$$p_\ell = (\ell + \tfrac{1}{2})dp,\ \ell = 1\dots N, \qquad \bar{\boldsymbol{\eta}}_{\ell n} = \sqrt{\frac{dp}{2\pi^2}}\eta_n(p_\ell)\Big|_{p_\ell < k}$$

$$\mathbf{C} = \boldsymbol{\eta}^T\boldsymbol{\eta}, \quad D_{nm} = \frac{C_{nm}}{\mathcal{K}_n - \mathcal{K}_m}.$$

Differential equation ($\alpha_k = 2k(\bar{E} + k^2)^{-2}$).

$$\partial_k\mathcal{K}_n = -\alpha_k C_{nn}, \quad \partial_k\boldsymbol{\eta} = -\alpha_k\boldsymbol{\eta}\mathbf{D}$$

Initial conditions at $\Lambda = Ndp$: eigensystem of potential

$$\mathbf{V}^{-1} - \frac{1}{\bar{E} + \Lambda^2}\mathbf{1}$$

Bound states if at $k \to 0:\ \mathcal{K}_n(\bar{E}_n) = 0$.

In present example: $N = 800,\ dp = 0.003$.

Numerical study

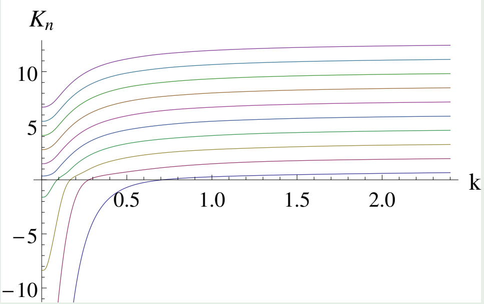
Discr ($|\mathbf{p}|$)

$p_\ell = ($ $\Big|_{p_\ell < k}$

Differ

Initial ential

Scale dependence of the eigenvalues



$E + K^2$

Bound states if at $k \to 0: \ \mathcal{K}_n(\bar{E}_n) = 0$.

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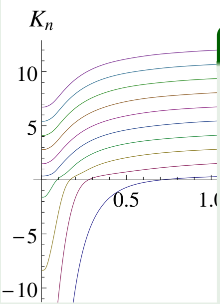
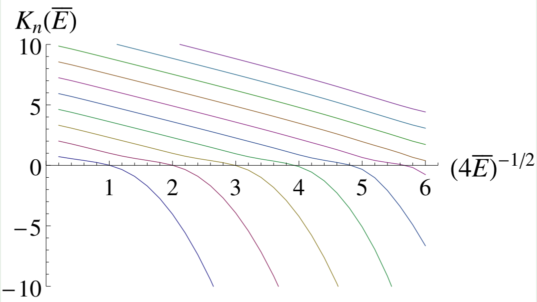


Spectrum at $k = 0$



Bound states if at $k \to ($

In present example: $N =$

Numerical study

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$(|\mathbf{p}|)$

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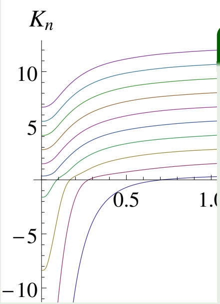
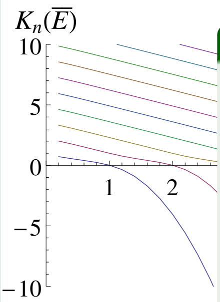
In present example: $N =$

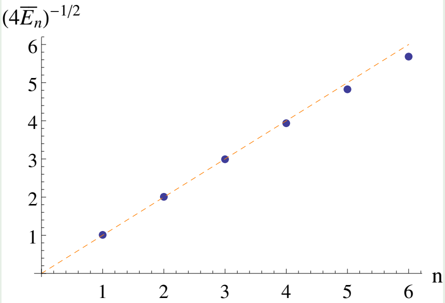
Scale dependence of the eigenvalues



Spectrum at $k = 0$



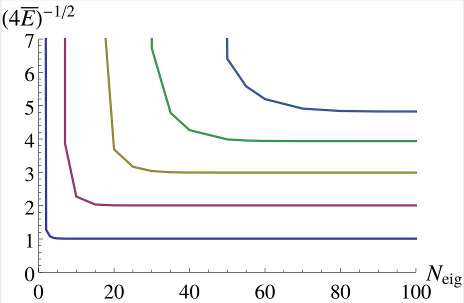
Bound states



How many bound states are needed?

We do not need to keep all the eigenvectors for a reasonable precision

Bound state energies



For the ground state:

- four eigenvectors: 1%
- three eigenvectors: 8%
- two eigenvectors: 28%
- one eigenvector (only the ground state) is not enough!

Generalization

We may hope that this remains true in the full relativistic case $\Rightarrow$ it is enough to use a reduced set of eigenvectors!

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1. Introduction
2. Equations for the relativistic bound states
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   - Dyson-Schwinger resummation
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5. Conclusions

The action

The fundamental action

$$\Gamma_{NJL} = \int d^4x \left[i\bar{\psi}\gamma_m\partial_m\psi + 2\lambda_\sigma \left(\bar{\psi}_L\psi_R\right)\left(\bar{\psi}_R\psi_L\right) - \frac{1}{2}\lambda_V(\bar{\psi}\gamma_m\psi)^2 \right].$$

Is there a bound state in the symmetric phase? The corresponding Ansatz:

$$\Gamma_{NL-NJL} = \int d^4x \left[\bar{\psi}i\gamma_m\partial_m\psi - \tfrac{1}{2}\lambda_V(\bar{\psi}\gamma_m\psi)^2 + 2\delta\lambda_\sigma(\bar{\psi}_R\psi_L)(\bar{\psi}_L\psi) + \tfrac{\lambda}{24}\left(\Phi_S^2(x) + \Phi_5^2(x)\right)^2 \right]$$
$$+\tfrac{1}{2}\int dx \int dy \left[\Phi_S(x)\Gamma_C^{(2)}(x-y)\Phi_S(y) + \Phi_5(x)\Gamma_C^{(2)}(x-y)\Phi_5(y) \right]$$
$$-i\int dx \int dx_1 \int dx_2 \Delta_C(x-x_1, x-x_2)\left[\Phi_S(x)\bar{\psi}(x_1)\psi(x_2) - i\Phi_5(x)\bar{\psi}(x_1)\gamma_5\psi(x_2) \right].$$

- scalar kernels: $\Gamma_S^{(2)}(q) = Z_C q^2 + M_C^2 = \Gamma_5^{(2)}(q)$.
- 4-fermion interaction (local) + nonlocal, resonant channel
- Ansatz for nonlocal vertex: $\Delta_C(q_1, q_2) = g_C e^{-\beta(q_1+q_2-Q)^2} e^{-\alpha(q_1-q_2)^2}$
  for simplicity $\alpha = \beta$; physically $\alpha \sim R^2$

FRG equations

Wetterich-equation for fermion + boson systems ($t = \log k/k_0$)

$$\partial_t \Gamma = -\mathrm{Tr}\log\Gamma_F^{(2)} + \frac{1}{2}\mathrm{Tr}\log\Gamma_B^{(2)} + \frac{1}{2}\mathrm{Tr}\log(I - G_B\Gamma_{BF}^{(2)}G_F\Gamma_{FB}^{(2)}).$$

- rescale parameters with appropriate powers of k
- equations for couplings ($g_c,\ \lambda,\ \lambda_\sigma,\ \lambda_V$), dimensionful quantities ($M,\ \alpha,\ Q$)
- wave function renormalization → anomalous dimensions $\eta_C,\ \eta_\psi$.

Fixed points:

- strongly coupled UV fixed point $(\delta\lambda^*_{\sigma r}, \lambda^*_{Vr}) = (6\pi^2, 2\pi^2)$
- IR fixed point: Gaussian; 4-fermion couplings are irrelevant, g_{Cr} and λ have mild scale dependence ($|t|^{-1},\ |t|^{-2}$).

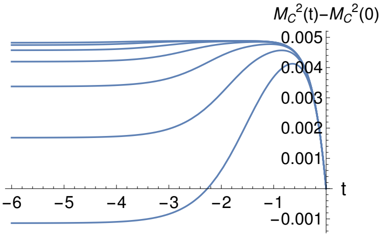
Evolution of scale dependent quantities

Renormalization condition:

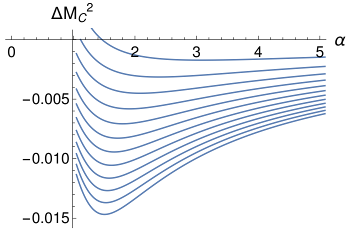
- changing $\alpha \quad \Rightarrow \quad$ change the separation of the two fundamental particles
- change in α should have an effect only on the IR quantities
- we should keep the UV evolution independent on the choice of α.

α dependence of the mass

- in UV the same behavior (renormalization)
- $\alpha = \infty$: no IR scale dependence
- decreasing α yields decreasing mass: binding energy



tunable parameters are $\alpha \sim R^2$ and dM_c^2/dt initial slope



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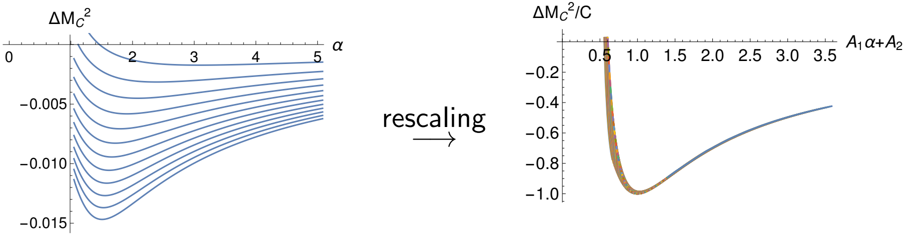

α dependence of the binding energy

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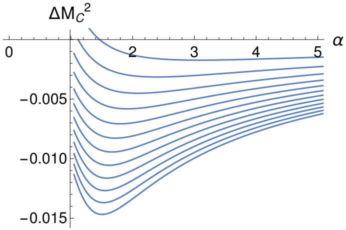
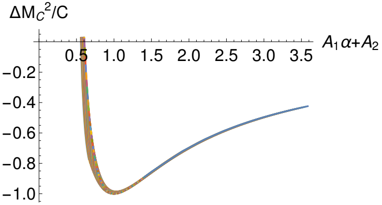


rescaling $\longrightarrow$



- physics remains invariant under changing the actual slope
- we find minimum of the binding energy at a nontrivial scale

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Conclusions

- Nonrelativistic "Schrödinger-equation"

$$\lambda \mathbf{v} = -G^{(H)}\mathbf{v},$$

where λ is the proper 4-fermion vertex, $\mathbf{v}$ is the Yukawa coupling between the bound states and fundamental fermions, $G^{(H)}$ is the bound state propagator

- calculation of λ: BS equations, or FRG equations
- reduced set of eigensates may be enough for a fair precision
- using these ideas in NJL model symmetric phase: convincing indications for an attractive potential

Multi-indices

- internal states are 2-particle states with momenta/spin: $(p, \alpha), (p - \ell, \beta)$
  $\Rightarrow \quad a = \{p\alpha\beta\}$ multi-index, ℓ "spectator" index
- rungs and 2-particle propagation depend on momenta
  $(p, \alpha), (p - \ell, \beta), (q, \gamma), (q - \ell, \delta) \quad \Rightarrow \quad$ can be represented as a matrix $M_{ab}^{(\ell)}$
- Scalar product: $f^{(\ell)} g^{(\ell)} = \sum_{\alpha\gamma} \int \frac{d^4 p}{(2\pi)^4} f_{p\alpha\gamma}^{(\ell)} g_{p\alpha\gamma}^{(\ell)}$

Numerical solution

Numerical study

equation:

$$(\triangle + \frac{1}{x})\Psi(x) = \bar{E}\Psi(x) \quad \Rightarrow \quad \bar{E}_n = \frac{1}{4n^2} \quad \to (4\bar{E}_n)^{-1/2} = n$$

Bethe-Salpeter version (Fourier-trf, $v = (p^2 + \bar{E})\Psi$)

$$v(q) = \int \frac{d^3p}{(2\pi)^3} \frac{v(p)}{(p-q)^2(p^2+\bar{E})}$$

s-channel: rotational invariance $\eta(p) = pv(|\mathbf{p}|)$

$$\eta(q) = \frac{1}{\pi} \int_0^\infty dp \log\left|\frac{q+p}{q-p}\right| \frac{1}{\bar{E}+p^2}\eta(p).$$

Numerical solution

Numerical study

equation:

$$(\triangle + \frac{1}{x})\Psi(x) = \bar{E}\Psi(x) \quad \Rightarrow \quad \bar{E}$$

Bethe-Salpeter version (Fourie

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Discretization

Discretization $p_m = (m + \frac{1}{2})dp$, $\eta_m = \eta(p_m)$ leads to the matrix equation

$$\boldsymbol{\eta} = \mathbf{V}\boldsymbol{\mathcal{G}}(\bar{E})\boldsymbol{\eta},$$

$$\mathcal{G}_{nm} = \frac{1}{\pi}\frac{dp}{\bar{E}+p_m^2}, \quad V_{mn} = (1-\delta_{nm}) \ln \left| \frac{n+m}{n-m} \right| + \delta_{nm} \ln(4n+2)$$

Strategy: eigenvalues λ_n, bound states if $\lambda_n(\bar{E}_n) = 1$.

Numerical solution

Numerical study

equation:

$(\triangle + \frac{1}{x})\Psi(x) = \bar{E}\Psi(x) \quad \Rightarrow \quad \bar{E}$

Be

S-(

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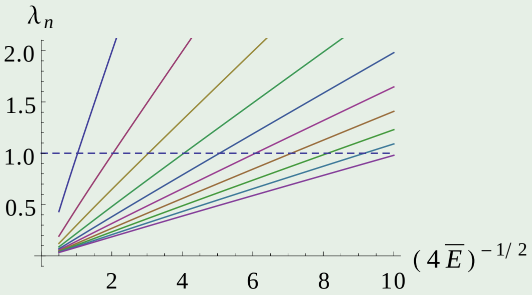
Eigenvalues



Numerical solution

Numerical study

equation:

$$(\triangle + \frac{1}{x})\Psi(x) = \bar{E}\Psi(x) \quad \Rightarrow \quad \bar{E}$$

Be

S-(

Discretization

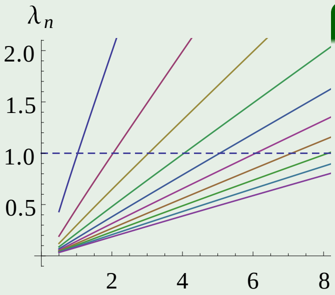
Discretization $p_m = (m + \frac{1}{2})dp, \; \eta_m = \eta(p_m)$ leads to the matrix equation

$$\boldsymbol{\eta} = \mathbf{V}\mathcal{G}(\bar{E})\boldsymbol{\eta},$$

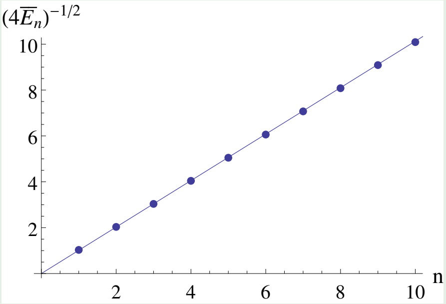
$+ \delta_{nm} \ln(4n + 2)$

if $\lambda_n(\bar{E}_n) = 1$.

Eigenvalues



Spectrum



Generating functional

$$e^{-W_k[J]} = \int \mathcal{D}\Phi\, e^{-S_E[\Phi] - \frac{1}{2}\int \Phi^* R_k \Phi + \int J\Phi}$$

differentiate wrt. k:

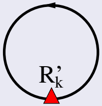
$$-\partial_k W_k[J] e^{-W_k[J]} = -\frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \partial_k R_k(p) \int \mathcal{D}\Phi\, \Phi^*(p)\Phi(p) e^{-S_E[\Phi] - \frac{1}{2}\int \Phi^* R_k \Phi + \int J\Phi}.$$

RHS is a 2-point function: $\langle \Phi\Phi \rangle \to \frac{\partial^2 W}{\partial J \partial J}$

Equation for the quantum effective action $\Gamma_k[\varphi]$:

Wetterich equation

$$\partial_k \Gamma_k = \frac{1}{2}$$



$$= \frac{1}{2}\,\mathrm{Tr}\,\partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1}$$

exact equation!

FRG equation

Peculiarities of the FRG approximation we will use:

- regularize only the matter sector, leave gauge sector unregularized (appropriate in QED: all diagrams are regularized)
- do not run QED couplings ($\alpha_{QED} \approx \text{const}$, at m_e scale; bound states are at α^2 smaller energies)
- **use nonlocal Ansätze**: it turns out to be crucial for representation of bound states.