Gravitational Waves and Ion Traps

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18-04-2019

Outline

Plane Gravitational Waves

$4d$ exact plane gravitational waves (GWs) in Brinkmann (B) coordinates $X^\mu = \{X^+, X^-, U, V\}$ [1],

$$g^B_{\mu\nu} dX^\mu dX^\nu = \delta_{ij} dX^i dX^j + 2dUdV + K_{ij}(U) X^i X^j dU^2, \tag{1a}$$

$$K_{ij} X^i X^j = \frac{1}{2} \mathcal{A}_+(U) \Big((X^+)^2 - (X^-)^2 \Big) + \mathcal{A}_\times X^+ X^-, \tag{1b}$$

where $i, j = \pm$ and U, V light cone coordinates.

- $K_{ij}(U)$ is the profile of GW.
- $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$: polarization state amplitudes.
- (1) is linearly polarized if $\mathcal{A}_\times$ (or $\mathcal{A}_+$) is vanishing.

[1] M. W. Brinkmann, 1925

Matrix $K(U)$ is symmetric and traceless:

$$(K_{ij}(U)) = \frac{1}{2}\begin{pmatrix} \mathcal{A}_{+}(U) & \mathcal{A}_{\times}(U) \\ \mathcal{A}_{\times}(U) & -\mathcal{A}_{+}(U) \end{pmatrix}, \tag{2}$$

- Non-vanishing components of the Riemann tensor for (1) are

$$R^{i}{}_{UjU} = -R^{V}{}_{ijU} = -K_{ij}. \tag{3}$$

- Since $\mathrm{tr}(\mathrm{K}) = 0$, (1) is Ricci flat, i.e., $R_{\mu\nu} = 0$.
  It is a vacuum solution of Einstein's equation.

Geodesic Lagrangian for a spinless, test particle is written as

$$L_B = \frac{1}{2} g^{B}_{\mu\nu} \frac{dX^{\mu}}{d\sigma}\frac{dX^{\nu}}{d\sigma} = \frac{1}{2}\left[\left(\frac{dX^{i}}{d\sigma}\right)^{2} + 2\frac{dU}{d\sigma}\frac{dV}{d\sigma} + K_{ij}X^{i}X^{j}\left(\frac{dU}{d\sigma}\right)^{2}\right] \tag{4}$$

where σ is an affine parameter.

Euler-Lagrange equations derived from (4)

$$\frac{d^2\boldsymbol{X}}{d\sigma^2} = K(U)\boldsymbol{X}, \quad \frac{d^2U}{d\sigma^2} = 0, \quad \frac{d^2V}{d\sigma^2} = \ldots \tag{5}$$

They match

$$\frac{d^2X^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\sigma}\frac{dX^\beta}{d\sigma} = 0. \tag{6}$$

- Transverse plane: $X^\pm = X^\pm(U)$, anisotropic, $U-$dependent attractive/repulsive oscillators.
- U itself can be considered as the affine parameter.
- $V-$ equation can be solved once $X^\pm(U)$ is found:

$$V(U) = -\frac{1}{2}\boldsymbol{X}\cdot\frac{d\boldsymbol{X}}{dU} + C_1U + C_2, \tag{7}$$

where C_1 and C_2 are constants.

Periodic GWs

- Particular periodic profile:
  $\mathcal{A}_+(U) = A_0 \cos \omega U, \quad \mathcal{A}_\times(U) = B_0 \cos(\omega U - \phi)$
- If $\mathcal{A}_\times = 0$: linearly polarized periodic (LPP) GW,
- If $A_0 = B_0$ and $\phi = \pi/2$: circularly polarized periodic (CPP) GW,

$$K = \frac{A_0}{2} \begin{pmatrix} \cos \omega U & \sin \omega U \\ \sin \omega U & -\cos \omega U \end{pmatrix} \tag{8}$$

- Periodic GWs are sought in inflationary models[2].
- They also exhibit exhibit memory effect[3].

[2] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations] 2017
[3] P.-M. Zhang, C. Duval and P. A. Horvathy, 2017

Symmetries

- General plane waves (2) admit a $5-$parameter isometry group.[4],
- Identified as the Carroll group with broken rotations [5],

$$\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial v}, H^{ij}\frac{\partial}{\partial x^j} - x^i\frac{\partial}{\partial v}\}, \tag{9}$$

in Baldwin-Jeffery-Rosen (BJR) coordinates.

- However, for CPP case (8), the broken symmetries combine to give a sixth symmetry,

$$\xi = \partial_U + \frac{\omega}{2}(X^+\partial_- - X^-\partial_+), \quad \mathcal{L}_\xi g_{\mu\nu} = 0, \tag{10}$$

called the screw symmetry.

[4] D. Kramer, H. Stephani, M. McCallum and E. Herlt 2003

[5] C. Duval, G. W. Gibbons, P. Horvathy and PM Zhang. 2017

Motivation

Ilderton[6]: Screw symmetric CPP GW (8) is the classical double copy of Bialynicki-Birula's electromagnetic vortex[7].

- Clue: Both systems are screw symmetric.
- Tool: Classical double copy.

Question: *Are there any other physical systems related to GWs?*
Answer : Ion traps.

- The clue: Both motions boil down to anisotropic, time dependent oscillators.
- The tool: Bargmann framework and the null geodesics therein.

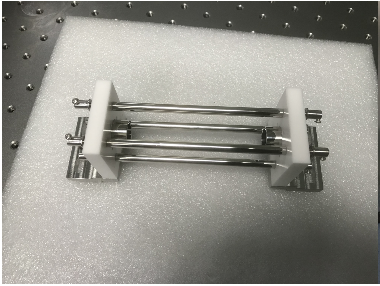
Strictly speaking, "$4d$ GW metric (1) is the Bargmann manifold of a $2d$ NR harmonic oscillator."

[6] A. Ilderton, 2018
[7] I. Bialynicki-Birula 2009

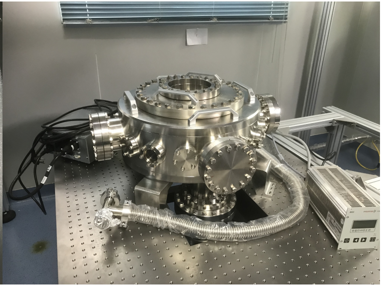
Planar Paul trap

Paul trap[8]:

- Charges and neutral particles can be trapped by electric or magnetic quadrupole potentials,
- Even a single ion can be trapped for high accuracy measurements,
- Functions as a mass spectrometer.



Linear Paul trap



MOT container

[8] W. Paul, 1990

$2d$ Paul trap quadrupole potential

$$\Phi(\boldsymbol{X}) = \frac{\Phi_0}{2}\left((X^+)^2 - (X^-)^2\right), \quad \nabla^2\Phi = 0. \tag{11}$$

where $X^\pm$ are plane coordinates and Φ_0 is a constant.

(11) allows harmonic motion in X^+ but escaping motion along X^- axis. No stable motion, particle is lost.

To stabilize, a periodic voltage $\Gamma_0 \cos\omega t$ is added:

$$\ddot{X}^\pm = \mp\left(a - 2q\cos\omega t\right) X^\pm, \quad \dot{X} = \frac{dX}{dt}. \tag{12}$$

t is NR time and ω is the oscillation frequency.

$a = \frac{e}{m}\Phi_0$ and $q = e\frac{\Gamma_0}{2m}$ constants.

Φ_0, Γ_0 at the order of few volts.

Experimentally interesting region: $0 < a,\ q < 1$.

Rescale in order to put into standard Mathieu form, $t \to \frac{2t}{\omega}$,

$$\ddot{X}^{\pm} = \mp\left(\hat{a} - 2\hat{q}\cos 2t\right) X^{\pm}\,, \tag{13}$$

with $\hat{a} = (4/\omega^2)a$, $\hat{q} = (4/\omega^2)q$.

(13) are called the Mathieu equations.

They have both bounded and unbounded solutions.

Stable motion depends on the parameters a and q. For instance

$$a = 0, \quad 0 < q < q_{max} = 0.92. \tag{14}$$

Bounded solutions of (13) are given in terms of Mathieu functions $C(\hat{a}, \hat{q}, t)$ and $S(\hat{a}, \hat{q}, t)$.

LPP GWs

Equations similar to (13) arise for particle motion in the spacetime of a LPP GW

$$ds^2 = \delta_{ij} dX^i dX^j + 2dUdV + \frac{1}{2}\mathcal{A}_+(U)\Big((X^+)^2 - (X^-)^2\Big) dU^2, \quad (15)$$

with

$$\mathcal{A}_+ = A_0 \cos 2U, \qquad K = \frac{A_0}{2}\begin{pmatrix} \cos 2U & 0 \\ 0 & -\cos 2U \end{pmatrix}. \qquad (16)$$

Transverse geodesic equations read

$$(X^\pm)'' = \pm\frac{A_0}{2}(\cos 2U) X^\pm, \quad (X^\pm)' = \frac{dX^\pm}{dU}, \qquad (17)$$

which are (13) with $a = 0$.

Bound motions are described by $C(A_0, U)$ and $S(A_0, U)$.

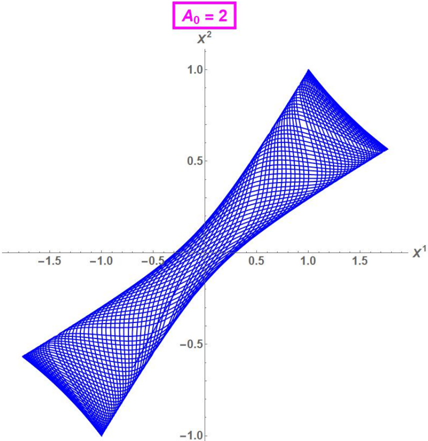


Figure: *In a weak linearly polarized periodic (LPP) wave, (16), the transverse coordinate* $\boldsymbol{X}(U)$ *oscillates in a bounded "bow tie"-shaped domain. The initial conditions are* $\dot{X}^{+}(U\!=\!0) = \dot{X}^{-}(U\!=\!0) = 0$ *(at rest for* $U\!=\!0$*), at initial position* $X^{+}(U\!=\!0) = 1$*,* $X^{-}(U\!=\!0) = 0$*.*

Bargmann framework[9]:

- $(d+1,1)$ dimensional relativistic framework endowed with a covariantly constant Killing vector ∂_V to discuss the symmetries of d dimensional NR systems
- Classical motions of the NR system will be the null geodesics.

We lift the $2d$ Paul trap (11) by putting $-2\Phi(\boldsymbol{X},t)$ to the UU component of the metric, adding an extra coordinate V and promoting $t \to U$

$$g^{P}_{\mu\nu}dX^{\mu}dX^{\nu} = d\boldsymbol{X}^2 + 2dUdV - 2\Phi(\boldsymbol{X},U)\,dU^2\,, \tag{18a}$$

$$\Phi(\boldsymbol{X},U) = \frac{1}{2}(a - 2q\cos 2U)\Big((X^{+})^2 - (X^{-})^2\Big)\,, \tag{18b}$$

Observe $\boldsymbol{\nabla}^2\Phi = 0 \implies R_{\mu\nu} = 0$.
Bargmann metric of Paul trap is Ricci flat and it is a LPP GW.

[9] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin 1985

Dynamics in $4d$ Bargmann manifold can be described via

$$L_P = \frac{1}{2} g^P_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}. \tag{19}$$

Identification of the canonical momenta

$$P_i = \frac{dX^i}{d\sigma}, \quad P_U = \frac{dV}{d\sigma} - 2\Phi \frac{dU}{d\sigma}, \quad P_V = \frac{dU}{d\sigma} = \text{cons.}, \tag{20}$$

leads to the Bargmann Hamiltonian as

$$H = \frac{P_i P_i}{2} + P_U P_V + \Phi(\boldsymbol{X}, U) P_V^2. \tag{21}$$

Recover NR motion by the null condition

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu \equiv 0, \quad g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu, \tag{22}$$

yields the $2d$ Paul trap Hamiltonian, H_{NR}

$$H_{NR} = \frac{P_i P_i}{2} + \Phi(\boldsymbol{X}, U) = -P_U, \quad P_V \equiv 1. \tag{23}$$

U becomes NR time in (23) and equations (13) can be obtained.
Null condition (22) projects $V(U) = V_0 - \int^U L_{NR}(\boldsymbol{X}, U') dU'$.

$3-d$ Paul trap

$3-$dimensional quadrupole potential for the Paul trap:

$$\Phi(X^{\pm}, z, t) = \frac{1}{2}(a + 2q\cos 2t)\Big((X^{+})^2 + (X^{-})^2 - 2z^2\Big), \qquad (24)$$

where a, q are the parameters.
The motion is described by three uncoupled Mathieu equations,

$$\ddot{X}^{\pm} + \big(a + 2q\cos 2t\big)\, X^{\pm} = 0, \qquad (25a)$$

$$\ddot{z} - 2\big(a + 2q\cos 2t\big)\, z = 0\,. \qquad (25b)$$

For a suitable range of parameters (a, q), bounded motions arise.

(24) can be lifted to $5d$ Bargmann space.

The quadratic form is traceless, $\nabla^2\Phi = 0 \implies R_{\mu\nu} = 0$:
Bargmann lift of the $3d$ Paul trap is a GW in $5d$.

CPP GWs

We consider CPP GWs

$$ds^2 = g_{ij} dX^i dX^j + 2dUdV + K_{ij}(U) X^i X^j dU^2, \tag{26}$$

with $\omega = 2$

$$K = (K_{ij}) = \frac{A_0}{2} \begin{pmatrix} \cos 2U & \sin 2U \\ \sin 2U & -\cos 2U \end{pmatrix} \qquad A_0 = \mathrm{const} > 0. \tag{27}$$

Geodesic equation for transverse plane (5):

$$(X^+)'' = \frac{A_0}{2}(X^+ \cos 2U + X^2 \sin 2U), \tag{28a}$$

$$(X^-)'' = \frac{A_0}{2}(X^+ \sin 2U - X^2 \cos 2U) \tag{28b}$$

Anisotropic and U dependent. Augmented with initial conditions,

$$\text{rest at } U = 0 \quad \text{i.e.,} \quad \boldsymbol{X}'(0) = 0. \tag{29}$$

Solve for the geodesic equations.

First step: Rotating frame trick

$$\begin{pmatrix} X^+ \\ X^- \end{pmatrix} = \begin{pmatrix} \cos U & -\sin U \\ \sin U & \cos U \end{pmatrix} \begin{pmatrix} Y^+ \\ Y^- \end{pmatrix}. \qquad (30)$$

It allows a U-independent potential together with the “magnetic” term $\mp(Y^\pm)'$:

$$(Y^\pm)'' \mp 2(Y^\mp)' - \Omega_\pm^2\, Y^\pm = 0 \quad \text{where} \quad \Omega_\pm^2 = 1 \pm A_0/2\,, \qquad (31)$$

(29) becomes

$$\boldsymbol{Y}'(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{Y}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{X}_0\,, \qquad (32)$$

Second step: Chiral decomposition[10]:

- a general method for solving Hill type equations (31),
- it also provides the decomposed Hamiltonian and the symplectic $2-$form.

Find smart phase space coordinates $\{Z^1_\pm,\ Z^2_\pm\}$

$$Y^+ = Z^1_+ + Z^1_-, \quad Y^- = Z^2_+ + Z^2_-, \tag{33a}$$
$$\Pi^+ = Z^2_+ + \Omega^2_- Z^2_-, \quad \Pi^- = -\Omega^2_+ Z^1_+ - Z^1_-, \tag{33b}$$

where $\Pi^\pm = (Y^\pm)'$. They decompose σ and H as

$$\sigma = \sigma_+ - \sigma_- = -\frac{A_0}{2}\Big[dZ^1_+ \wedge dZ^2_+ - dZ^1_- \wedge dZ^2_-\Big], \tag{34a}$$
$$H = H_+ - H_- = \frac{A_0}{4}\Big[\big(\Omega^2_+ Z^1_+ Z^1_+ + Z^2_+ Z^2_+\big) - \big(Z^1_- Z^1_- + \Omega^2_- Z^2_- Z^2_-\big)\Big]. \tag{34b}$$

Note the relative minus sign in (34).

[10] P. D. Alvarez, J. Gomis, K. Kamimura and M. S. Plyushchay 2007, 2008

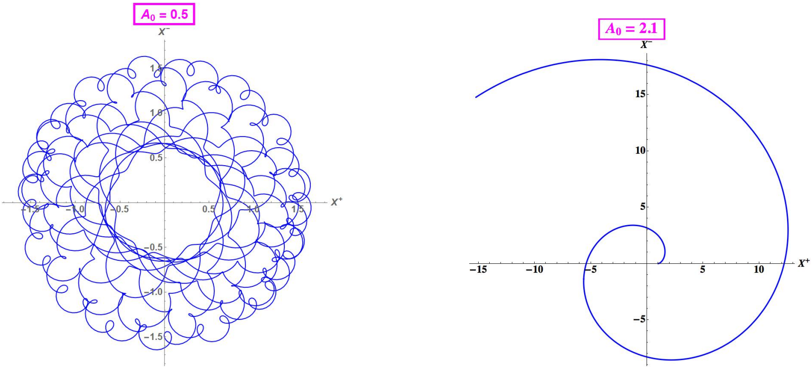
Solutions are found as

$$Y^{+} = A\cos\Omega_{+}U + B\sin\Omega_{+}U + C\cos\Omega_{-}U + D\sin\Omega_{-}U, \tag{35a}$$

$$Y^{-} = -\Omega_{+}(A\sin\Omega_{+}U - B\cos\Omega_{+}U) - \frac{1}{\Omega_{-}}(C\sin\Omega_{-}U - D\cos\Omega_{-}U). \tag{35b}$$

They match with Bialynicki - Birula's EM vortex solutions.

- $A_0 < 2$ both frequencies $\Omega_{\pm}$ are real, motions are bounded.
- $A_0 > 2$, Ω_{-} becomes imaginary and the motion is unbounded.
- $A_0 = 2$, one of the motions is free.



(i) (ii)

Figure: *(i) In a sufficiently weak CPP GW (27) the transverse trajectory of a particle initially at rest remains confined in a toroidal region. (ii) For a strong wave the trajectory becomes unbounded. The initial conditions are* $(X^+)'(0) = (X^-)'(0) = 0$ *and* $X^+(0) = 1$, $X^-(0) = 0$.

3d Penning trap

Penning trap was developed by Dehmelt[11].

An anisotropic but time independent quadrupole potential and a uniform $\boldsymbol{B} = B\hat{z}$ as stabilizer,

$$\Psi = -\left(\frac{\omega_z}{2}\right)^2 \left((Y^+)^2 + (Y^-)^2 - 2z^2\right), \quad \boldsymbol{\nabla}^2\Psi = 0, \tag{36a}$$

$$A_+ = -\frac{1}{2}BY^-, \quad A_- = \frac{1}{2}BY^+, \quad A_z = 0\,. \tag{36b}$$

The Lagrangian of a charged particle inside the Penning trap:

$$L = \frac{1}{2}\dot{\boldsymbol{Y}}^2 + \frac{\omega_c}{2}(\dot{Y}^-Y^+ - \dot{Y}^+Y^-) + \frac{1}{4}\omega_z^2(\boldsymbol{Y}^2 - 2z^2)\,, \tag{37}$$

where $\omega_c = B$ is the cyclotron frequency. $\dot{()} = d/dt$, t is NR time.

[11] H. G. Dehmelt 1989

Equations of motion

$$\ddot{Y}^{\pm} \mp \omega_c \dot{Y}^{\mp} - \frac{1}{2}\omega_z^2 Y^{\pm} = 0 \tag{38a}$$

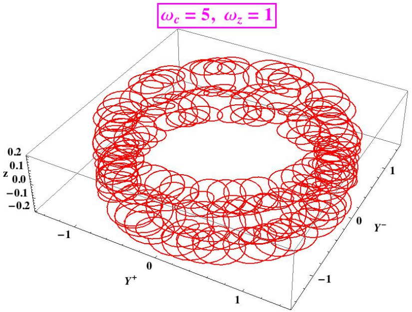
$$\ddot{z} + \omega_z^2 z = 0 \tag{38b}$$

(38) are similar to the CPP case (31) and can be solved analytically.

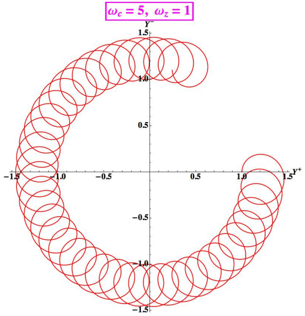
Bounded, periodic solutions require $\omega_z^2 > 0$ and $\omega_c^2 - 2\omega_z^2 > 0$.

Typically in an experiment $\omega_c >> \omega_z$.

For example: $V_0 = 10V$, $\omega_z \sim 400MHz$ and $\omega_c \sim 10^3\omega_z$.



(i)



(ii)

Figure: *Trajectory of a charged particle in a Penning Trap*
(i) in $3D$
(ii) its projection on the $Y^{\pm}$ *plane.*
Initial conditions are $Y^{+}(0) = 1.0,\ \ \dot{Y}^{+}(0) = 0.0, Y^{-}(0) = 0,\ \ \dot{Y}^{-}(0) = 1.0,\ \ z(0) = 0,\ \ \dot{z}(0) = 0.2$.

Bargmann lift of Penning trap

Form of (38a) is promising for a relation to CPP GWs.

We lift the non-relativistic motions in Penning trap to a $5d$ Bargmann space with $(\boldsymbol{Y}, z, U, V)$

$$ds^2 = d\boldsymbol{Y}^2 + dz^2 + 2dU\big(dV + A_i dY^i\big) - 2\Psi\, dU^2, \tag{39}$$

whose null geodesics project consistently with (38).

To have a clear understanding, we eliminate the magnetic term

$$\begin{pmatrix} Y^+ \\ Y^- \\ z \end{pmatrix} = \begin{pmatrix} \cos\frac{\omega_c}{2}U & \sin\frac{\omega_c}{2}U & 0 \\ -\sin\frac{\omega_c}{2}U & \cos\frac{\omega_c}{2}U & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X^+ \\ X^- \\ z \end{pmatrix}. \tag{40}$$

We obtained the Bargmann metric of an axially symmetric oscillator.

$$ds^2 = d\boldsymbol{X}^2 + 2dUdV - 2\Phi\, dU^2, \tag{41a}$$

$$\Phi = \frac{1}{8}(\omega_c^2 - 2\omega_z^2)\Big[(X^+)^2 + (X^-)^2\Big] + \frac{1}{2}\omega_z^2 z^2\,, \tag{41b}$$

(41) is *not* a CPP GW: it is not Ricci-flat $\boldsymbol{\nabla}^2\Phi = \omega_c^2 \equiv \boldsymbol{B}^2$.

(38a) has the same frequency ω_z for both $Y^{\pm}$ which means $A_0 = 0$ in CPP case (26).

Modified Penning trap

A modified version of the Penning trap with the same A (36b),

$$\widetilde{\Psi} = -(\frac{\omega_z}{2})^2 \left((1+\frac{\epsilon}{2})(Y^+)^2 + (1-\frac{\epsilon}{2})(Y^-)^2 - 2z^2\right) \qquad (42a)$$

$$A_\pm = \mp\frac{1}{2}\omega_c Y^\mp, \quad A_z = A_t = 0\,. \qquad (42b)$$

Perturbations $\epsilon << 1$ breaks the axial symmetry[12].

The equations of a charged particle in the modified trap:

$$\ddot{Y}^\pm \mp \omega_c \dot{Y}^\mp - \frac{\omega_z^2}{2}\left(1 \pm \frac{\epsilon}{2}\right) Y^\pm = 0, \qquad \ddot{z} + \omega_z^2 z = 0. \qquad (43)$$

Exactly in the same form as (31).

[12] Such imperfections are possible. See L.S. Brown and G. Gabrielse 1986.

Lifting to $5D$ Bargmann space (Y^+, Y^-, z, U, V), we obtain

$$ds^2 = (dY^+)^2 + (dY^-)^2 + dz^2 + 2dU(dV + A_i dY^i) - 2\widetilde{\Psi} dU^2 \quad (44)$$

To remove the magnetic term, we perform the $3d$ rotation (40).

We also rescale $U \to 2U/\omega_c$, $V \to \omega_c V/2$ and finally get

$$ds^2 = (dX^+)^2 + (dX^-)^2 + (dz)^2 + 2dUdV - 2\widetilde{\Phi} dU^2\,, \qquad (45a)$$

$$\widetilde{\Phi} = \left(\frac{1}{2} - \big(\frac{\omega_z}{\omega_c}\big)^2\right)\Big[(X^+)^2 + (X^-)^2\Big] + 2\big(\frac{\omega_z}{\omega_c}\big)^2 z^2$$
$$- \big(\frac{\omega_z}{\omega_c}\big)^2 \frac{\epsilon}{2}\Big[\cos 2U\,\big((X^+)^2 - (X^-)^2\big) + 2\sin 2U (X^+X^-)\Big], \qquad (45b)$$

A complicated mixture of an anisotropic oscillator together with a periodic correction term.

However, if we choose

$$\Delta = \left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{1}{2} = 0\,, \tag{46}$$

we are left with a CPP profile (26):

$$\widetilde{\Phi}_s = -\frac{\epsilon}{4}\Big[\cos 2U\Big((X^+)^2 - (X^-)^2\Big) + 2\sin 2U\Big(X^+X^-\Big)\Big] + z^2\,. \tag{47}$$

The Bargman lift of the modified Penning trap is a CPP GW in $5d$.

In this special case, chiral decomposition of (43) yields

$$\begin{aligned} Y^{+} = {} & A\cos\left(\frac{\omega_c}{2}\sqrt{1+\frac{\epsilon}{2}}\,t\right) + B\sin\left(\frac{\omega_c}{2}\sqrt{1+\frac{\epsilon}{2}}\,t\right) \\ & + C\cos\left(\frac{\omega_c}{2}\sqrt{1-\frac{\epsilon}{2}}\,t\right) + D\sin\left(\frac{\omega_c}{2}\sqrt{1-\frac{\epsilon}{2}}t\right) \end{aligned} \tag{48a}$$

$$\begin{aligned} Y^{-} = {} & \sqrt{1+\frac{\epsilon}{2}}\left[B\cos\left(\frac{\omega_c}{2}\sqrt{1+\frac{\epsilon}{2}}\,t\right) - A\sin\left(\frac{\omega_c}{2}\sqrt{1+\frac{\epsilon}{2}}\,t\right)\right] \\ & + \frac{1}{\sqrt{1-\frac{\epsilon}{2}}}\left[D\cos\left(\frac{\omega_c}{2}\sqrt{1-\frac{\epsilon}{2}}t\right) - C\sin\left(\frac{\omega_c}{2}\sqrt{1-\frac{\epsilon}{2}}\,t\right)\right]. \end{aligned} \tag{48b}$$

Therefore, one recovers the same motion in CPP GW (35).

Conclusions

- There exist similarities between quite distant systems like GWs and ion traps.
- The motion of test particles in both systems boils down to that of anisotropic, time dependent oscillators.
- The Bargmann lift of a Paul trap is a LPP GW.
- Analytical solutions for the motion in CPP GW is found.
- The Bargmann lift of the Penning trap does not correspond to a CPP GW but its modified version does.
- Associated Sturm-Liouville problem is solved.
- Similar arguments apply for the stability of Lagrange points in celestial mechanics.